17.7 Stokes' Theorem

With the divergence, the curl, and surface integrals in hand, we are ready to present two of the crowning results of calculus. Fortunately, all of the heavy lifting has been done. In this section, you will see Stokes' Theorem, and in the next section we present the Divergence Theorem.

Note »

> Born in Ireland, George Gabriel Stokes (1819–1903) led a long and distinguished life as one of the prominent mathematicians and physicists of his day. He entered Cambridge University as a student and remained there as a professor for most of his life, taking the Lucasian chair of mathematics, once held by Sir Isaac Newton. The first statement of Stokes' Theorem was given by William Thomson (Lord Kelvin).

Stokes' Theorem »

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a region R and $\mathbf{F} = \langle f, g \rangle$ is a differentiable vector field on R, Green's Theorem says that

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}} \, dA.$$

The line integral on the left gives the circulation along the boundary of R. The double integral on the right sums the curl of the vector field over all points of R. If $\mathbf{F}$ represents a fluid flow, the theorem says the cumulative rotation of the flow within R equals the circulation along the boundary.

In Stokes' Theorem, the plane region R in Green's Theorem becomes an oriented surface S in $\mathbb{R}^3$. The circulation integral in Green's Theorem remains a circulation integral, but now over the closed simple piecewise-smooth oriented curve C that forms the boundary of S. The double integral of the curl in Green's Theorem becomes a surface integral of the three-dimensional curl (**Figure 17.59**).

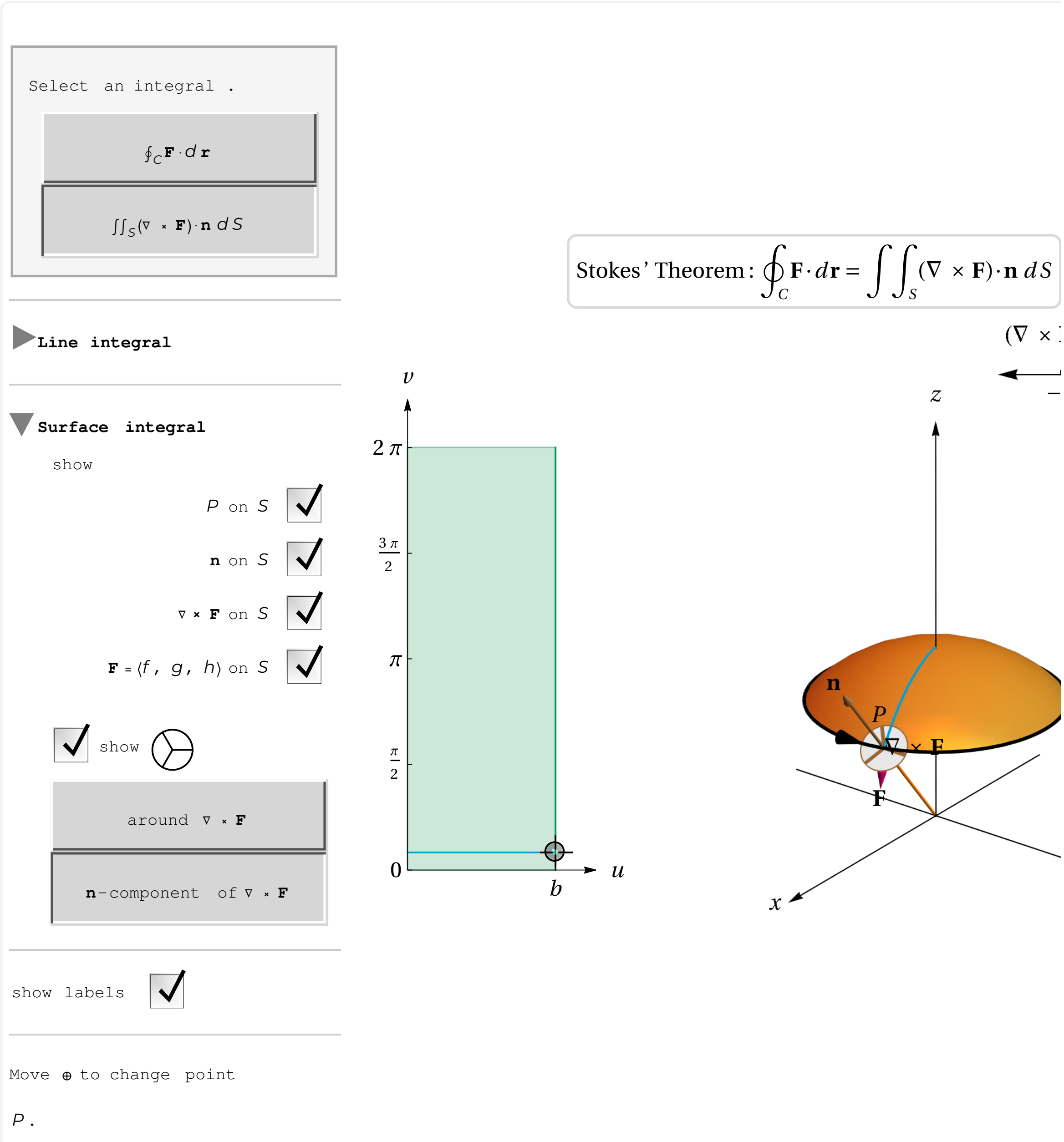


Figure 17.59

Stokes' Theorem involves an oriented curve C and an oriented surface S on which there are two unit normal vectors at every point. These orientations must be consistent and the normal vectors must be chosen correctly. Here is the right-hand rule that relates the orientations of S and C, and determines the choice of the normal vectors:

> If the fingers of your right hand curl in the positive direction around C, then your right thumb points in the (general) direction of the vectors normal to S (**Figure 17.60**).

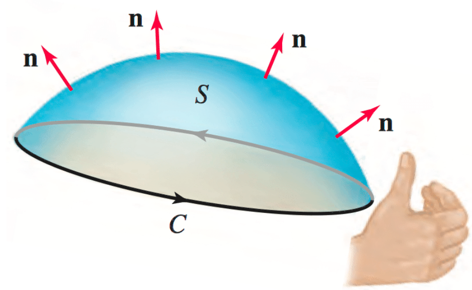


Figure 17.60

A common situation occurs when C has a counterclockwise orientation when viewed from above; then, the vectors normal to S point upward.

Note »

THEOREM 17.15 **Stokes' Theorem**

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Quick Check 1 Suppose S is a region in the xy-plane with a boundary oriented counterclockwise. What is the normal to S? Explain why Stokes' Theorem becomes the circulation form of Green's Theorem. ◆

Answer »

The meaning of Stokes' Theorem is much the same as for the circulation form of Green's Theorem: Under the proper conditions, the accumulated rotation of the vector field over the surface S (as given by the normal component of the curl) equals the net circulation on the boundary of S. An outline of the proof of Stokes' Theorem is given at the end of this section. First, we look at some special cases that give further insight into the theorem.

If $\mathbf{F}$ is a conservative vector field on a domain D, then it has a potential function φ such that $\mathbf{F} = \nabla\varphi$. Because $\nabla \times \nabla\phi = \mathbf{0}$, it follows that $\nabla \times \mathbf{F} = \mathbf{0}$ (Theorem 17.11); therefore, the circulation integral is zero on all closed curves in D. Recall that the circulation integral is also a work integral for the force field $\mathbf{F}$, which emphasizes the fact that no work is done in moving an object on a closed path in a conservative force field. Among the important conservative vector fields are the radial fields $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, which generally have zero curl and zero circulation on closed curves.

EXAMPLE 1 Verifying Stokes' Theorem

Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y, x, -x \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

SOLUTION »

The orientation of C implies that the vectors normal to S point in the outward direction. The vector field is a rotation field $\mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle 0, 1, 1 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$; so the axis of rotation points in the direction of the vector $\langle 0, 1, 1\rangle$ (**Figure 17.61**). We first compute the circulation integral in Stokes' Theorem. The curve C with the given orientation is parametrized as $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$, for $0 \le t \le 2\pi$; therefore, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$.

Note »

Recall that for a constant nonzero vector $\mathbf{a}$ and the position vector $\mathbf{r} = \langle x, y, z\rangle$, the field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ is a rotational field. In Example 1,

$$\mathbf{F} = \langle 0, 1, 1\rangle \times \langle x, y, z\rangle.$$

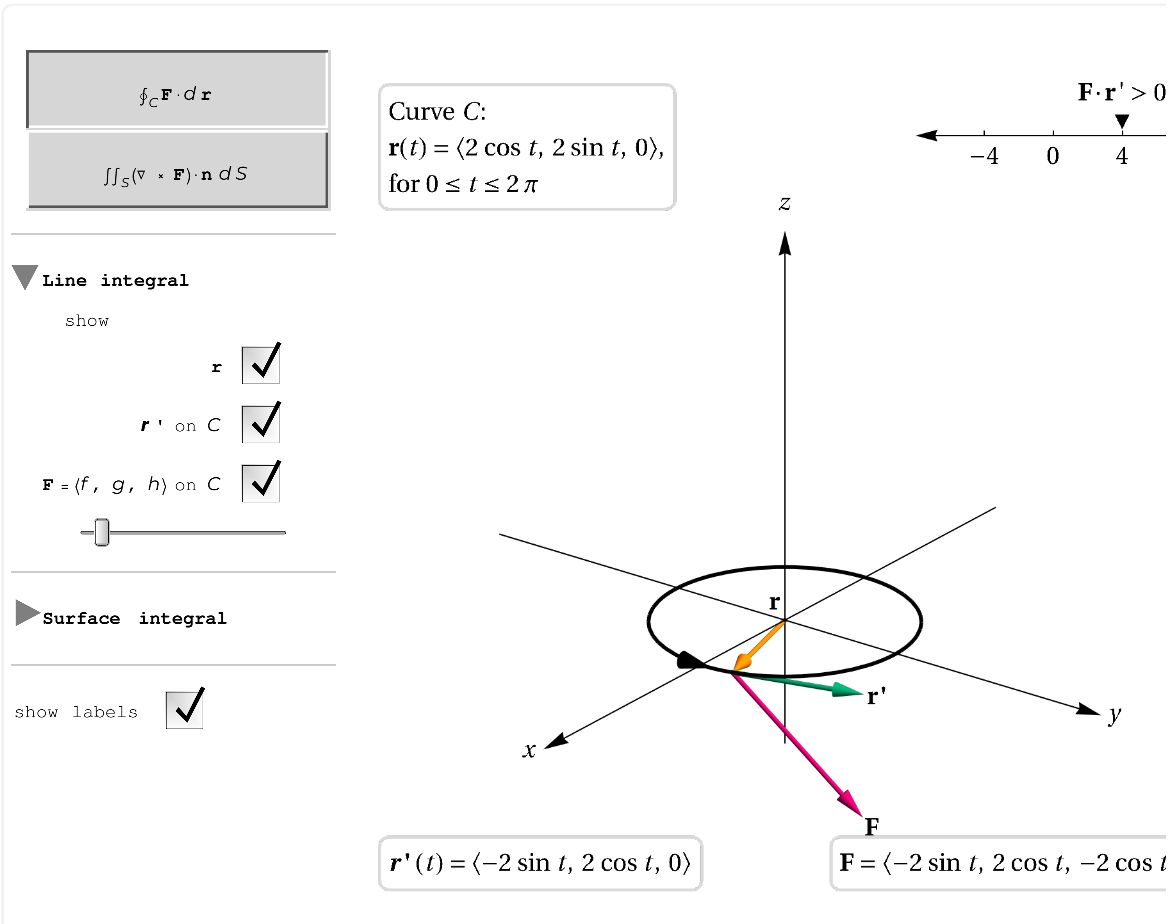


Figure 17.61

The circulation integral is

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt && \text{Definition of line integral}\\
&= \int_0^{2\pi} \Big\langle \underbrace{z-y}_{-2\sin t}, x, -x\Big\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\,dt && \text{Substitute.}\\
&= \int_0^{2\pi} 4\left(\sin^2 t + \cos^2 t\right)dt && \text{Simplify.}\\
&= 4\int_0^{2\pi} dt && \sin^2 t + \cos^2 t = 1\\
&= 8\pi. && \text{Evaluate integral.}
\end{aligned}$$

The surface integral requires computing the curl of the vector field:

$$\nabla \times \mathbf{F} = \nabla \times \langle z - y, x, -x\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix} = \langle 0, 2, 2\rangle.$$

Recall from Section 17.6 (Table 17.3) that an outward normal to the hemisphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. The region of integration is the base of the hemisphere in the *xy*-plane, which is

$$R = \left\{(x, y) : x^2 + y^2 \le 4\right\}, \text{ or, in polar coordinates}, \{(r, \theta) : 0 \le r \le 2, 0 \le \theta \le 2\pi\}.$$

Combining these results, the surface integral in Stokes' Theorem is

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 0, 2, 2\rangle} \cdot \mathbf{n}\, dS = \iint_R \langle 0, 2, 2\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dA \qquad \text{Substitute and convert to a double integral over } R.$$

$$= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2\right) dA \qquad \text{Simplify and use } z = \sqrt{4 - x^2 - y^2}.$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{2r \sin\theta}{\sqrt{4 - r^2}} + 2\right) r\, dr\, d\theta. \qquad \text{Convert to polar coordinates.}$$

We integrate first with respect to θ because the integral of $\sin\theta$ from 0 to 2π is zero and the first term in the integral is eliminated. Therefore, the surface integral reduces to

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^2 \int_0^{2\pi} \left(\frac{2r^2 \sin\theta}{\sqrt{4 - r^2}} + 2r\right) d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} 2r\, d\theta\, dr \qquad \int_0^{2\pi} \sin\theta\, d\theta = 0$$

$$= 4\pi \int_0^2 r\, dr \qquad \text{Evaluate inner integral.}$$

$$= 8\pi. \qquad \text{Evaluate outer integral.}$$

Note »

> In eliminating the first term of this double integral, we note that the improper integral $\int_0^2 \frac{r^2}{\sqrt{4 - r^2}}\, dr$ has a finite value.

Computed either as a line integral or a surface integral, the vector field has a positive circulation along the boundary of S, which is produced by the net rotation of the field over the surface S.

Related Exercises 5–6 ◆

In Example 1, it was possible to evaluate both sides of Stokes' Theorem. Often the theorem provides an easier way to evaluate difficult line integrals.

EXAMPLE 2 Using Stokes' Theorem to evaluate a line integral

Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = z\,\mathbf{i} - z\,\mathbf{j} + \left(x^2 - y^2\right)\mathbf{k}$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown in **Figure 17.62** .

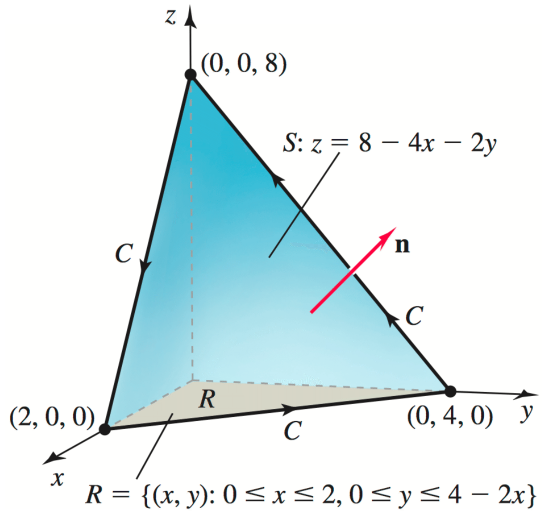


Figure 17.62

SOLUTION »

Evaluating the line integral directly involves parameterizing the three line segments. Instead, we use Stokes' Theorem to convert the line integral to a surface integral, where S is that portion of the plane $z = 8 - 4x - 2y$ that lies in the first octant. The curl of the vector field is

$$\nabla \times \mathbf{F} = \nabla \times \left\langle z, -z, x^2 - y^2 \right\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix} = \langle 1 - 2y, 1 - 2x, 0 \rangle.$$

The appropriate vector normal to the plane $z = 8 - 4x - 2y$ is $\left\langle -z_x, -z_y, 1 \right\rangle = \langle 4, 2, 1 \rangle$, which points upward, consistent with the orientation of C. The triangular region R in the xy-plane beneath the plane is found by setting $z = 0$ in the equation of the plane; we find $R = \{(x, y) : 0 \le x \le 2, 0 \le y \le 4 - 2x\}$. The surface integral in Stokes' Theorem may now be evaluated:

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 1-2y, 1-2x, 0\rangle} \cdot \mathbf{n}\, dS = \iint_R \langle 1 - 2y, 1 - 2x, 0 \rangle \cdot \langle 4, 2, 1 \rangle\, dA \quad \text{Substitute and convert to a double integral over } R.$$

$$= \int_0^2 \int_0^{4-2x} (6 - 4x - 8y)\, dy\, dx \quad \text{Simplify.}$$

$$= -\frac{88}{3}. \quad \text{Evaluate integrals.}$$

Note »

Recall that for an explicitly defined surface S given by $z = s(x, y)$ over a region R with $\mathbf{F} = \langle f, g, h \rangle$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \left(-f z_x - g z_y + h\right) dA.$$

In Example 2, $\mathbf{F}$ is replaced by $\nabla \times \mathbf{F}$.

The circulation around the boundary of R is negative, indicating a net circulation in the clockwise direction on C (looking from above).

Related Exercises 13, 16 ◆

In other situations, Stokes' Theorem may be used to convert a difficult surface integral into a relatively easy line integral, as illustrated in the next example.

EXAMPLE 3 Using Stokes' Theorem to evaluate a surface integral

Evaluate the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$, in the following cases.

a. S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ that lies within the paraboloid $z = 3x^2 + y^2$ (the blue surface in **Figure 17.63**). Assume $\mathbf{n}$ points in the upward direction on S.

b. S is the part of the paraboloid $z = 3x^2 + y^2$ that lies within the paraboloid $z = 4 - x^2 - 3y^2$, with $\mathbf{n}$ pointing in the upward direction on S.

c. S is the surface in part (b), with $\mathbf{n}$ pointing in the downward direction on S.

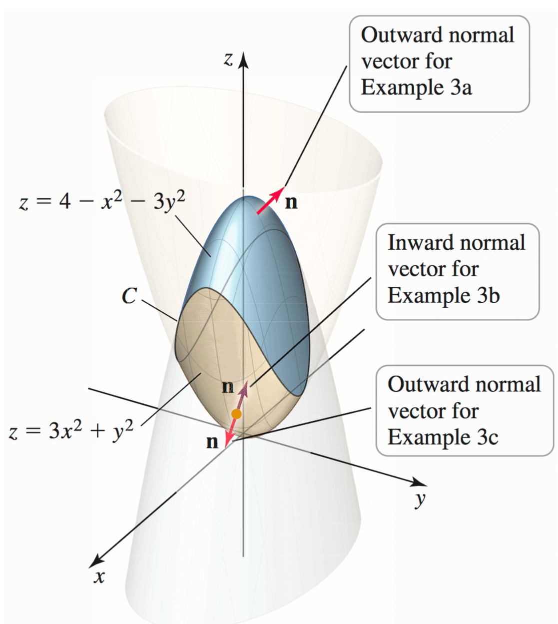


Figure 17.63

SOLUTION »

a. Finding a parametric description for S is challenging, so we use Stokes' Theorem to convert the surface integral into a line integral along the curve C that bounds S. Note that C is the intersection between the paraboloids $z = 4 - x^2 - 3y^2$ and $z = 3x^2 + y^2$. Eliminating z from these equations, we find that the projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, which suggests that we choose $x = \cos t$ and $y = \sin t$ for the x- and y-components of the equations for C. To find the z-component, we substitute x and y into the equation of either paraboloid. Choosing $z = 3x^2 + y^2$, we find that a parametric description of C is $\mathbf{r}(t) = \left\langle \cos t, \sin t, 3\cos^2 t + \sin^2 t \right\rangle$; note that C is oriented in the counterclockwise direction, consistent with the orientation of S.

To evaluate the line integral in Stokes' Theorem, it is helpful to first compute $\mathbf{F}\cdot\mathbf{r}'(t)$. Along C, the vector field is $\mathbf{F} = \langle -y, x, z\rangle = \left\langle -\sin t, \cos t, 3\cos^2 t + \sin^2 t\right\rangle$. Differentiating $\mathbf{r}$ yields $\mathbf{r}'(t) = \langle -\sin t, \cos t, -4\cos t \sin t\rangle$, which leads to

$$\begin{aligned}\mathbf{F}\cdot\mathbf{r}'(t) &= \left\langle -\sin t, \cos t, 3\cos^2 t + \sin^2 t\right\rangle \cdot \langle -\sin t, \cos t, -4\cos t\sin t\rangle \\ &= \underbrace{\sin^2 t + \cos^2 t}_{1} - 12\cos^3 t\sin t - 4\sin^3 t\cos t.\end{aligned}$$

Noting that $\sin^2 t + \cos^2 t = 1$, we are ready to evaluate the integral:

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS &= \oint_C \mathbf{F}\cdot d\mathbf{r} && \text{Stokes' Theorem} \\ &= \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'(t)\, dt && \text{Definition of line integral} \\ &= \int_0^{2\pi} \left(1 - 12\cos^3 t\sin t - 4\sin^3 t\cos t\right) dt && \text{Substitute.} \\ &= \int_0^{2\pi} 1\, dt - \underbrace{\int_0^{2\pi} 12\cos^3 t\sin t\, dt}_{0} - \underbrace{\int_0^{2\pi} 4\sin^3 t\cos t\, dt}_{0} && \text{Split integral.} \\ &= 2\pi. && \text{Evaluate integrals.}\end{aligned}$$

A standard substitution in the last two integrals of the final step shows that both integrals equal 0.

b. Because the lower surface ($z = 3x^2 + y^2$) shares the same boundary C with the upper surface ($z = 4 - x^2 - 3y^2$), and because both surfaces have an upward-pointing normal vector, the line integral resulting from an application of Stokes' Theorem is identical to the integral in part (a). For this surface S with its associated normal vector, we conclude that $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$. In fact, the value of this integral is 2π for *any* surface whose boundary is C and whose normal vectors point in the upward direction.

c. In this case, $\mathbf{n}$ points downward. We use the parametrization $\mathbf{r}(t) = \left\langle \sin t, \cos t, 3\cos^2 t + \sin^2 t\right\rangle$ for C so that C is oriented in the clockwise direction, consistent with the orientation of S. You should verify that, when duplicating the calculations in part (a) with a new description for C, we have

$$\mathbf{F}\cdot\mathbf{r}'(t) = \underbrace{-\sin^2 t - \cos^2 t}_{-1} - 12\cos^3 t\sin t - 4\sin^3 t\cos t.$$

Note »

> Recall that $x = \cos t$, $y = \sin t$ is a standard parameterization for the unit circle centered at the origin with counterclockwise orientation. The parameterization $x = \sin t$, $y = \cos t$ reverses the orientation.

Therefore, the required integral is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$
$$= \int_0^{2\pi} \left(-1 - 12\cos^3 t \sin t - 4\sin^3 t \cos t\right) dt$$
$$= -2\pi.$$

This result is perhaps not surprising when compared to parts (a) and (b): The reversal of the orientation of S requires a reversal of the orientation of C, and we know from Section 17.2 that $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$. As we discuss at the end of this section, it follows that the surface integral over the closed surface enclosed by *both* paraboloids (with normal vectors everywhere outward) has the value $2\pi - 2\pi = 0$.

Related Exercises 21–22 ◆

Quick Check 2 In Example 3a, we used the parameterization $\mathbf{r}(t) = \left\langle \cos t, \sin t, 3\cos^2 t + \sin^2 t \right\rangle$ for C. Confirm that the parameterization $C : \mathbf{r}(t) = \left\langle \cos t, \sin t, 4 - \cos^2 t - 3\sin^2 t \right\rangle$ also results in an answer of 2π. ◆

Interpreting the Curl »

Stokes' Theorem leads to another interpretation of the curl at a point in a vector field. We need the idea of the **average circulation**. If C is the boundary of an oriented surface S, we define the average circulation of $\mathbf{F}$ over S as

$$\frac{1}{\text{area of } S} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where Stokes' theorem is used to convert the circulation integral to a surface integral.

First consider a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$. Recall that $\mathbf{F}$ describes the rotation about an axis in the direction of $\mathbf{a}$ with angular speed $\omega = |\mathbf{a}|$. We also showed that $\mathbf{F}$ has a constant curl, $\nabla \times \mathbf{F} = \nabla \times (\mathbf{a} \times \mathbf{r}) = 2\,\mathbf{a}$. We now take S to be a small circular disk centered at a point P, whose normal vector $\mathbf{n}$ makes an angle θ with the axis $\mathbf{a}$ (**Figure 17.64**). Let C be the boundary of S with a counterclockwise orientation.

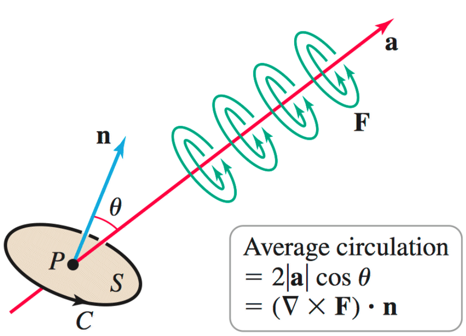


Figure 17.64

The average circulation of this vector field on S is

$$\begin{aligned}
\frac{1}{\text{area of } S}\iint_S \underbrace{(\nabla \times \mathbf{F})\cdot\mathbf{n}}_{\text{constant}}\, dS & \qquad \text{Definition} \\
= \frac{1}{\text{area of } S}(\nabla \times \mathbf{F})\cdot\mathbf{n}\cdot\text{area}(S) & \qquad \iint_S dS = \text{area of } S \\
= \underbrace{(\nabla \times \mathbf{F})}_{2\,\mathbf{a}}\cdot\mathbf{n} & \qquad \text{Simplify.} \\
= 2\,|\mathbf{a}|\cos\theta. & \qquad |\mathbf{n}| = 1,\ |\nabla \times \mathbf{F}| = 2\,|\mathbf{a}|
\end{aligned}$$

Note »

> Recall that $\mathbf{n}$ is a unit normal vector with $|\mathbf{n}| = 1$. By definition, the dot product gives $\mathbf{a}\cdot\mathbf{n} = |\mathbf{a}|\cos\theta$.

If the normal vector $\mathbf{n}$ is aligned with $\nabla \times \mathbf{F}$ (which is parallel to $\mathbf{a}$), then $\theta = 0$ and the average circulation has its maximum value of $2\,|\mathbf{a}|$. However, if the vector normal to the surface S is orthogonal to the axis of rotation $(\theta = \frac{\pi}{2})$, the average circulation is zero.

We see that for a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, the curl of $\mathbf{F}$ has the following interpretations, where S is a small disk centered at a point P with a normal vector $\mathbf{n}$.

- The scalar component of $\nabla \times \mathbf{F}$ at P in the direction of $\mathbf{n}$, which is $(\nabla \times \mathbf{F})\cdot\mathbf{n} = 2\,|\mathbf{a}|\cos\theta$, is the average circulation on S.
- The direction of $\nabla \times \mathbf{F}$ at P is the direction that maximizes the average circulation on S. Equivalently, it is the direction in which you should orient the axis of a paddle wheel to obtain the maximum angular speed.

A similar argument may be applied to a general vector field (with a variable curl) to give an analogous interpretation of the curl at a point (Exercise 48).

EXAMPLE 4 Horizontal channel flow

Consider the velocity field $\mathbf{v} = \langle 0, 1 - x^2, 0\rangle$, for $|x| \le 1$ and $|z| \le 1$, which represents a horizontal flow in the y-direction (**Figure 17.65**).

a. Suppose you place a paddle wheel at the point $P\left(\frac{1}{2}, 0, 0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2}, 0, 0\right)$?

b. Compute and graph the curl of $\mathbf{v}$ and provide an interpretation.

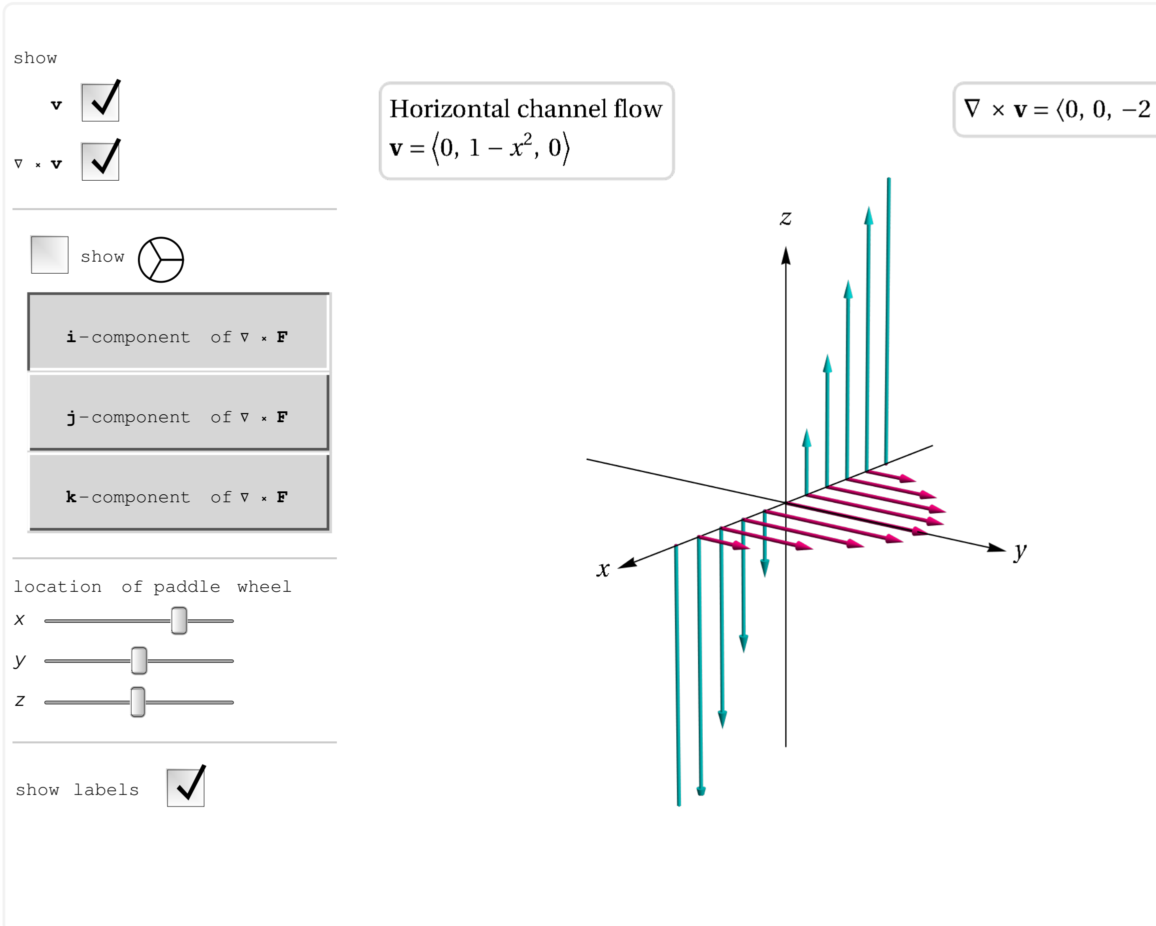


Figure 17.65

SOLUTION »

a. If the axis of the wheel is aligned with the x-axis at P, the flow strikes the upper and lower halves of the wheel symmetrically and the wheel does not spin. If the axis of the wheel is aligned with the y-axis, the flow strikes the face of the wheel and it does not spin. If the axis of the wheel is aligned with the z-axis at P, the flow in the y-direction is greater for $x < \frac{1}{2}$ than it is for $x > \frac{1}{2}$. Therefore, a wheel located at $\left(\frac{1}{2}, 0, 0\right)$ spins in the clockwise direction, looking from above. Using a similar argument, we conclude that a vertically oriented paddle wheel placed at $Q\left(-\frac{1}{2}, 0, 0\right)$ spins in the counterclockwise direction (when viewed from above).

b. A short calculation shows that

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 1 - x^2 & 0 \end{vmatrix} = -2x\,\mathbf{k}.$$

As shown in Figure 17.65, the curl points in the z-direction, which is the direction of the paddle wheel axis that gives the maximum angular speed of the wheel. Consider the z-component of the curl, which is

$(\nabla \times \mathbf{v})\cdot\mathbf{k} = -2x$. At $x = 0$, this component is zero, meaning the wheel does not spin at any point along the y-axis when its axis of the wheel is aligned with the z-axis. For $x > 0$, we see that $(\nabla \times \mathbf{v})\cdot\mathbf{k} < 0$, which corresponds to clockwise rotation of the vector field. For $x < 0$, we have $(\nabla \times \mathbf{F})\cdot\mathbf{v} > 0$, corresponding to counterclockwise rotation.

Related Exercise 26 ◆

Quick Check 3 In Example 4, explain why a paddle wheel with its axis aligned with the z-axis does not spin when placed on the y-axis. ◆

Answer »

Proof of Stokes' Theorem »

The proof of the most general case of Stokes' Theorem is intricate. However, a proof of a special case is instructive and it relies on several previous results.

Consider the case in which the surface S is the graph of the function $z = s(x, y)$, defined on a region in the xy-plane. Let C be the curve that bounds S with a counterclockwise orientation, let R be the projection of S in the xy-plane, and let C' be the projection of C in the xy-plane (**Figure 17.66**).

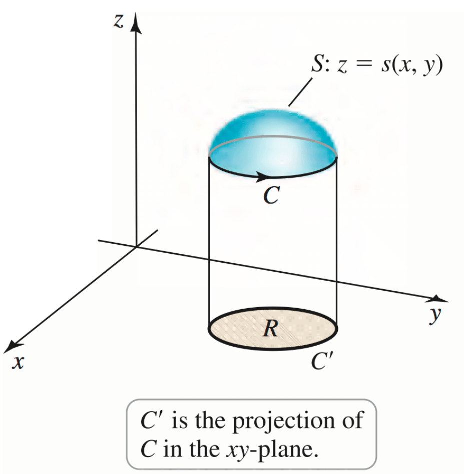


Figure 17.66

Letting $\mathbf{F} = \langle f, g, h\rangle$, the line integral in Stokes' Theorem is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C f\,dx + g\,dy + h\,dz.$$

The key observation for this integral is that along C (which is the boundary of S), $dz = z_x\,dx + z_y\,dy$. Making this substitution, we convert the line integral on C to a line integral on C' in the xy-plane:

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_{C'} f\,dx + g\,dy + h\underbrace{\left(z_x\,dx + z_y\,dy\right)}_{dz}\\ &= \oint_{C'} \underbrace{\left(f + h\,z_x\right)}_{M(x,y)}dx + \underbrace{\left(g + h\,z_y\right)}_{N(x,y)}dy.\end{aligned}$$

We now apply the circulation form of Green's Theorem to this line integral with $M(x, y) = f + h\,z_x$ and $N(x, y) = g + h\,z_y$; the result is

$$\oint_{C'} M\,dx + N\,dy = \iint_R \left(N_x - M_y\right) dA.$$

A careful application of the Chain Rule (remembering that z is a function of x and y, Exercise 49) reveals that

$$\begin{aligned} M_y &= f_y + f_z\, z_y + h\, z_{xy} + z_x\left(h_y + h_z\, z_y\right) \quad \text{and} \\ N_x &= g_x + g_z\, z_x + h\, z_{yx} + z_y(h_x + h_z\, z_x). \end{aligned}$$

Making these substitutions in the line integral and simplifying (note that $z_{xy} = z_{yx}$ is needed), we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(z_x\left(g_z - h_y\right) + z_y(h_x - f_z) + \left(g_x - f_y\right)\right) dA. \tag{1}$$

Now let's look at the surface integral in Stokes' Theorem. The upward vector normal to the surface is $\left\langle -z_x, -z_y, 1\right\rangle$. Substituting the components of $\nabla \times \mathbf{F}$ the surface integral takes the form

$$\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \left(\left(h_y - g_z\right)(-z_x) + (f_z - h_x)\left(-z_y\right) + \left(g_x - f_y\right)\right) dA,$$

which upon rearrangement becomes the integral in (1). ◆

Two Final Notes on Stokes' Theorem »

1. Stokes' Theorem allows a surface integral $\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$ to be evaluated using only the values of the vector field on the boundary C. This means that if a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2, which both have an orientation consistent with that of C, then the integrals of $(\nabla \times \mathbf{F})\cdot\mathbf{n}$ on the two surfaces are equal; that is,

$$\iint_{S_1} (\nabla \times \mathbf{F})\cdot\mathbf{n}_1\,dS = \iint_{S_2} (\nabla \times \mathbf{F})\cdot\mathbf{n}_2\,dS,$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the respective unit normal vectors consistent with the orientation of the surfaces (**Figure 17.67a**; see Example 3).

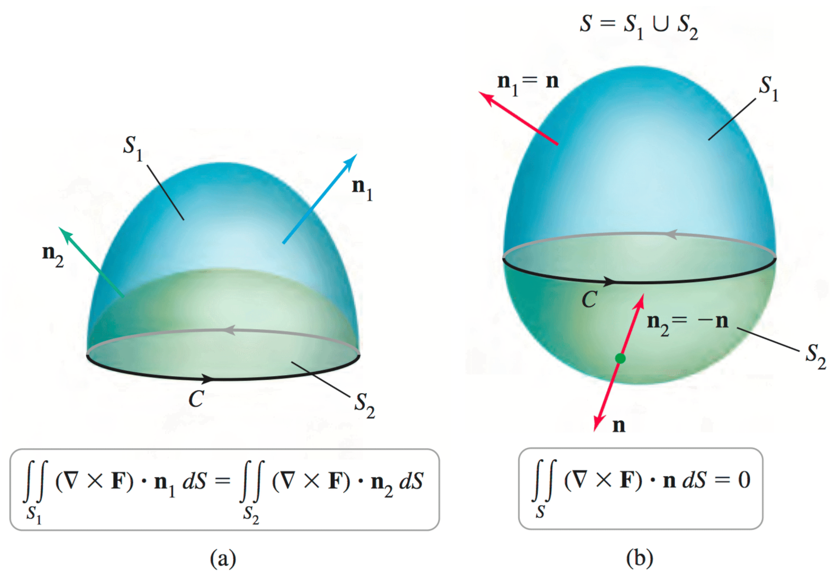


Figure 17.67

Now let's take a different perspective. Suppose S is a *closed* surface consisting of S_1 and S_2 with a common boundary curve C (**Figure 17.67b**). Let $\mathbf{n}$ represent the outward unit normal vector for the entire surface S. It follows that $\mathbf{n}$ points in the same direction as $\mathbf{n}_1$ and in the direction opposite to that of $\mathbf{n}_2$ (Figure 17.67b). Therefore, $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$ and $\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$ are equal in magnitude and of opposite sign, from which we conclude that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0.$$

This argument can be adapted to show that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0$ over any closed oriented surface S (Exercise 50).

2. We can now resolve an assertion made in Section 17.5. There we proved (Theorem 17.11) that if $\mathbf{F}$ is a conservative vector field, then $\nabla \times \mathbf{F} = \mathbf{0}$; we claimed, but did not prove, that the converse is true. The converse follows directly from Stokes' Theorem.

THEOREM 17.16 **Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ is Conservative**

Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

Proof: Given a closed simple smooth curve C, an advanced result states that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F})}_{\mathbf{0}} \cdot \mathbf{n} \, dS = 0$$

Because the line integral equals zero over all such curves in D, the vector field is conservative on D by Theorem 17.6. ◆

Exercises »

Getting Started »

Practice Exercises »

5–10. Verifying Stokes' Theorem *Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume C has counterclockwise orientation and S has a consistent orientation.*

5. $\mathbf{F} = \langle y, -x, 10\rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circle $x^2 + y^2 = 1$ in the xy-plane.

6. $\mathbf{F} = \langle 0, -x, y\rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and C is the circle $x^2 + y^2 = 4$ in the xy-plane.

7. $\mathbf{F} = \langle x, y, z\rangle$; S is the paraboloid $z = 8 - x^2 - y^2$, for $0 \le z \le 8$, and C is the circle $x^2 + y^2 = 8$ in the xy-plane.

8. $\mathbf{F} = \langle 2z, -4x, 3y\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and C is the boundary of S.

9. $\mathbf{F} = \langle y - z, z - x, x - y\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$ and C is the boundary of S.

10. $\mathbf{F} = \langle -y, -x - z, y - x\rangle$; S is the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$ and C is the boundary of S.

11–16. Stokes' Theorem for evaluating line integrals *Evaluate the line integral* $\oint_C \mathbf{F}\cdot d\mathbf{r}$ *by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume C has a counterclockwise orientation.*

11. $\mathbf{F} = \langle 2y, -z, x\rangle$; C is the circle $x^2 + y^2 = 12$ in the plane $z = 0$.

12. $\mathbf{F} = \langle y, xz, -y\rangle$; C is the ellipse $x^2 + \dfrac{y^2}{4} = 1$ in the plane $z = 1$.

13. $\mathbf{F} = \left\langle x^2 - z^2, y, 2xz\right\rangle$; C is the boundary of the plane $z = 4 - x - y$ in the first octant.

14. $\mathbf{F} = \left\langle x^2 - y^2, z^2 - x^2, y^2 - z^2\right\rangle$; C is the boundary of the square $|x| \le 1$, $|y| \le 1$ in the plane $z = 0$.

15. $\mathbf{F} = \left\langle y^2, -z^2, x\right\rangle$; C is the circle $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle$, for $0 \le t \le 2\pi$.

16. $\mathbf{F} = \left\langle 2xy\sin z, x^2\sin z, x^2 y\cos z\right\rangle$; C is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

17–24. Stokes' Theorem for evaluating surface integrals *Evaluate the line integral in Stokes' Theorem to determine the value of the surface integral* $\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\, dS$. *Assume* **n** *points in an upward direction.*

17. $\mathbf{F} = \langle x, y, z\rangle$; S is the upper half of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$.

18. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$; S is the paraboloid $x = 9 - y^2 - z^2$, for $0 \le x \le 9$ (excluding its base), and $\mathbf{r} = \langle x, y, z\rangle$.

19. $\mathbf{F} = \langle 2y, -z, x - y - z\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25$, for $3 \le x \le 5$ (excluding its base).

20. $\mathbf{F} = \langle x + y, y + z, z + x\rangle$; S is the tilted disk enclosed by $\mathbf{r}(t) = \left\langle \cos t, 2\sin t, \sqrt{3}\cos t\right\rangle$.

21. $\mathbf{F} = \langle y, z - x, -y\rangle$; S is the part of the paraboloid $z = 2 - x^2 - 2y^2$ that lies within the cylinder $x^2 + y^2 = 1$.

22. $\mathbf{F} = \langle 4x, -8z, 4y\rangle$; S is the part of the paraboloid $z = 1 - 2x^2 - 3y^2$ that lies within the paraboloid $z = 2x^2 + y^2$.

23. $\mathbf{F} = \langle y, 1, z\rangle$; S is the part of the surface $z = 2\sqrt{x}$ that lies within the cone $z = \sqrt{x^2 + y^2}$.

24. $\mathbf{F} = \left\langle e^x, \dfrac{1}{z}, y\right\rangle$; S is the part of the surface $z = 4 - 3y^2$ that lies within the paraboloid $z = x^2 + y^2$.

25–28. Interpreting and graphing the curl *For the following velocity fields, compute the curl, make a sketch of the curl, and interpret the curl.*

25. $\mathbf{v} = \langle 0, 0, y\rangle$

26. $\mathbf{v} = \left\langle 1 - z^2, 0, 0\right\rangle$

27. $\mathbf{v} = \langle -2z, 0, 1\rangle$

28. $\mathbf{v} = \langle 0, -z, y\rangle$

29. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** A paddle wheel with its axis in the direction $\langle 0, 1, -1\rangle$ would not spin when put in the vector field $\mathbf{F} = \langle 1, 1, 2\rangle \times \langle x, y, z\rangle$.
- **b.** Stokes' Theorem relates the flux of a vector field **F** across a surface to values of **F** on the boundary of the surface.
- **c.** A vector field of the form $\mathbf{F} = \langle a + f(x), b + g(y), c + h(z)\rangle$, where a, b, and c are constants, has zero circulation on a closed curve.
- **d.** If a vector field has zero circulation on all simple closed smooth curves C in a region D, then **F** is conservative on D.

30–33. Conservative fields *Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C.*

30. $\mathbf{F} = \langle 2x, -2y, 2z\rangle$

31. $\mathbf{F} = \nabla(x \sin y e^z)$

32. $\mathbf{F} = \left\langle 3x^2 y, x^3 + 2yz^2, 2y^2 z\right\rangle$

33. $\mathbf{F} = \left\langle y^2 z^3,\ 2xyz^3,\ 3xy^2 z^2 \right\rangle$

34–38. Tilted disks *Let S be the disk enclosed by the curve* $C : \mathbf{r}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$, *for* $0 \le t \le 2\pi$, *where* $0 \le \varphi \le \frac{\pi}{2}$ *is a fixed angle.*

34. What is the area of S? Find a vector normal to S.

35. What is the length of C?

36. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\mathbf{F} = \langle -y,\ x,\ 0 \rangle$ as a function of φ. For what value of φ is the circulation a maximum?

37. What is the circulation on C of the vector field $\mathbf{F} = \langle -y,\ -z,\ x \rangle$ as a function of φ? For what value of φ is the circulation a maximum?

38. Consider the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1,\ a_2,\ a_3 \rangle$ is a constant nonzero vector and $\mathbf{r} = \langle x,\ y,\ z \rangle$. Show that the circulation is a maximum when $\mathbf{a}$ points in the direction of the normal to S.

39. Circulation in a plane A circle C in the plane $x + y + z = 8$ has a radius of 4 and center $(2,\ 3,\ 3)$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 0,\ -z,\ 2y \rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

40. No integrals Let $\mathbf{F} = \langle 2z,\ z,\ 2y + x \rangle$ and let S be the hemisphere of radius a with its base in the xy-plane and center at the origin.

a. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ by computing $\nabla \times \mathbf{F}$ and appealing to symmetry.

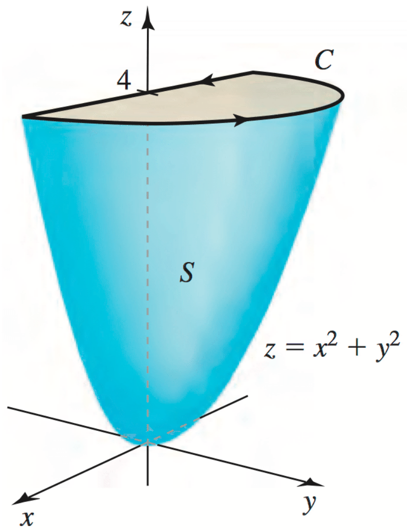
b. Evaluate the line integral using Stokes' Theorem to check part (a).

41. Compound surface and boundary Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$ (including the planar surface in the xz-plane) (see figure). Let C be the semicircle and line segment that bound the cap of S in the plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$.

a. Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

b. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

c. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and check for agreement with part (b).



42. **Ampère's Law** The French physicist André-Marie Ampère (1775-1836) discovered that an electrical current I in a wire produces a magnetic field $\mathbf{B}$. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume the current I is given in terms of the current density $\mathbf{J}$ as $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \mathbf{B} = \mu \mathbf{J}$.

43. **Maximum surface integral** Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\mathbf{F} = \langle x - y, y + z, z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ have its maximum value?

Explorations and Challenges »

44. **Area of a region in a plane** Let R be a region in a plane that has a unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and boundary C. Let $\mathbf{F} = \langle bz, cx, ay \rangle$.

a. Show that $\nabla \times \mathbf{F} = \mathbf{n}$.

b. Use Stokes' Theorem to show that

$$\text{area of } R = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

c. Consider the curve C given by $\mathbf{r} = \langle 5 \sin t, 13 \cos t, 12 \sin t \rangle$, for $0 \leq t \leq 2\pi$. Prove that C lies in a plane by showing that $\mathbf{r} \times \mathbf{r}'$ is constant for all t.

d. Use part (b) to find the area of the region enclosed by C in part (c). (*Hint:* Find the unit normal vector that is consistent with the orientation of C.)

45. **Choosing a more convenient surface** The goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle yz, -xz, xy \rangle$ and S is the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1$ $(z \geq 0)$.

a. Evaluate a surface integral over a more convenient surface to find the value of A.

b. Evaluate A using a line integral.

46. Radial fields and zero circulation Consider the radial vector fields $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number and $\mathbf{r} = \langle x, y, z \rangle$. Let C be any circle in the xy-plane centered at the origin.

- **a.** Evaluate a line integral to show that the field has zero circulation on C.
- **b.** For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

47. Zero curl Consider the vector field $\mathbf{F} = -\dfrac{y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j} + z\,\mathbf{k}$.

- **a.** Show that $\nabla \times \mathbf{F} = \mathbf{0}$.
- **b.** Show that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is not zero on a circle C in the xy-plane enclosing the origin.
- **c.** Explain why Stokes' Theorem does not apply in this case.

48. Average circulation Let S be a small circular disk of radius R centered at the point P with a unit normal vector $\mathbf{n}$. Let C be the boundary of S.

- **a.** Express the average circulation of the vector field $\mathbf{F}$ on S as a surface integral of $\nabla \times \mathbf{F}$.
- **b.** Argue that for small R, the average circulation approaches $(\nabla \times \mathbf{F})|_P \cdot n$ (the component of $\nabla \times \mathbf{F}$ in the direction of $\mathbf{n}$ evaluated at P) with the approximation improving as $R \to 0$.

49. Proof of Stokes' Theorem Confirm the following step in the proof of Stokes' Theorem. If $z = s(x, y)$ and f, g, and h are functions of x, y, and z, with $M = f + h\,z_x$ and $N = g + h\,z_y$, then

$$M_y = f_y + f_z\,z_y + h\,z_{xy} + z_x\left(h_y + h_z\,z_y\right) \quad \text{and}$$
$$N_x = g_x + g_z\,z_x + h\,z_{yx} + z_y(h_x + h_z\,z_x).$$

50. Stokes' Theorem on closed surfaces Prove that if $\mathbf{F}$ satisfies the conditions of Stokes' Theorem, then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$, where S is a smooth surface that encloses a region.

51. Rotated Green's Theorem Use Stokes' Theorem to write the circulation form of Green's Theorem in the yz-plane.