17.8 Divergence Theorem

Vector fields can represent electric or magnetic fields, air velocities in hurricanes, or blood flow in an artery. These and other vector phenomena suggest movement of a "substance." A frequent question concerns the amount of a substance that flows across a surface—for example, the amount of water that passes across the membrane of a cell per unit time. Such flux calculations may be done using flux integrals as in Section 17.6. The Divergence Theorem offers an alternative method. In effect, it says that instead of integrating the flow in and out of a region across its boundary, you may also add up all the sources (or sinks) of the flow throughout the region.

Note »

$$\begin{aligned}\text{Circulation form of Green's Theorem} &\longrightarrow \text{Stokes' Theorem}\\ \text{Flux form of Green's Theorem} &\longrightarrow \text{Divergence Theorem}\end{aligned}$$

Divergence Theorem »

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem. Recall that if R is a region in the xy-plane, C is the simple closed piecewise-smooth oriented boundary of R, and $\mathbf{F} = \langle f, g\rangle$ is a vector field, Green's Theorem says that

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\, ds}_{\text{flux across } C} = \iint_R \underbrace{\left(f_x + g_y\right)}_{\text{divergence}} dA.$$

The line integral on the left gives the flux across the boundary of R. The double integral on the right measures the net expansion or contraction of the vector field within R. If $\mathbf{F}$ represents a fluid flow or the transport of a material, the theorem says that the cumulative effect of the sources (or sinks) of the flow within R equals the net flow across its boundary.

The Divergence Theorem is a direct extension of Green's Theorem. The plane region in Green's Theorem becomes a solid region D in $\mathbb{R}^3$, and the closed curve in Green's Theorem becomes the oriented surface S that encloses D. The flux integral in Green's Theorem becomes a surface integral over S, and the double integral in Green's Theorem becomes a triple integral over D of the three-dimensional divergence (**Figure 17.68**).

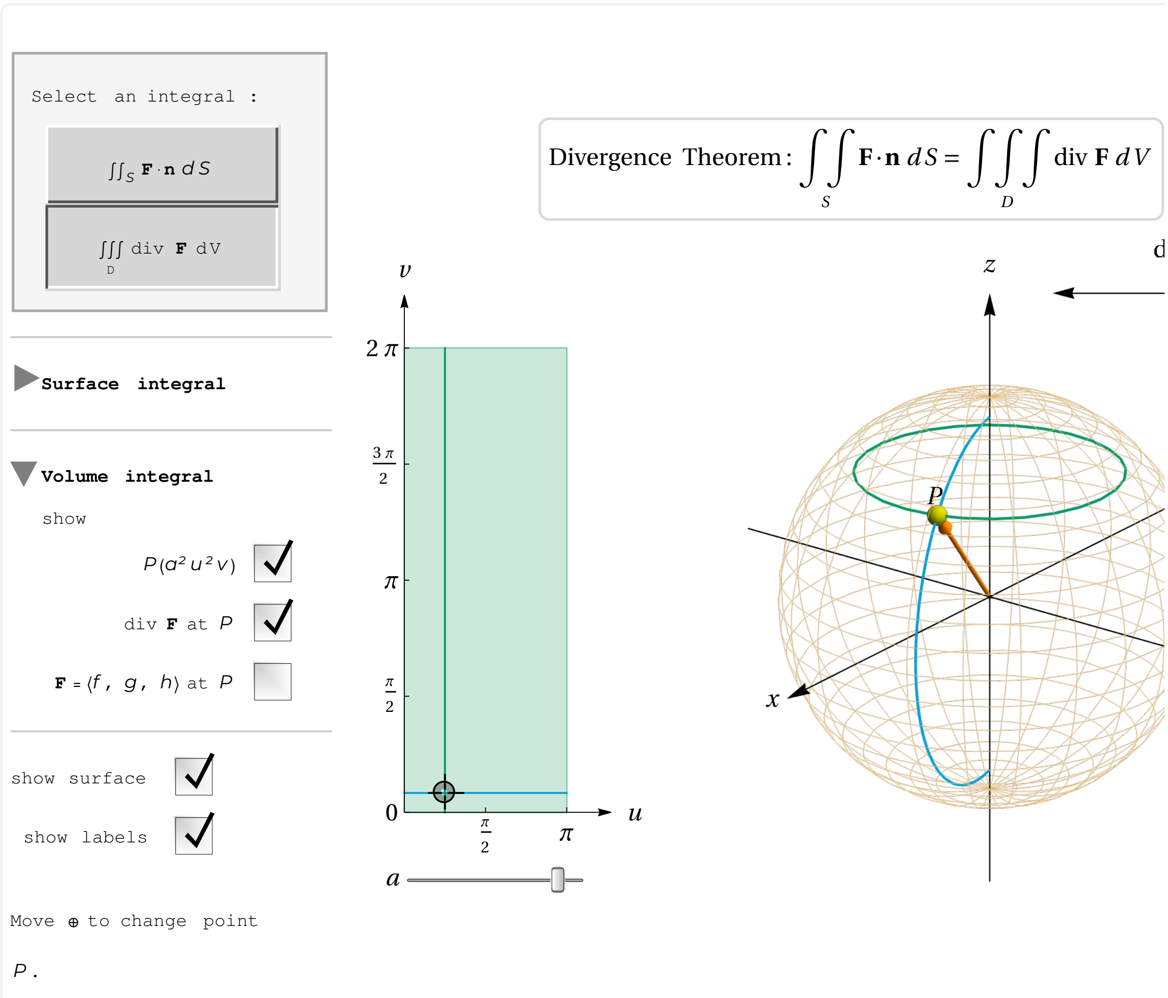


Figure 17.68

THEOREM 17.17 Divergence Theorem

Let **F** be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

where **n** is the outward unit normal vector on S.

The surface integral on the left gives the flux of the vector field across the boundary; a positive flux integral means there is a net flow of the field out of the region. The triple integral on the right is the cumulative expansion or contraction of the field over the region D. The proof of a special case of the theorem is given later in this section.

Quick Check 1 Interpret the Divergence Theorem in the case that $\mathbf{F} = \langle a, b, c \rangle$ is a constant vector field and D is a ball. ◆

Answer »

If $\mathbf{F}$ is constant, then div $(\mathbf{F}) = 0$, so $\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$. This means that all the "material" that flows into one side of D flows out of the other side of D.

EXAMPLE 1 Verifying the Divergence Theorem

Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

SOLUTION »

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$$

Integrating over D, we have

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = 3 \times \text{volume of } D = 3 \cdot \frac{4}{3} \pi a^3 = 4 \pi a^3.$$

To evaluate the surface integral, we parametrize the sphere (Section 17.6, Table 17.3) in the form

$$\mathbf{r} = \langle x, y, z \rangle = \langle a \sin u \cos v,\ a \sin u \sin v,\ a \cos u \rangle,$$

where $R = \{(u, v) : 0 \le u \le \pi,\ 0 \le v \le 2\pi\}$ (u and v are the spherical coordinates φ and θ, respectively). The surface integral is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

where the required vector normal to the surface is

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \right\rangle.$$

Substituting for $\mathbf{F} = \langle x, y, z \rangle$ and $\mathbf{t}_u \times \mathbf{t}_v$, we find after simplifying that $\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = a^3 \sin u$. Therefore, the surface integral becomes

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \underbrace{\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)}_{a^3 \sin u}\, dA \\ &= \int_0^{2\pi} \int_0^{\pi} a^3 \sin u\, du\, dv && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{t}_u \times \mathbf{t}_v. \\ &= 4 \pi a^3. && \text{Evaluate integrals.} \end{aligned}$$

The two integrals of the Divergence Theorem are equal.

Note »

See Exercise 32 for an alternative evaluation of the surface integral.

Related Exercise 9 ◆

EXAMPLE 2 Divergence Theorem with a rotation field

Consider the rotation field

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle 1, 0, 1 \rangle \times \langle x, y, z \rangle = \langle -y, x - z, y \rangle.$$

Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, together with its base in the xy-plane. Find the net outward flux across S.

SOLUTION »

To find the flux using surface integrals, two surfaces must be considered (the hemisphere and its base). The Divergence Theorem gives a simpler solution. Note that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y) = 0.$$

We see that the flux across the hemisphere is zero.

Related Exercise 13 ◆

With Stokes' Theorem, rotation fields are noteworthy because they have a nonzero curl. With the Divergence Theorem, the situation is reversed. As suggested by Example 2, pure rotation fields of the form $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ have zero divergence (Exercise 16). However, with the Divergence Theorem, radial fields are interesting and have many physical applications.

EXAMPLE 3 Computing flux with the Divergence Theorem

Find the net outward flux of the field $\mathbf{F} = xyz\,\langle 1, 1, 1 \rangle$ across the boundaries of the cube $D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$.

SOLUTION »

Computing a surface integral involves the six faces of the cube. The Divergence Theorem gives the outward flux with a single integral over D. The divergence of the field is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz) = yz + xz + xy.$$

The integral over D is a standard triple integral:

$$\begin{aligned}
\iiint_D \nabla \cdot \mathbf{F}\, dV &= \iiint_D (yz + xz + xy)\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\, dx\, dy\, dz && \text{Convert to a triple integral.} \\
&= \frac{3}{4}. && \text{Evaluate integrals.}
\end{aligned}$$

On three faces of the cube (those that lie in the coordinate planes), we see that $\mathbf{F}(0, y, z) = \mathbf{F}(x, 0, z) = \mathbf{F}(x, y, 0) = \mathbf{0}$, so there is no contribution to the flux on these faces (**Figure 17.69**). On the other three faces, the vector field has components out of the cube. Therefore, the net outward flux is positive, as calculated.

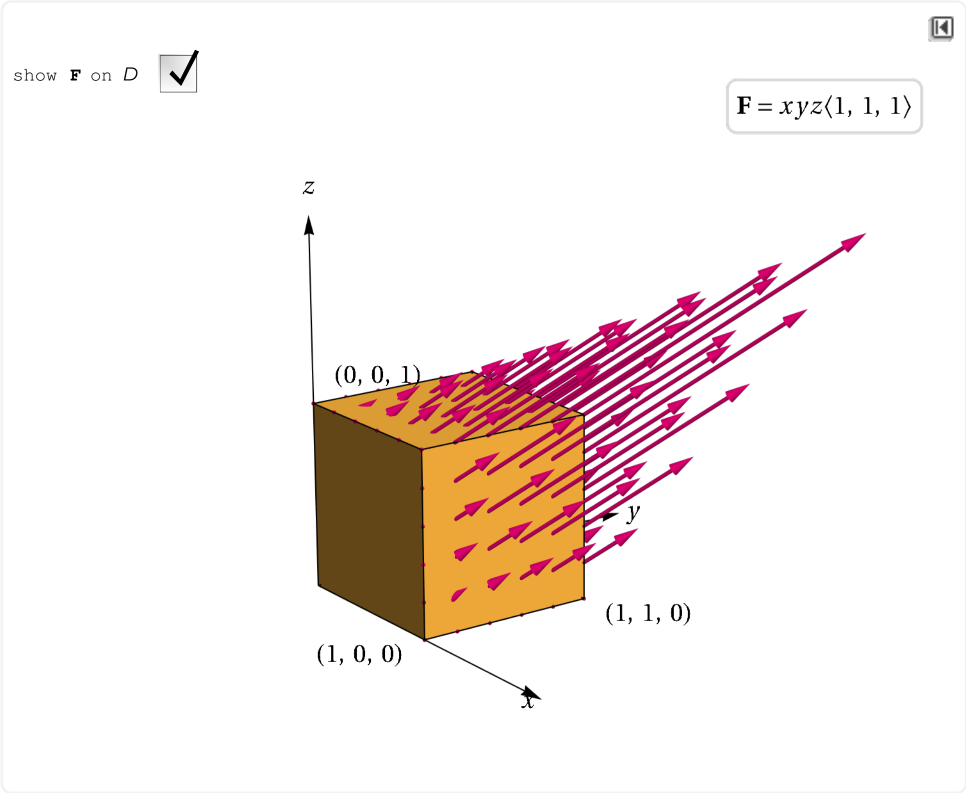


Figure 17.69

Related Exercises 18–19 ◆

Quick Check 2 In Example 3, does the vector field have negative components anywhere in the cube D? Is the divergence negative anywhere in D? ◆

Answer »

> The vector field and the divergence are positive throughout D.

Interpretation of the Divergence Using Mass Transport

Suppose $\mathbf{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density. The vector field $\mathbf{F} = \rho\,\mathbf{v} = \langle f, g, h\rangle$ describes the **mass transport** of the material, with units of (mass / vol) × (length / time) = mass / (area · time); typical units of mass transport are $\mathrm{g}/\mathrm{m}^2/\mathrm{s}$. This means that $\mathbf{F}$ gives the mass of material flowing past a point (in each of the three coordinate directions) per unit of surface area per unit of time. When $\mathbf{F}$ is multiplied by an area, the result is the **flux**, with units of mass / unit time.

Note »

> The mass transport is also called the *flux density*; when multiplied by an area, it gives the flux. We use the convention that flux has units of mass per unit time.

Now consider a small cube located in the vector field with its faces parallel to the coordinate planes. One vertex is located at $(0, 0, 0)$, the opposite vertex is at $(\Delta x, \Delta y, \Delta z)$, and (x, y, z) is an arbitrary point in the cube (**Figure 17.70**). The goal is to compute the approximate flux of material across the faces of the cube. We begin with the flux across the two parallel faces $x = 0$ and $x = \Delta x$.

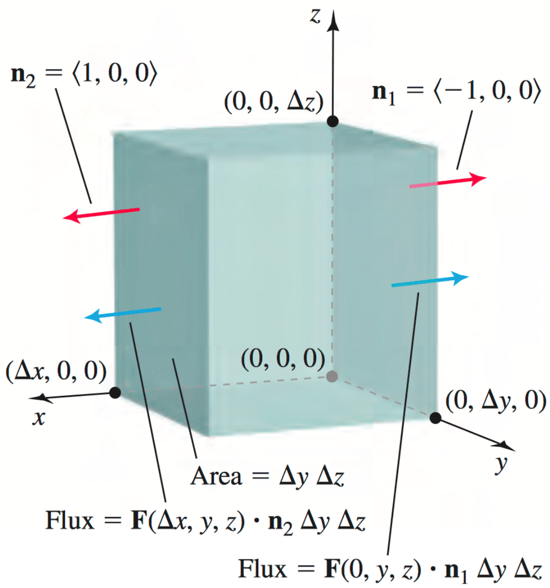


Figure 17.70

The outward unit vectors normal to the faces $x = 0$ and $x = \Delta x$ are $\mathbf{n}_1 = \langle -1, 0, 0\rangle$ and $\mathbf{n}_2 = \langle 1, 0, 0\rangle$, respectively. Each face has area $\Delta y\, \Delta z$, so the approximate net flux across these faces is

$$\underbrace{\mathbf{F}(\Delta x, y, z)}_{x=\Delta x \text{ face}} \cdot \underbrace{\mathbf{n}_2}_{\langle 1,0,0\rangle} \Delta y\, \Delta z + \underbrace{\mathbf{F}(0, y, z)}_{x=0 \text{ face}} \cdot \underbrace{\mathbf{n}_1}_{\langle -1,0,0\rangle} \Delta y\, \Delta z = (f(\Delta x, y, z) - f(0, y, z))\, \Delta y\, \Delta z.$$

Note »

Check the units: if $\mathbf{F}$ has units of mass / (area · time), then the flux has units of mass / time ($\mathbf{n}$ has no units).

Note that if $f(\Delta x, y, z) > f(0, y, z)$, the net flux across these two faces of the cube is positive, which means the net flow is *out* of the cube. Letting $\Delta V = \Delta x\, \Delta y\, \Delta z$ be the volume of the cube, we rewrite the net flux as

$$\begin{aligned}(f(\Delta x, y, z) - f(0, y, z))\, \Delta y\, \Delta z &= \frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}\, \Delta x\, \Delta y\, \Delta z \quad \text{Multiply by } \frac{\Delta x}{\Delta x}.\\ &= \frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}\, \Delta V. \qquad \Delta V = \Delta x\, \Delta y\, \Delta z\end{aligned}$$

A similar argument can be applied to the other two pairs of faces. The approximate net flux across the faces $y = 0$ and $y = \Delta y$ is

$$\frac{g(x, \Delta y, z) - g(x, 0, z)}{\Delta y}\, \Delta V,$$

and the approximate net flux across the faces $z = 0$ and $z = \Delta z$ is

$$\frac{h(x, y, \Delta z) - h(x, y, 0)}{\Delta z}\, \Delta V.$$

Adding these three individual fluxes gives the approximate net flux out of the cube:

$$\text{net flux out of cube} \approx \left(\underbrace{\frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}}_{\approx \frac{\partial f}{\partial x}(0,0,0)} + \underbrace{\frac{g(x, \Delta y, z) - g(x, 0, z)}{\Delta y}}_{\approx \frac{\partial g}{\partial y}(0,0,0)} + \underbrace{\frac{h(x, y, \Delta z) - h(x, y, 0)}{\Delta z}}_{\approx \frac{\partial h}{\partial z}(0,0,0)} \right) \Delta V$$
$$\approx \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right)\Bigg|_{(0,0,0)} \Delta V$$
$$= (\nabla \cdot \mathbf{F})(0, 0, 0)\, \Delta V.$$

Notice how the three quotients approximate partial derivatives when Δx, Δy, and Δz are small. A similar argument may be made at any point in the region.

Taking one more step, we show informally how the Divergence Theorem arises. Suppose the small cube we just analyzed is one of many small cubes of volume ΔV that fill a region D. We label the cubes $k = 1, \ldots, n$ and apply the preceding argument to each cube, letting $(\nabla \cdot \mathbf{F})_k$ be the divergence evaluated at a point in the kth cube. Adding the individual contributions to the net flux from each cube, we obtain the approximate net flux across the boundary of D:

$$\text{net flux out of } D \approx \sum_{k=1}^{n} (\nabla \cdot \mathbf{F})_k \, \Delta V.$$

Note »

> In making this argument, notice that for two adjacent cubes the flux into one cube equals the flux out of the other cube across the common face. Thus, there is a cancellation of fluxes throughout the interior of D.

Letting the volume of the cubes ΔV approach 0 and letting the number of cubes n increase, we obtain an integral over D:

$$\text{net flux out of } D = \lim_{n \to \infty} \sum_{k=1}^{n} (\nabla \cdot \mathbf{F})_k \, \Delta V = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

The net flux across the boundary of D is also given by $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Equating the surface integral and the volume integral gives the Divergence Theorem. Now we look at a formal proof.

Quick Check 3 Draw the unit cube $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ and sketch the vector field $\mathbf{F} = \langle x, -y, 2z \rangle$ on the six faces of the cube. Compute and interpret div $\mathbf{F}$. ◆

Answer »

> The vector field has no flow into or out of the cube on the faces $x = 0$, $y = 0$, and $z = 0$ because the vectors of $\mathbf{F}$ on these faces are parallel to the faces. The vector field points out of the cube on the $x = 1$ and $z = 1$ faces and into the cube on the $y = 1$ face. div $(\mathbf{F}) = 2$, so there is a net flow out of the cube.

Proof of the Divergence Theorem »

We prove the Divergence Theorem under special conditions on the region D. Let R be the projection of D in the xy-plane (**Figure 17.71**); that is

$$R = \{(x, y) : (x, y, z) \text{ is in } D\}.$$

Assume the boundary of D is S and let $\mathbf{n}$ be the unit vector normal to S that points outward.

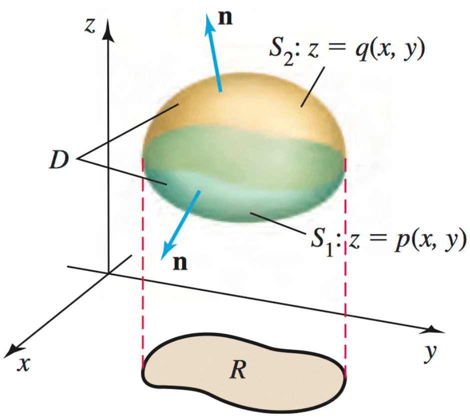


Figure 17.71

Letting $\mathbf{F} = \langle f, g, h\rangle = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$, the surface integral in the Divergence Theorem is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k})\cdot\mathbf{n}\,dS$$
$$= \iint_S f\,\mathbf{i}\cdot\mathbf{n}\,dS + \iint_S g\,\mathbf{j}\cdot\mathbf{n}\,dS + \iint_S h\,\mathbf{k}\cdot\mathbf{n}\,dS.$$

The volume integral in the Divergence Theorem is

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dV.$$

Matching terms of the surface and volume integrals, the theorem is proved by showing that

$$\iint_S f\,\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial f}{\partial x}\,dV, \tag{1}$$

$$\iint_S g\,\mathbf{j}\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial g}{\partial y}\,dV, \text{ and} \tag{2}$$

$$\iint_S h\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial h}{\partial z}\,dV. \tag{3}$$

We work on equation (3) assuming special properties for D. Suppose D is bounded by two surfaces $S_1; z = p(x, y)$ and $S_2 : z = q(x, y)$, where $p(x, y) \le q(x, y)$ on R (Figure 17.71). The Fundamental Theorem of Calculus is used in the triple integral to show that

$$\iiint_D \frac{\partial h}{\partial z}\, dV = \iint_R \int_{p(x,y)}^{q(x,y)} \frac{\partial h}{\partial z}\, dz\, dx\, dy$$
$$= \iint_R (h(x, y, q(x, y)) - h(x, y, p(x, y)))\, dx\, dy. \quad \text{Evaluate inner integral.}$$

Now let's turn to the surface integral in equation (3), $\iint_S h\,\mathbf{k}\cdot\mathbf{n}\, dS$, and note that S consists of three pieces: the lower surface S_1, the upper surface S_2, and the vertical sides S_3 of the surface (if they exist). The normal to S_3 is everywhere orthogonal to $\mathbf{k}$, so $\mathbf{k}\cdot\mathbf{n} = 0$ and the S_3 integral makes no contribution. What remains is to compute the surface integrals over S_1 and S_2.

The required outward normal to S_2 (which is the graph of $z = q(x, y)$) is $\langle -q_x, -q_y, 1\rangle$. The outward normal to S_1 (which is the graph of $z = p(x, y)$) points *downward*, so it is given by $\langle p_x, p_y, -1\rangle$. The surface integral of (3) becomes

$$\iint_S h\,\mathbf{k}\cdot\mathbf{n}\, dS = \iint_{S_2} h(x, y, z)\,\mathbf{k}\cdot\mathbf{n}\, dS + \iint_{S_1} h(x, y, z)\,\mathbf{k}\cdot\mathbf{n}\, dS$$
$$= \iint_R h(x, y, q(x, y))\,\underbrace{\mathbf{k}\cdot\langle -q_x, -q_y, 1\rangle}_{1}\, dx\, dy$$
$$+ \iint_R h(x, y, p(x, y))\,\underbrace{\mathbf{k}\cdot\langle p_x, p_y, -1\rangle}_{-1}\, dx\, dy$$
$$= \iint_R h(x, y, q(x, y))\, dx\, dy - \iint_R h(x, y, p(x, y))\, dx\, dy. \quad \text{Simplify.}$$

Observe that both the volume integral and the surface integral of (3) reduce to the same integral over R. Therefore, $\iint_S h\,\mathbf{k}\cdot\mathbf{n}\, dS = \iiint_D \frac{\partial h}{\partial z}\, dV$.

Equations (1) and (2) are handled in a similar way.

- To prove (1), we make the special assumption that D is also bounded by two surfaces, $S_1 : x = s(y, z)$ and $S_2 : x = t(y, z)$, where $s(x, y) \le t(x, y)$.
- To prove (2), we assume that D is bounded by two surfaces, $S_1 : y = u(x, z)$ and $S_2 : y = v(x, z)$, where $u(x, y) \le v(x, y)$.

When combined, equations (1), (2), and (3) yield the Divergence Theorem. ◆

Divergence Theorem for Hollow Regions »

The Divergence Theorem may be extended to more general solid regions. Here we consider the important case of hollow regions. Suppose D is a region consisting of all points inside a closed oriented surface S_2 and outside a closed oriented surface S_1, where S_1 lies within S_2 (**Figure 17.72**). Therefore, the boundary of D consists of S_1 and S_2. (Note that D is simply connected.)

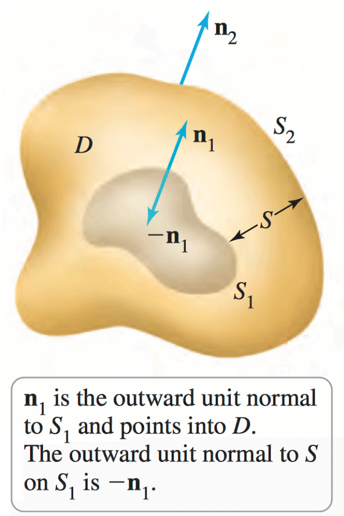


Figure 17.72

We let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Note that $\mathbf{n}_1$ points into D, so the outward normal to S on S_1 is $-\mathbf{n}_1$. With this observation, the Divergence Theorem takes the following form.

THEOREM 17.18 **Divergence Theorem for Hollow Regions**

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D $(S = S_1 \cup S_2)$ and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$

Note »

It's important to point out again that $\mathbf{n}_1$ is the unit normal that we would use for S_1 alone, independent of S. It is the outward unit normal to S_1, but it points into D.

This form of the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields.

EXAMPLE 4 Flux for an inverse square field

Consider the inverse square vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x, y, z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}.$$

a. Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \left\{(x, y, z) : a^2 \le x^2 + y^2 + z^2 \le b^2\right\}$ that lies between concentric spheres with radii a and b.

b. Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.

Note »

Recall that an inverse square force is proportional to $\frac{1}{|\mathbf{r}|^2}$ multiplied by a unit vector in the radial direction, which is $\frac{\mathbf{r}}{|\mathbf{r}|}$. Combining these two factors gives $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$.

SOLUTION »

a. Although the vector field is undefined at the origin, it is defined and differentiable in D, which excludes the origin. In Section 17.5 (Exercise 73) it was shown that the divergence of the radial field $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$ with $p = 3$ is 0. We let S be the union of S_2, the larger sphere of radius b, and S_1, the smaller sphere of radius a. Because $\iiint_D \nabla \cdot \mathbf{F}\, dV = 0$, the Divergence Theorem implies that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\, dS = 0.$$

Therefore, the net flux across S is zero.

b. Part (a) implies that

$$\underbrace{\iint_{S_2} \mathbf{F}\cdot\mathbf{n}_2\, dS}_{\text{out of } D} = \underbrace{\iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\, dS}_{\text{into } D}.$$

We see that the flux out of D across S_2 equals the flux into D across S_1. To find that flux, we evaluate the surface integral over S_1 on which $|\mathbf{r}| = a$. (Because the fluxes are equal, S_2 could also be used.)

The easiest way to evaluate the surface integral is to note that on the sphere S_1, the unit outward normal vector is $\mathbf{n}_1 = \frac{\mathbf{r}}{|\mathbf{r}|}$. Therefore, the surface integral is

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\, dS &= \iint_{S_1} \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\, dS && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{n}_1.\\
&= \iint_{S_1} \frac{|\mathbf{r}|^2}{|\mathbf{r}|^4}\, dS && \mathbf{r}\cdot\mathbf{r} = |\mathbf{r}|^2\\
&= \iint_{S_1} \frac{1}{a^2}\, dS && |\mathbf{r}| = a\\
&= \frac{4\pi a^2}{a^2} && \text{Surface area} = 4\pi a^2\\
&= 4\pi.
\end{aligned}$$

The same result is obtained using S_2 or any smooth surface enclosing the origin. The flux of the inverse square field across any surface enclosing the origin is 4π. As shown in Exercise 46, among radial fields, this property is held only by the inverse square field ($p = 3$).

Related Exercises 26–27 ◆

Gauss' Law »

Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge Q located at the origin is given by the inverse square law,

$$\mathbf{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r} = \langle x, y, z\rangle$ and ϵ_0 is a physical constant called the *permittivity of free space*.

According to the calculation of Example 4, the flux of the field $\frac{\mathbf{r}}{|\mathbf{r}|^3}$ across any surface that encloses the origin is 4π. Therefore, the flux of the electric field across any surface enclosing the origin is $\frac{Q}{4\pi\epsilon_0}\cdot 4\pi = \frac{Q}{\epsilon_0}$ (**Figure 17.73a**). This is one statement of Gauss' Law: If S is a surface that encloses a point charge Q, then the flux of the electric field across S is

$$\iint_S \mathbf{E}\cdot\mathbf{n}\, dS = \frac{Q}{\epsilon_0}.$$

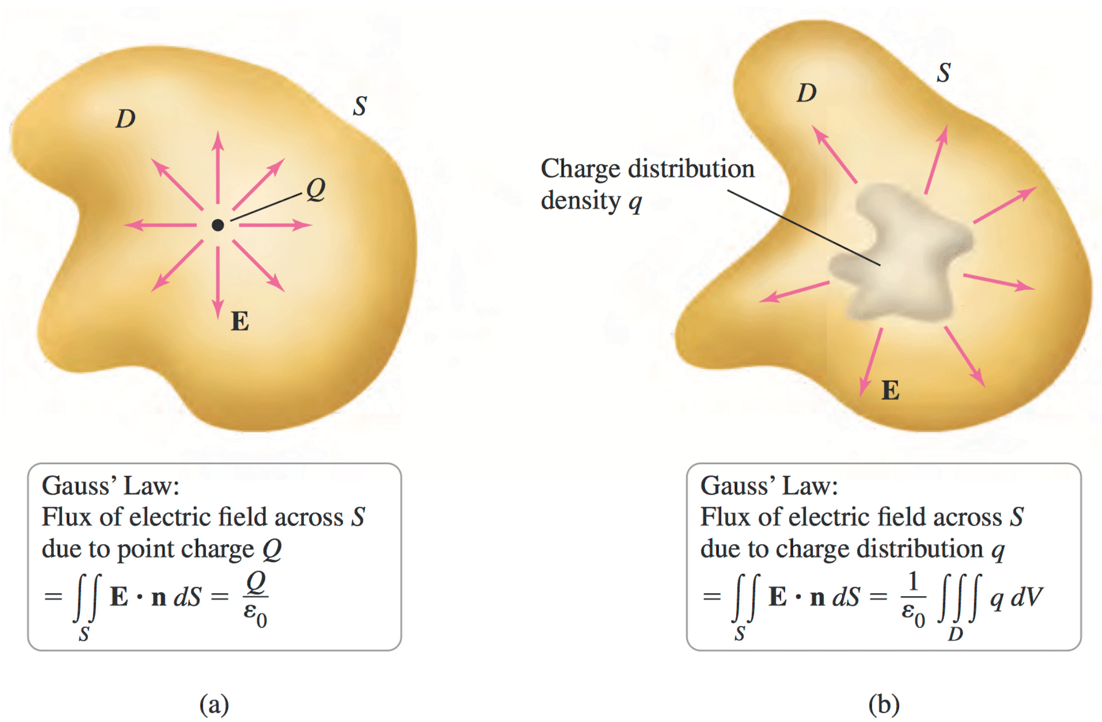


Figure 17.73

In fact, Gauss' Law applies to more general charge distributions (Exercise 39). If $q(x, y, z)$ is a charge density (charge per unit volume) defined on a region D enclosed by S, then the total charge within D is

$Q = \iiint_D q(x, y, z)\, dV$ (**Figure 17.73b**). Replacing Q with this triple integral, Gauss' Law takes the form

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\epsilon_0} \underbrace{\iiint_D q(x, y, z)\, dV}_{Q}.$$

Gauss' Law applies to other inverse square fields. In a slightly different form, it also governs heat transfer. If T is the temperature distribution in a solid body D, then the heat flow vector field is $\mathbf{F} = -k\, \nabla T$. (Heat flows down the temperature gradient.) If $q(x, y, z)$ represents the sources of heat within D, Gauss' Law says

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -k \iint_S \nabla T \cdot \mathbf{n}\, dS = \iiint_D q(x, y, z)\, dV.$$

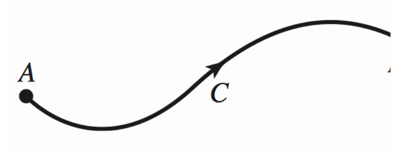
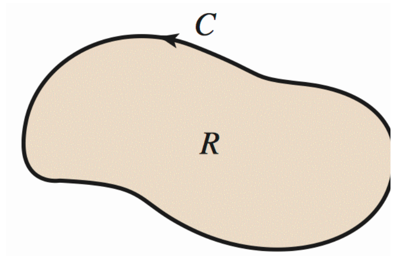
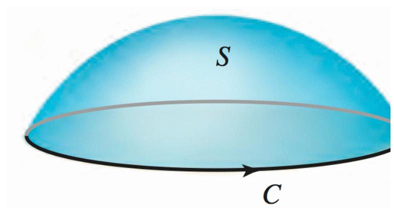
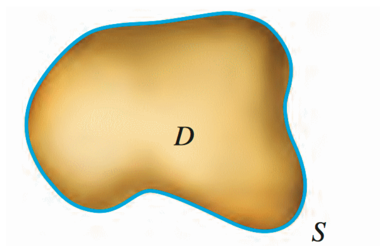
We see that, in general, the flux of material (fluid, heat, electric field lines) across the boundary of a region is the cumulative effect of the sources within the region.

A Final Perspective »

Table 17.4 offers a look at the progression of fundamental theorems of calculus that have appeared throughout this text. Each theorem builds on its predecessors, extending the same basic idea to a different situation or to higher dimensions.

In all cases, the statement is effectively the same: The cumulative (integrated) effect of the *derivatives* of a function throughout a region is determined by the values of the function on the boundary of that region. This principle underlies much of our understanding of the world around us.

Table 17.4

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	
Green's Theorem (Circulation form)	$\int\int_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\int\int\int_D \nabla \cdot \mathbf{F}\,dV = \int\int_S \mathbf{F} \cdot \mathbf{n}\,dS$	

Exercises »

Getting Started »

Practice Exercises »

9–12. Verifying the Divergence Theorem *Evaluate both integrals of the Divergence Theorem for the following vector fields and regions. Check for agreement.*

9. $\mathbf{F} = \langle 2x, 3y, 4z \rangle$; $D = \left\{(x, y, z) : x^2 + y^2 + z^2 \le 4\right\}$

10. $\mathbf{F} = \langle -x, -y, -z \rangle$; $D = \{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$

11. $\mathbf{F} = \langle z - y, x, -x \rangle$; $D = \left\{(x, y, z) : \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \le 1\right\}$

12. $\mathbf{F} = \left\langle x^2, y^2, z^2 \right\rangle$; $D = \{(x, y, z) : |x| \le 1, |y| \le 2, |z| \le 3\}$

13–16. Rotation fields

13. Find the net outward flux of the field $\mathbf{F} = \langle 2z - y, x, -2x \rangle$ across the sphere of radius 1 centered at the origin.

14. Find the net outward flux of the field $\mathbf{F} = \langle z - y, x - z, y - x \rangle$ across the boundary of the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$.

15. Find the net outward flux of the field $\mathbf{F} = \langle bz - cy, cx - az, ay - bx \rangle$ across any smooth closed surface in $\mathbb{R}^3$, where a, b, and c are constants.

16. Find the net outward flux of $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ across any smooth closed surface in $\mathbb{R}^3$, where $\mathbf{a}$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$.

17–24. Computing flux *Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface S.*

17. $\mathbf{F} = \langle x, -2y, 3z \rangle$; S is the sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 6\right\}$.

18. $\mathbf{F} = \left\langle x^2, 2xz, y^2 \right\rangle$; S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

19. $\mathbf{F} = \langle x, 2y, z \rangle$; S is the boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$.

20. $\mathbf{F} = \left\langle x^2, y^2, z^2 \right\rangle$; S is the sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 25\right\}$.

21. $\mathbf{F} = \left\langle y - 2x, x^3 - y, y^2 - z \right\rangle$; S is the sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 4\right\}$.

22. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$; S consists of the faces of the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$.

23. $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the xy-plane.

24. $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$.

25–30. Divergence Theorem for more general regions *Use the Divergence Theorem to compute the net outward flux of the following vector fields across the boundary of the given regions D.*

25. $\mathbf{F} = \langle z - x, x - y, 2y - z \rangle$; D is the region between the spheres of radius 2 and 4 centered at the origin.

26. $\mathbf{F} = \mathbf{r}\,|\mathbf{r}| = \langle x, y, z \rangle \sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

27. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

28. $\mathbf{F} = \langle z - y, x - z, 2y - x \rangle$; D is the region between two cubes: $\{(x, y, z) : 1 \le |x| \le 3, 1 \le |y| \le 3, 1 \le |z| \le 3\}$.

29. $\mathbf{F} = \langle x^2, -y^2, z^2 \rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$.

30. $\mathbf{F} = \langle x, 2y, 3z \rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, for $0 \le z \le 8$.

31. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

- **a.** If $\nabla \cdot \mathbf{F} = 0$ at all points of a region D, then $\mathbf{F} \cdot \mathbf{n} = 0$ at all points of the boundary of D.
- **b.** If $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$ on all closed surfaces in $\mathbb{R}^3$, then $\mathbf{F}$ is constant.
- **c.** If $|\mathbf{F}| < 1$, then $\left| \int\int\int_D \nabla \cdot \mathbf{F}\, dV \right|$ is less than the area of the surface of D.

32. **Flux across a sphere** Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere of radius a centered at the origin. Compute the outward flux of $\mathbf{F}$ across S using the representation $z = \pm\sqrt{a^2 - x^2 - y^2}$ for the sphere (either symmetry or two surfaces must be used).

33–35. Flux integrals *Compute the outward flux of the following vector fields across the given surfaces S. You should decide which integral of the Divergence Theorem to use.*

33. $\mathbf{F} = \langle x^2 e^y \cos z, -4x e^y \cos z, 2x e^y \sin z \rangle$; S is the boundary of the ellipsoid $\dfrac{x^2}{4} + y^2 + z^2 = 1$.

34. $\mathbf{F} = \langle -yz, xz, 1 \rangle$; S is the boundary of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + z^2 = 1$.

35. $\mathbf{F} = \langle x \sin y, -\cos y, z \sin y \rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \dfrac{\pi}{2}$, $z = 0$, and $z = x$.

36. **Radial fields** Consider the radial vector field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{p/2}}$. Let S be the sphere of radius a centered at the origin.

- **a.** Use a surface integral to show that the outward flux of $\mathbf{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to the sphere is $\dfrac{\mathbf{r}}{|\mathbf{r}|}$.
- **b.** For what values of p does $\mathbf{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact (Theorem 17.10) that $\nabla \cdot \mathbf{F} = \dfrac{3-p}{|\mathbf{r}|^p}$ to compute the flux across S using the Divergence Theorem.

37. **Singular radial field** Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}}$.

- **a.** Evaluate a surface integral to show that $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b. Note that the first partial derivatives of the components of **F** are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (a) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate div **F** over the region between two spheres of radius a and $0 < \epsilon < a$. Then let $\epsilon \to 0^+$ to obtain the flux computed in part (a).

38. Logarithmic potential Consider the potential function $\varphi(x, y, z) = \frac{1}{2} \ln\left(x^2 + y^2 + z^2\right) = \ln |\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z \rangle$.

a. Show that the gradient field associated with φ is $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^2} = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$.

b. Show that $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a$, where S is the surface of a sphere of radius a centered at the origin.

c. Compute div **F**.

d. Note that **F** is undefined at the origin, so the Divergence Theorem does not apply directly. Evaluate the volume integral as described in Exercise 37.

39. Gauss' Law for electric fields The electric field due to a point charge Q is $\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = \langle x, y, z \rangle$, and ϵ_0 is a constant.

a. Show that the flux of the field across a sphere of radius a centered at the origin is $\int\int_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0}$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b, respectively, with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of charge within a region D. Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\int\int_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\epsilon_0} \int\int\int_D q(x, y, z)\, dV.$$

d. Assuming **E** satisfies the conditions of the Divergence Theorem on D, conclude from part (c) that $\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}$.

e. Because the electric force is conservative, it has a potential function φ. From part (d), conclude that $\nabla^2 \varphi = \nabla \cdot \nabla \varphi = \frac{q}{\epsilon_0}$.

40. Gauss' Law for gravitation The gravitational force due to a point mass M at the origin is proportional to $\mathbf{F} = \frac{GM\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = \langle x, y, z \rangle$ and G is the gravitational constant.

a. Show that the flux of the force field across a sphere of radius a centered at the origin is $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi GM$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b, respectively, with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of mass within a region D. Let $\rho(x, y, z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi G \iiint_D \rho(x, y, z)\, dV.$$

d. Assuming $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on D, conclude from part (c) that $\nabla \cdot \mathbf{F} = 4\pi G \rho$.

e. Because the gravitational force is conservative, it has a potential function φ. From part (d), conclude that $\nabla^2 \varphi = 4\pi G \rho$.

41–45. Heat transfer *Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector* $\mathbf{F}$ *at a point is proportional to the negative gradient of the temperature; that is,* $\mathbf{F} = -k \nabla T$*, which means that heat energy flows from hot regions to cold regions. The constant* $k > 0$ *is called the conductivity, which has metric units of J/(m-s-K). A temperature function for a region D is given. Find the net outward heat flux* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -k \iint_S \nabla T \cdot \mathbf{n}\, dS$ *across the boundary S of D. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume* $k = 1$.

41. $T(x, y, z) = 100 + x + 2y + z$; $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

42. $T(x, y, z) = 100 + x^2 + y^2 + z^2$; $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

43. $T(x, y, z) = 100 + e^{-z}$; $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

44. $T(x, y, z) = 100 + x^2 + y^2 + z^2$; D is the unit sphere centered at the origin.

T **45.** $T(x, y, z) = 100\, e^{-x^2 - y^2 - z^2}$; D is the sphere of radius a centered at the origin.

Explorations and Challenges »

46. Inverse square fields are special Let $\mathbf{F}$ be a radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number and $\mathbf{r} = \langle x, y, z\rangle$. With $p = 3$, $\mathbf{F}$ is an inverse square field.

a. Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$.

b. Explain the observation in part (a) by finding the flux of $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$ across the boundaries of a spherical box $\{(\rho, \varphi, \theta) : a \le \rho \le b,\ \varphi_1 \le \varphi \le \varphi_2,\ \theta_1 \le \theta \le \theta_2\}$ for various values of p.

47. A beautiful flux integral Consider the potential function $\varphi(x, y, z) = G(\rho)$, where G is any twice differentiable function and $\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, G depends only on the distance from the origin.

a. Show that the gradient vector field associated with φ is $\mathbf{F} = \nabla \varphi = G'(\rho)\dfrac{\mathbf{r}}{\rho}$, where $\mathbf{r} = \langle x, y, z\rangle$ and $\rho = |\mathbf{r}|$.

b. Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S. Show that the flux of **F** across S is $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi a^2\,G'(a)$.

c. Show that $\nabla\cdot\mathbf{F} = \nabla\cdot\nabla\varphi = \dfrac{2\,G'(\rho)}{\rho} + G''(\rho)$.

d. Use part (c) to show that the flux across S (as given in part (b)) is also obtained by the volume integral $\iiint_D \nabla\cdot\mathbf{F}\,dV$. (*Hint:* use spherical coordinates and integrate by parts.)

48. Integration by parts (Gauss' formula) Recall the Product Rule of Theorem 17.13: $\nabla\cdot(u\,\mathbf{F}) = \nabla u\cdot F + u\,(\nabla\cdot\mathbf{F})$.

a. Integrate both sides of this identity over a solid region D with a closed boundary S and use the Divergence Theorem to prove an integration by parts rule:

$$\iiint_D u\,(\nabla\cdot\mathbf{F})\,dV = \iint_S u\,\mathbf{F}\cdot\mathbf{n}\,dS - \iiint_D \nabla u\cdot\mathbf{F}\,dV.$$

b. Explain the correspondence between this rule and the integration by parts rule for single-variable functions.

c. Use integration by parts to evaluate $\iiint_D \left(x^2\,y + y^2\,z + z^2\,x\right)dV$, where D is the cube in the first octant cut by the planes $x = 1$, $y = 1$, and $z = 1$.

49. Green's Formula Write Gauss' Formula of Exercise 48 in two dimensions—that is, where $\mathbf{F} = \langle f, g\rangle$, D is a plane region R and C is the boundary of R. Show that the result is Green's Formula:

$$\iint_R u\left(f_x + g_y\right)dA = \oint_C u\,(\mathbf{F}\cdot\mathbf{n})\,ds - \iint_R \left(f\,u_x + g\,u_y\right)dA.$$

Show that with $u = 1$, one form of Green's Theorem appears. Which form of Green's Theorem is it?

50. Green's First Identity Prove Green's First Identity for twice differentiable scalar-valued functions u and v defined on a region D:

$$\iiint_D \left(u\,\nabla^2 v + \nabla u\cdot\nabla v\right)dV = \iint_S u\,\nabla v\cdot\mathbf{n}\,dS,$$

where $\nabla^2 v = \nabla\cdot\nabla v$. You may apply Gauss' Formula in Exercise 48 to $\mathbf{F} = \nabla v$ or apply the Divergence Theorem to $\mathbf{F} = u\,\nabla v$.

51. Green's Second Identity Prove Green's Second Identity for scalar-valued functions u and v defined on a region D:

$$\iiint_D \left(u\,\nabla^2 v - v\,\nabla^2 u\right)dV = \iint_S (u\,\nabla v - v\,\nabla u)\cdot\mathbf{n}\,dS.$$

(*Hint:* Reverse the roles of u and v in Green's First Identity.)

52–54. Harmonic functions *A scalar-valued function φ is harmonic on a region D if $\nabla^2\varphi = \nabla\cdot\nabla\varphi = 0$ at all points of D.*

52. Show that the potential function $\varphi(x, y, z) = |\mathbf{r}|^{-p}$ is harmonic provided $p = 0$ or $p = 1$, where $\mathbf{r} = \langle x, y, z\rangle$. To what vector fields do these potentials correspond?

53. Show that if φ is harmonic on a region D enclosed by a surface S, then $\iint_S \nabla\varphi \cdot \mathbf{n}\, dS = 0$.

54. Show that if u is harmonic on a region D enclosed by a surface S, then
$\iint_S u\, \nabla u \cdot \mathbf{n}\, dS = \iiint_D |\nabla u|^2\, dV$.

55. **Miscellaneous integral identities** Prove the following identities.

a. $\iiint_D \nabla \times \mathbf{F}\, dV = \iint_S (\mathbf{n} \times \mathbf{F})\, dS$ (*Hint:* Apply the Divergence Theorem to each component of the identity.)

b. $\iint_S (\mathbf{n} \times \nabla\varphi)\, dS = \oint_C \varphi\, d\mathbf{r}$ (*Hint:* Apply Stokes' Theorem to each component of the identity.)