Practice Problems I

Math. 262, Spring 2024

Problem 1.

(a) For the system

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 4 \\ 3x_1 + 6x_2 + 5x_3 + 10x_4 &= 0 \\ 5x_1 + 10x_2 + 7x_3 + 17x_4 &= 23 \end{aligned} \tag{1}$$

write down the augmented matrix

(b) Use Gauss elimination to obtain an equivalent system to (1) whose coefficient matrix is in the reduced row echelon form. Indicate **ALL** row operations.

(c) Is the system (1) consistent or inconsistent. If it is consistent, find its solution set.

Solution

(a) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 3 & 6 & 5 & 10 & 0 \\ 5 & 10 & 7 & 17 & 23 \end{array}\right]$$

(b)

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 3 & 6 & 5 & 10 & 0 \\ 5 & 10 & 7 & 17 & 23 \end{array}\right] \begin{array}{l} \rho_2 - 3\rho_1 \to \rho_2 \\ \rho_3 - 5\rho_1 \to \rho_3 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 0 & 2 & 7 & -12 \\ 0 & 0 & 2 & 12 & 3 \end{array}\right] \ (1/2)\rho_2 \to \rho_2 \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7/2 & -6 \\ 0 & 0 & 2 & 12 & 3 \end{array}\right] \ \rho_3 - 2\rho_2 \to \rho_3$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7/2 & -6 \\ 0 & 0 & 0 & 5 & 15 \end{array}\right] \ (1/5)\rho_3 \to \rho_3 \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7/2 & -6 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \ \rho_1 - \rho_2 \to \rho_1 \left[\begin{array}{cccc|c} 1 & 2 & 0 & -5/2 & 10 \\ 0 & 0 & 1 & 7/2 & -6 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} \rho_1 + (5/2)\rho_3 \to \rho_1 \\ \rho_2 - (7/2)\rho_3 \to \rho_2 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 35/2 \\ 0 & 0 & 1 & 0 & -33/2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \longleftarrow \text{reduced row echelon form}$$

(c) The system is consistent. The variables x_1, x_3, and x_4 are lead variables and x_2 is the free variable. Denoting $x_2 = t$, the solution is $x_1 = \frac{35}{2} - 2t, x_2 = t, x_3 = -\frac{33}{2}, x_4 = 3$, or the solution set is

$$\left\{ \left(\frac{35}{2} - 2t, t, -\frac{33}{2}, 3 \right) : t \in \mathbb{R} \right\}.$$

Problem 2.

(a) For the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 1 \\ 2x_1 + 4x_2 &= 5 \\ 3x_1 + 3x_2 &= 6 \end{aligned} \tag{2}$$

use Gauss-Jordan elimination to obtain an equivalent system to (2) whose augmented matrix is in reduced row echelon form. Indicate **ALL** row operations.

(b) Is the system (2) consistent or inconsistent. If it is consistent, find its solution set.

Solution

(a) The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right].$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right] \begin{array}{l} \rho_2 + \rho_1 \to \rho_2 \\ \rho_3 - 2\rho_1 \to \rho_3 \\ \rho_4 - 3\rho_1 \to \rho_4 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{array}\right] \ \rho_3 - \rho_1 \to \rho_3 \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right] \ (1/2)\rho_2 \to \rho_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right] \quad \rho_4 - 3\rho_3 \to \rho_4 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \rho_1 - \rho_2 \to \rho_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \rho_1 - \rho_3 \to \rho_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \longleftarrow \text{reduced row echelon form}$$

(b) The third row of the reduced echelon form corresponds to the equation $0 \cdot x_1 + 0 \cdot x_2 = 1$, thus the system is inconsistent.

Problem 3.

(a) For the system

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= -1 \\ 5x_1 - 12x_2 - 6x_3 &= 3 \end{aligned} \tag{3}$$

use Gauss-Jordan elimination to obtain an equivalent system to (3) whose augmented matrix is in reduced row echelon form.

(b) Is the system (3) consistent or inconsistent. If it is consistent, find its solution set.

Solution

(a) The augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ -2 & 5 & 5 & -1 \\ 5 & -12 & -6 & 3 \end{array}\right]$. The reduced row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 30 & 3 \\ 0 & 1 & 13 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

(b) The system is consistent. The variables x_1 and x_2 are lead variables and x_3 is the free variable. Denoting $x_3 = t$, the solution is $x_1 = 3 - 30t, x_2 = 1 - 13tx_3 = t$, or the solution set is

$$\{(3 - 30t, 1 - 13t, t) : t \in \mathbb{R}\}.$$

Problem 4.

The augmented matrix in reduced row echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find all solutions.

Solution

For the augmented matrix in its reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

the lead variables are x_1 and x_3. The free variables are x_2, x_4 and x_5. Denoting $x_2 = t$, $x_4 = s$, and $x_5 = u$, the solution is $x_1 = -2 - 2t - 3s - u, x_2t, x_3 = 5 - 2s - 4u, x_4 = s, x_5 = u$, or the solution set is

$$\{(-2 - 2t - 3s - u, t, 5 - 2s - 4u, s, u) : t \in \mathbb{R}, s \in \mathbb{R}, u \in \mathbb{R}\}.$$

Problem 5.

(a) For the system

$$\begin{aligned} 3x_1 + 4x_2 - x_3 &= 8 \\ 6x_1 + 8x_2 - 2x_3 &= 3 \end{aligned} \tag{4}$$

use Gauss-Jordan elimination to transform the augmented matrix of system (4) to its reduced row echelon form.

(b) Is the system (4) consistent or inconsistent? If the system (4) is consistent find the solution.

Solution

(a)

$$\left[\begin{array}{ccc|c} 3 & 4 & -1 & 8 \\ 6 & 8 & -2 & 3 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{reduced row echelon form}$$

(b) The second row of the reduced row echelon form corresponds to the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$, thus the system is inconsistent.

Problem 6.

a) For the system

$$\begin{aligned} x_2 + 2x_4 + 3x_5 &= 0 \\ 4x_4 + 8x_5 &= 3 \end{aligned} \tag{5}$$

use Gauss-Jordan elimination to transform the augmented matrix of system (5) to its reduced row echelon form.

(b) Is the system (5) consistent or inconsistent? If the system (5) is consistent find the solution.

Solution

(a)

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 & 8 & 3 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & -3/2 \\ 0 & 0 & 0 & 1 & 2 & 3/4 \end{array}\right] \longleftarrow \text{reduced row echelon form}$$

(b) Consistent. Leading variables: x_2 and x_4. Free variables: x_1, x_3, and x_5. With $x_1 = \alpha$, $x_3 = \beta$, and $x_5 = \gamma$, the solution is $\{(\alpha, -3/2 + \gamma, 3/4 - 2\beta, \gamma)\}$.

Problem 7.

Consider the system

$$\begin{aligned} y + 2kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2z &= 1, \end{aligned} \tag{6}$$

where k is an arbitrary constant.

(a) For what values of k does (6) have unique solution?
(b) When is there no solution?
(c) When are there infinitely many solutions?

Solution

The reduced row echelon of the augmented matrix for $k \neq 1$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/(k-1) \\ 0 & 1 & 0 & k/(k-1) \\ 0 & 0 & 1 & -1/[2(k-1)] \end{array}\right]$$

and the reduced row echelon form of the augmented matrix for $k = 1$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

(a) For $k \neq 1$ system (6) has a unique solution given by $x = 1/(k-1), y = k/(k-1), z = -1/[2(k-1)]$.

(b) For $k = 1$ there is no solution.

(c) The system (6) does not have infinite solutions.

Problem 8.

Consider the system

$$\begin{aligned} 2x + y &= C \\ 3y + z &= C \\ x + 4z &= C, \end{aligned} \tag{7}$$

where C is a constant. Find the smallest positive integer C such that xy, and z are all integers.

Solution

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9C/25 \\ 0 & 1 & 0 & 7C/25 \\ 0 & 0 & 1 & 4C/25 \end{array}\right]$$

The solution $x = 9C/25$, $y = 7C/25$, and $z = 4C/25$ are all integers if C is a multiple of 25. The smallest such value is $C = 25$.

Problem 9.

Find the values α such that the system

$$\begin{aligned} \alpha x_1 + x_2 + x_3 &= 1 \\ x_1 + \alpha x_2 + x_3 &= 1 \\ x_1 + x_2 + \alpha x_3 &= 1 \end{aligned} \tag{8}$$

has
(a) no solution,
(b) one solution,
(c) infinitely many solutions.

Solution

The augmented matrix is

$$\left[\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \end{array}\right]$$

Using elementary row operations we obtain

$$\left[\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \end{array}\right] \begin{array}{l} \rho_1 \leftrightarrow \rho_3 \\ \rho_2 - \rho_1 \to \rho_2 \\ \rho_3 - \alpha\rho_1 \to \rho_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha - 1 & 1 - \alpha & 0 \\ 0 & 1 - \alpha & 1 - \alpha^2 & 1 - \alpha \end{array}\right] \tag{9}$$

If $\alpha = 1$ then system (8) has infinitely many solutions, with x_1 the leading variable, while x_2 and x_3 being free variables.

Now assume that $\alpha \neq 1$ and reduce further to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha - 1 & 1 - \alpha & 0 \\ 0 & 1 - \alpha & 1 - \alpha^2 & 1 - \alpha \end{array}\right] \quad \rho_3 + \rho_2 \to \rho_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha - 1 & 1 - \alpha & 0 \\ 0 & 0 & (1 - \alpha)(2 + \alpha) & 1 - \alpha \end{array}\right] \tag{10}$$

noting that $1 - \alpha^2 + 1 - \alpha = 2 - \alpha - \alpha^2 = (1 - \alpha)(2 + \alpha)$.

Thus, if $\alpha \neq -2$ and $\alpha \neq 1$, system (8) has a unique solution given by $x_1 = x_2 = x_3 = 1/(\alpha + 2)$ (*Check it!*)

Summary

(a) $\alpha = -2$: no solutions; The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(b) $\alpha \neq 1$ and $\alpha \neq -2$: unique solution; The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/(a + 2) \\ 0 & 1 & 0 & 1/(a + 2) \\ 0 & 0 & 1 & 1/(a + 2) \end{array}\right]$$

(c) $\alpha = 1$: infinitely many solutions. The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$