17.6 Surface Integrals

We have studied integrals on intervals on the real line, on regions in the plane, on solid regions in space, and along curves in space. One situation is still unexplored. Suppose a sphere has a known temperature distribution; perhaps it is cold near the poles and warm near the equator. How do you find the average temperature over the entire sphere? In analogy with other average value calculations, we should expect to "add up" the temperature values over the sphere and divide by the surface area of the sphere. Because the temperature varies continuously over the sphere, adding up means integrating. How do you integrate a function over a surface? This question leads to *surface integrals*.

It helps to keep curves, arc length, and line integrals in mind as we discuss surfaces, surface area, and surface integrals. What we discover about surfaces parallels what we already know about curves—all "lifted" up one dimension.

Parallel Concepts

Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces »

A curve in $\mathbb{R}^2$ is defined parametrically by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$; it requires one parameter and two dependent variables. Stepping up one dimension, to define a surface in $\mathbb{R}^3$ we need *two* parameters and *three* dependent variables. Letting u and v be parameters, the general parametric description of a surface has the form

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

We make the assumption that the parameters vary over a rectangle $R = \{(u, v) : a \le u \le b, c \le v \le d\}$ (**Figure 17.43**). As the parameters (u, v) vary over R, the vector $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ sweeps out a surface S in $\mathbb{R}^3$.

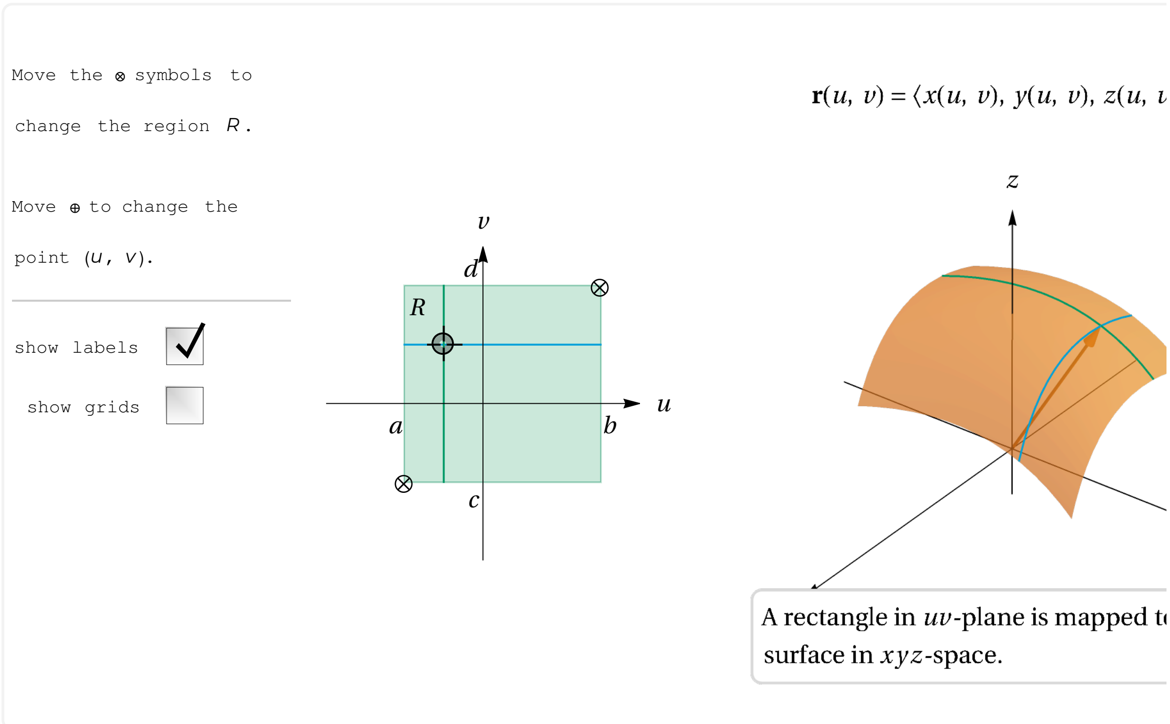


Figure 17.43

We work extensively with three surfaces that are easily described in parametric form. As with parameter-ized curves, a parametric description of a surface is not unique.

Cylinders

In Cartesian coordinates, the set

$$\{(x, y, z) : x = a\cos\theta,\ y = a\sin\theta,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\},$$

where $a > 0$, is a cylindrical surface of radius a and height h with its axis along the z-axis. Using the parameters $u = \theta$ and $v = z$, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle = \langle a\cos u,\ a\sin u,\ v\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$ (**Figure 17.44**).

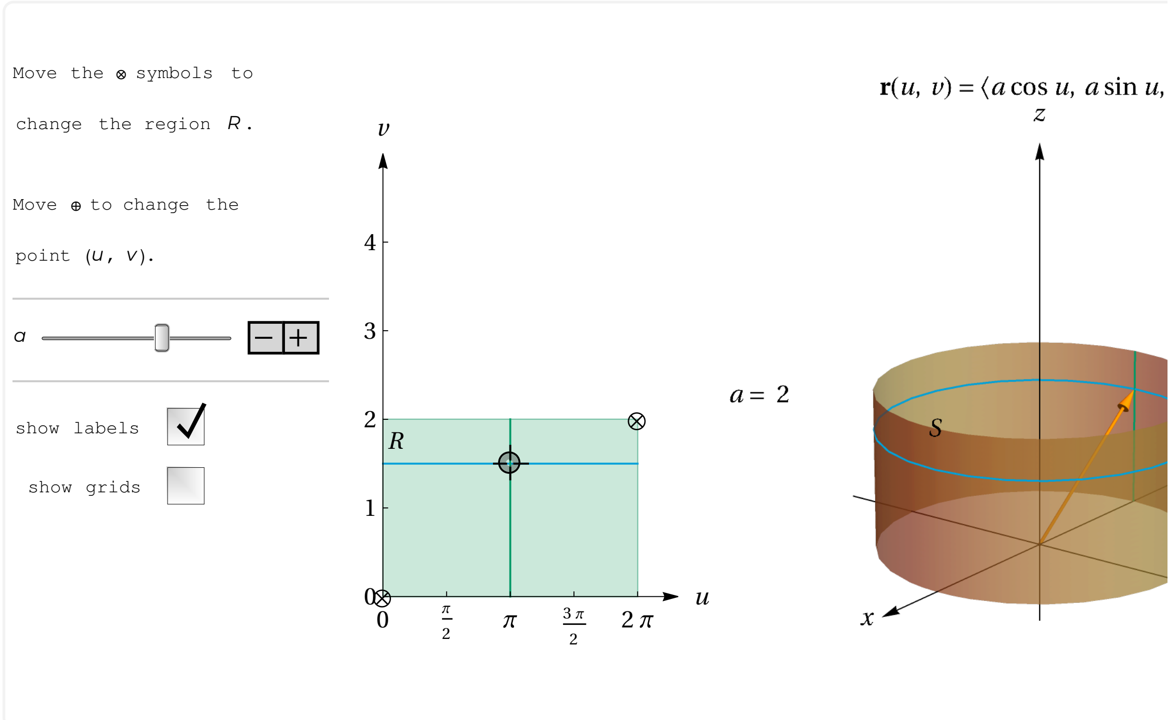


Figure 17.44

Quick Check 1 Describe the surface $\mathbf{r}(u, v) = \langle 2\cos u, 2\sin u, v\rangle$, for $0 \le u \le \pi$ and $0 \le v \le 1$. ◆

Answer »

A half cylinder with height 1 and radius 2 with its axis along the z-axis.

Cones

The surface of a cone of height h and radius a with its vertex at the origin is described in cylindrical coordinates by

$$\left\{(r, \theta, z) : 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ z = \frac{r\,h}{a}\right\}.$$

Note »

Note that when $r = 0$, $z = 0$ and when $r = a$, $z = h$.

For a fixed value of z, we have $r = \dfrac{a\,z}{h}$; therefore, on the surface of the cone

$$x = r\cos\theta = \frac{a\,z}{h}\cos\theta \quad \text{and} \quad y = r\sin\theta = \frac{a\,z}{h}\sin\theta.$$

Note »

Recall the relationships among polar and rectangular coordinates:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \text{ and }\ x^2 + y^2 = r^2.$$

Using the parameters $u = \theta$ and $v = z$, the parametric description of the conical surface is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \left\langle \frac{a\,v}{h} \cos u, \frac{a\,v}{h} \sin u, v \right\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$ (**Figure 17.45**).

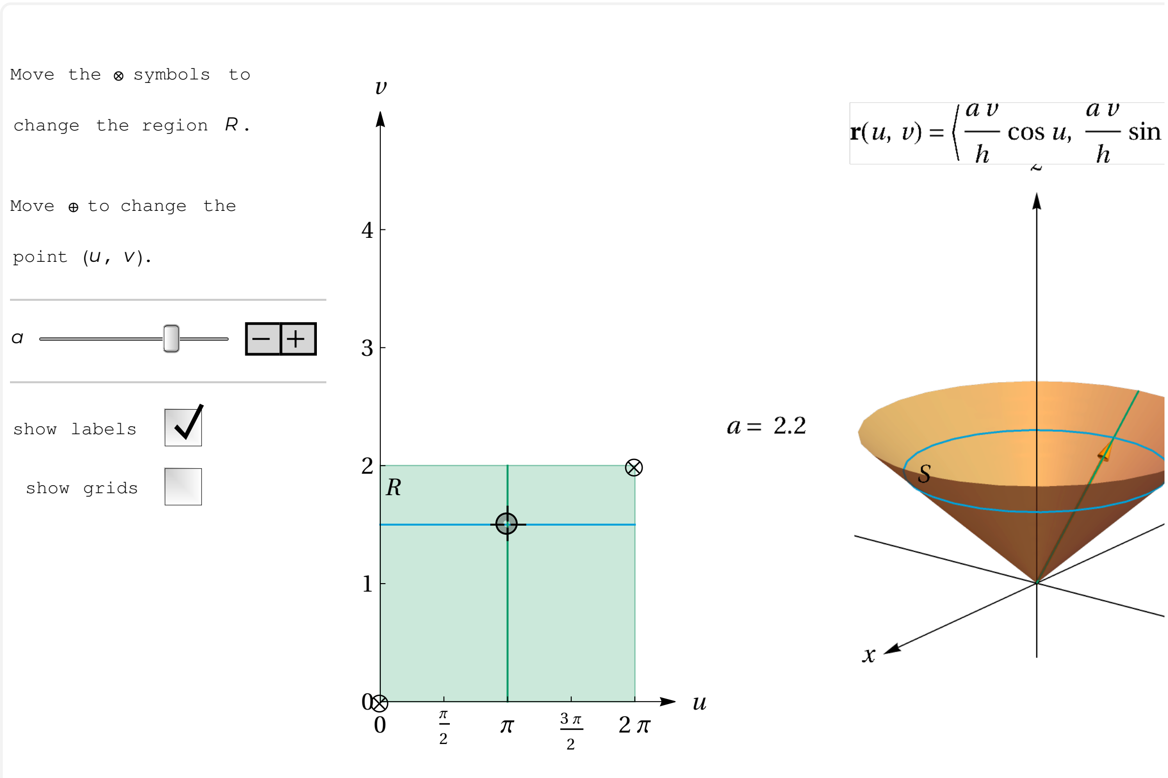


Figure 17.45

Quick Check 2 Describe the surface $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v\rangle$, for $0 \le u \le \pi$ and $0 \le v \le 10$. ◆

Answer »

Spheres

The parametric description of a sphere of radius a centered at the origin comes directly from spherical coordinates:

$$\{(\rho, \varphi, \theta) : \rho = a, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}.$$

Note »

> The complete cylinder, cone, and sphere are generated as the angle variable θ varies over the half-open interval $[0, 2\pi)$. As in previous chapters, we will use the closed interval $[0, 2\pi]$.

Recall the following relationships among spherical and rectangular coordinates (Section 16.5):

$$x = a \sin\varphi \cos\theta, \qquad y = a \sin\varphi \sin\theta, \qquad z = a \cos\varphi.$$

When we define the parameters $u = \varphi$ and $v = \theta$, a parametric description of the sphere is

$$\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u\rangle,$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$ (**Figure 17.46**).

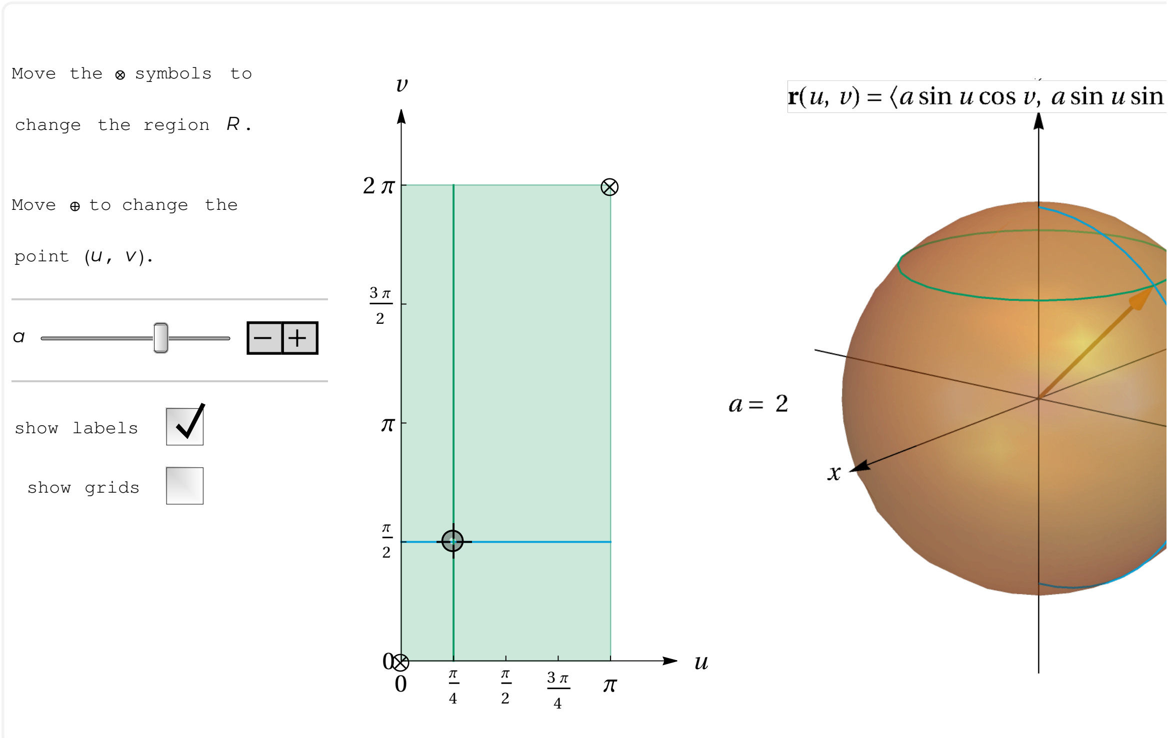


Figure 17.46

Quick Check 3 Describe the surface $\mathbf{r}(u, v) = \langle 4\sin u \cos v, 4\sin u \sin v, 4\cos u\rangle$, for $0 \le u \le \frac{\pi}{2}$ and $0 \le v \le \pi$. ◆

Answer »

EXAMPLE 1 Parametric surfaces

Find parametric descriptions for the following surfaces.

a. The plane $3x - 2y + z = 2$

b. The paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

SOLUTION »

a. Defining the parameters $u = x$ and $v = y$, we find that

$$z = 2 - 3x + 2y = 2 - 3u + 2v.$$

Therefore, a parametric description of the plane is

$$\mathbf{r}(u, v) = \langle u, v, 2 - 3u + 2v\rangle,$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

b. Thinking in terms of polar coordinates, we let $u = \theta$ and $v = \sqrt{z}$, which means that $z = v^2$. The equation of the paraboloid is $x^2 + y^2 = z = v^2$, so v plays the role of the polar coordinate r. Therefore, $x = v\cos\theta = v\cos u$

and $y = v \sin\theta = v \sin u$. A parametric description for the paraboloid is

$$\mathbf{r}(u, v) = \langle v \cos u,\ v \sin u,\ v^2 \rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 3$.

Alternatively, we could choose $u = \theta$ and $v = z$. The resulting description is

$$\mathbf{r}(u, v) = \langle \sqrt{v} \cos u,\ \sqrt{v} \sin u,\ v \rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 9$.

Related Exercises 9, 12 ◆

Surface Integrals of Scalar-Valued Functions »

We now develop the surface integral of a scalar-valued function f on a smooth parameterized surface S described by the equation

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

where the parameters vary over a rectangle $R = \{(u, v) : a \le u \le b,\ c \le v \le d\}$. The functions x, y, and z are assumed to have continuous partial derivatives with respect to u and v. The rectangular region R in the uv-plane is partitioned into rectangles, with sides of length Δu and Δv, that are ordered in some convenient way, for $k = 1, \ldots, n$. The kth rectangle R_k corresponds to a curved patch S_k on the surface S (**Figure 17.47**), which has area ΔS_k.

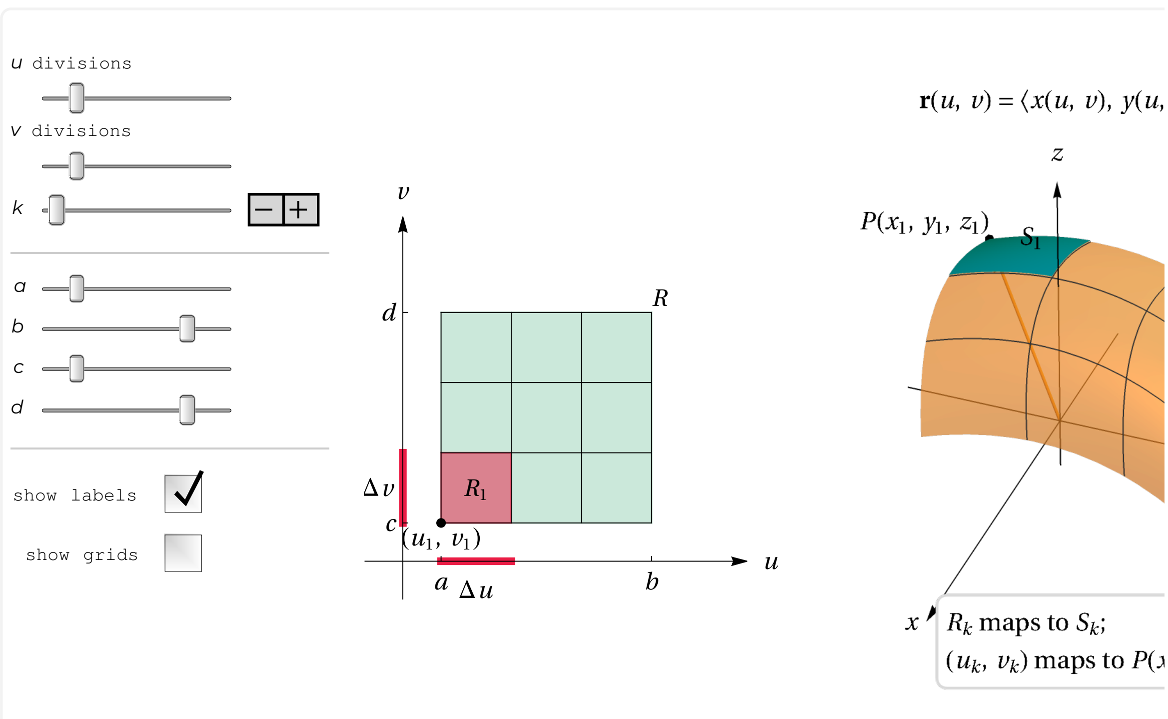


Figure 17.47

We let (u_k, v_k) be the lower-left corner point of R_k. The parameterization then assigns (u_k, v_k) to a point $P(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))$, or more simply, $P(x_k, y_k, z_k)$, on S_k. To construct the surface integral we define a Riemann sum, which adds up function values multiplied by areas of the respective patches:

$$\sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))\,\Delta S_k.$$

The crucial step is computing ΔS_k, the area of the kth patch S_k.

Note »

Figure 17.48 shows the patch S_k and the point $P(x_k, y_k, z_k)$. Two special vectors are tangent to the surface at P; these vectors lie in the plane tangent to S at P.

- $\mathbf{t}_u$ is a vector tangent to the surface corresponding to a change in u with v constant in the uv-plane.
- $\mathbf{t}_v$ is a vector tangent to the surface corresponding to a change in v with u constant in the uv-plane.

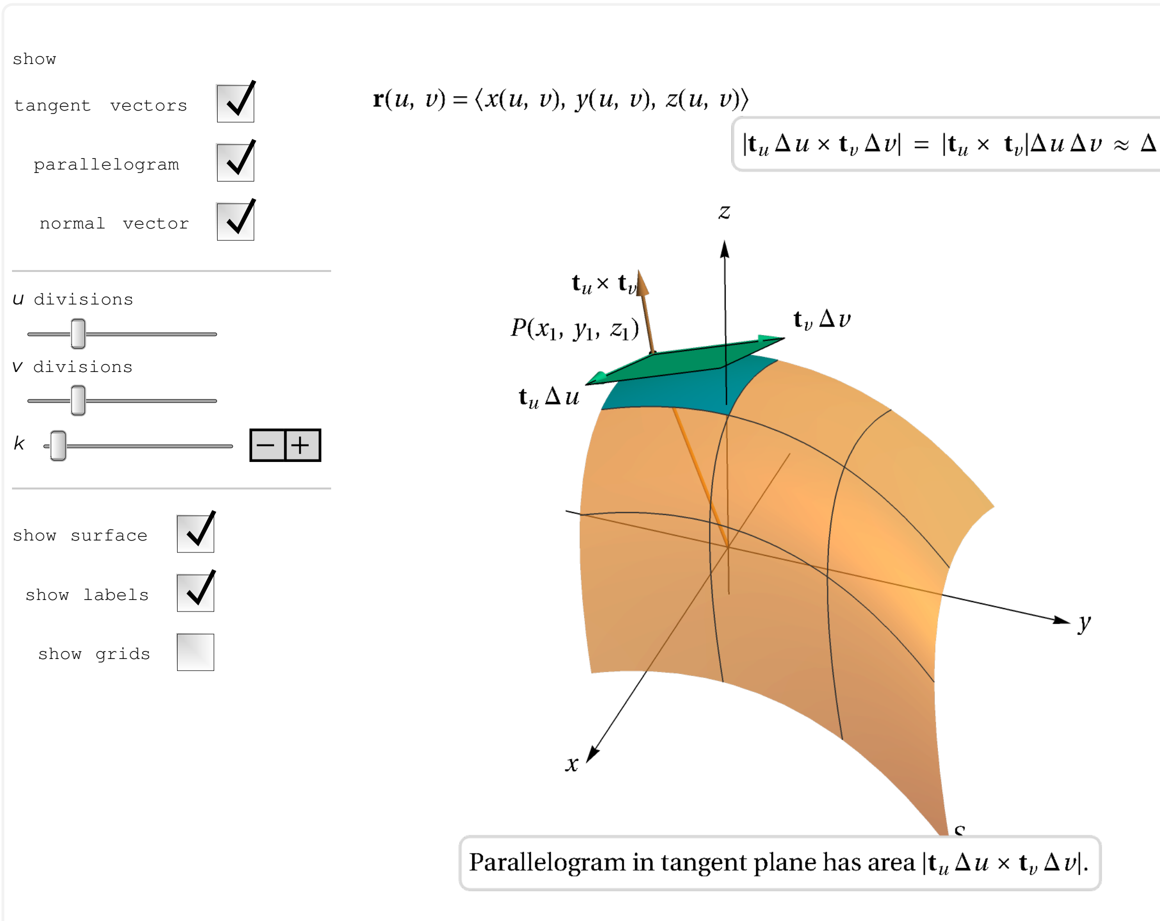


Figure 17.48

Note »

> In general, the vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ are different for each patch, so they should carry a subscript k. To keep the notation as simple as possible, we have suppressed the subscripts on these vectors with the understanding that they change with k. These tangent vectors are given by partial derivatives because in each case, either u or v is held constant, while the other variable changes.

Because the surface S may be written $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, a tangent vector corresponding to a change in u with v fixed is

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle.$$

Similarly, a tangent vector corresponding to a change in v with u fixed is

$$\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Now consider an increment Δu in u with v fixed; the corresponding change in $\mathbf{r}$, which is $\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)$, can be approximated using the definition of the partial derivative of $\mathbf{r}$ with respect to u. Specifically, when Δu is small, we have

$$\frac{\partial \mathbf{r}}{\partial u} \approx \frac{1}{\Delta u} (\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)).$$

Multiplying both sides of this equation by Δu and recalling that $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u}$, we see that the change in $\mathbf{r}$ corresponding to the increment Δu is approximated by the vector

$$\mathbf{t}_u \Delta u \approx \underbrace{\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)}_{\text{change in } \mathbf{r} \text{ corresponding to } \Delta u}.$$

Using a similar line of reasoning, the change in $\mathbf{r}$ corresponding to the increment Δv (with u fixed) is approximated by the vector

$$\mathbf{t}_v \Delta v \approx \underbrace{\mathbf{r}(u, v + \Delta v) - r(u, v)}_{\text{change in } \mathbf{r} \text{ corresponding to } \Delta v}.$$

As nonzero scalar multiples of $\mathbf{t}_u$ and $\mathbf{t}_v$, the vectors $\mathbf{t}_u \Delta u$ and $\mathbf{t}_v \Delta v$ are also tangent to the surface. They determine a parallelogram that lies in the plane tangent to S at P (Figure 17.48); the area of this parallelogram approximates the area of the kth patch S_k, which is ΔS_k.

Appealing to the cross product (Section 13.4), the area of the parallelogram is

$$|\mathbf{t}_u \Delta u \times \mathbf{t}_v \Delta v| = |\mathbf{t}_u \times \mathbf{t}_v| \Delta u \Delta v \approx \Delta S_k.$$

Note that $\mathbf{t}_u \times \mathbf{t}_v$ is evaluated at (u_k, v_k) and is a vector normal to the surface at P, which we assume to be nonzero at all points of S.

We write the Riemann sum with the observation that the areas of the parallelograms approximate the areas of the patches S_k:

$$\sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)) \Delta S_k \approx \sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)) \underbrace{|\mathbf{t}_u \times \mathbf{t}_v| \Delta u \Delta v}_{\approx \Delta S_k}.$$

We now assume f is continuous on S. As Δu and Δv approach zero, the areas of the parallelograms approach the areas of the corresponding patches on S. We define the limit of this Riemann sum to be the surface integral of f over S, which we write $\displaystyle\iint_S f(x, y, z)\, dS$. The surface integral is evaluated as an ordinary double integral over the region R in the uv-plane:

$$\iint_S f(x, y, z)\, dS = \lim_{\Delta u, \Delta v \to 0} \sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)) \,|\mathbf{t}_u \times \mathbf{t}_v|\, \Delta u\, \Delta v$$

$$= \iint_R f(x(u, v), y(u, v), z(u, v)) \,|\mathbf{t}_u \times \mathbf{t}_v|\, dA.$$

Note »

The factor $|\mathbf{t}_u \times \mathbf{t}_v|\, dA$ plays an analogous role in surface integrals as the factor $|\mathbf{r}'(t)|\, dt$ in line integrals.

If R is a rectangular region, as we have assumed, the double integral becomes an iterated integral with respect to u and v with constant limits. In the special case that $f(x, y, z) = 1$, the integral gives the surface area of S.

DEFINITION **Surface Integral of Scalar-Valued Functions on Parameterized Surfaces**

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where u and v vary over $R = \{(u, v) : a \le u \le b,\ c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$ are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v)) \,|\mathbf{t}_u \times \mathbf{t}_v|\, dA.$$

If $f(x, y, z) = 1$, the integral equals the surface area of S.

Note »

The condition that $\mathbf{t}_u \times \mathbf{t}_v$ be nonzero means $\mathbf{t}_u$ and $\mathbf{t}_v$ are nonzero and not parallel. If $\mathbf{t}_u \times \mathbf{t}_v \ne \mathbf{0}$ at all points, then the surface is *smooth*. The value of the integral is independent of the parameterization of S.

EXAMPLE 2 Surface area of a cylinder and sphere

Find the surface area of the following surfaces.

a. A cylinder with radius $a > 0$ and height h (excluding the circular ends)

b. A sphere of radius a

SOLUTION »

The critical step is evaluating the normal vector $\mathbf{t}_u \times \mathbf{t}_v$. It needs to be done only once for any given surface.

a. As shown before, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \langle a\cos u, a\sin u, v\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$. A normal vector is

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \quad \text{Definition of cross product}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Evaluate derivatives.}$$

$$= \langle a \cos u, a \sin u, 0 \rangle. \quad \text{Compute cross product.}$$

Notice that the normal vector points outward from the cylinder, away from the z-axis (**Figure 17.49**). It now follows that

$$|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{a^2 \cos^2 u + a^2 \sin^2 u} = a.$$

Setting $f(x, y, z) = 1$, the surface area of the cylinder is

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a}\, dA = \int_0^{2\pi} \int_0^h a\, dv\, du = 2\pi a h,$$

confirming the formula for the surface area of a cylinder (excluding the ends).

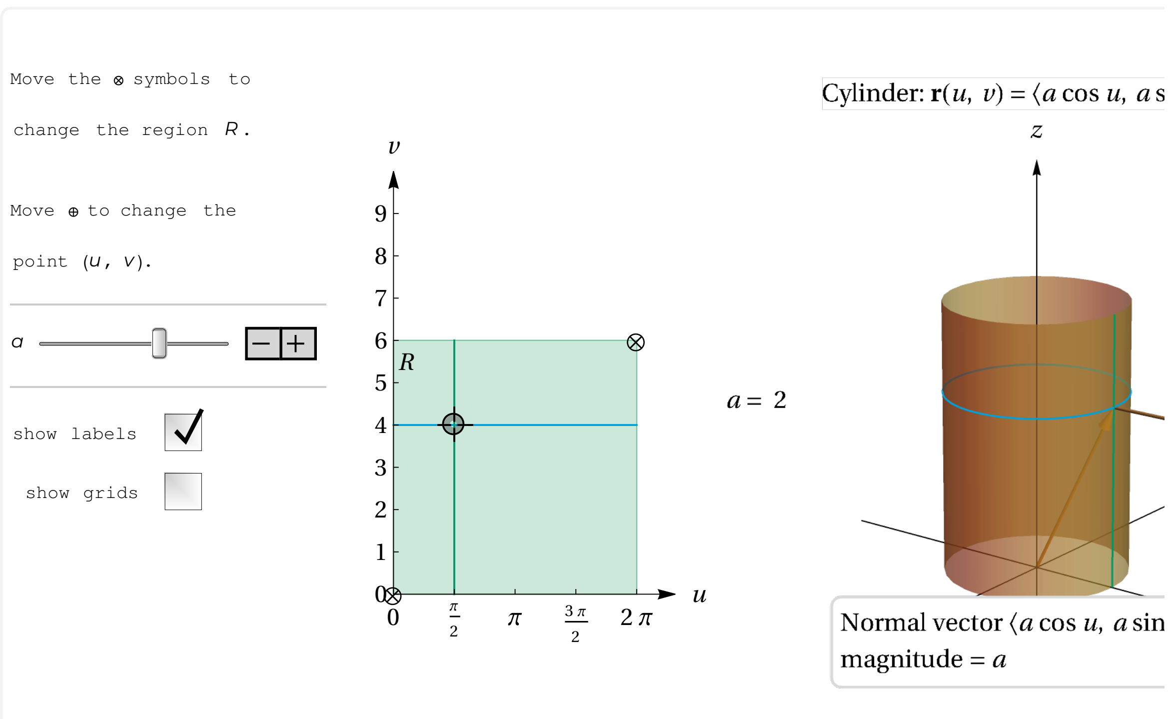


Figure 17.49

b. A parametric description of the sphere is

$$\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle,$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$. A normal vector is

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u \cos v & a\cos u \sin v & -a\sin u \\ -a\sin u \sin v & a\sin u \cos v & 0 \end{vmatrix}$$
$$= \langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \rangle.$$

Computing $|\mathbf{t}_u \times \mathbf{t}_v|$ requires several steps (Exercise 70). However, the needed result is quite simple: $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ points outward from the surface of the sphere (**Figure 17.50**). With $f(x, y, z) = 1$, the surface area of the sphere is

$$\iint\limits_S 1\, dS = \iint\limits_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2 \sin u}\, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin u\, du\, dv = 4\pi a^2,$$

confirming the formula for the surface area of a sphere.

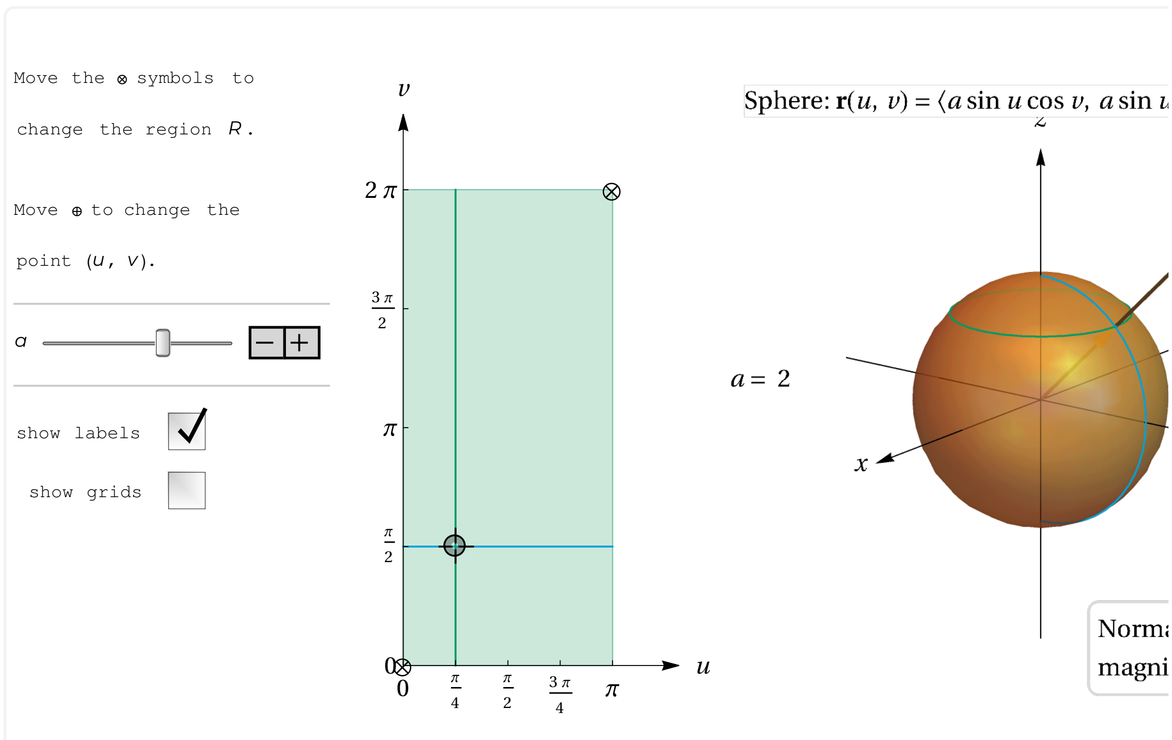


Figure 17.50

Note »

Recall that for the sphere, $u = \varphi$ and $v = \theta$, where φ and θ are spherical coordinates. The element of surface area in spherical coordinates is $dS = a^2 \sin \varphi\, d\varphi\, d\theta$.

Related Exercises 19, 22 ◆

EXAMPLE 3 Surface area of a partial cylinder

Find the surface area of the cylinder $\{(r, \theta): r = 4,\ 0 \le \theta \le 2\pi\}$ between the planes $z = 0$ and $z = 16 - 2x$ (excluding the top and bottom surfaces).

SOLUTION »

Figure 17.51 shows the cylinder bounded by the two planes.

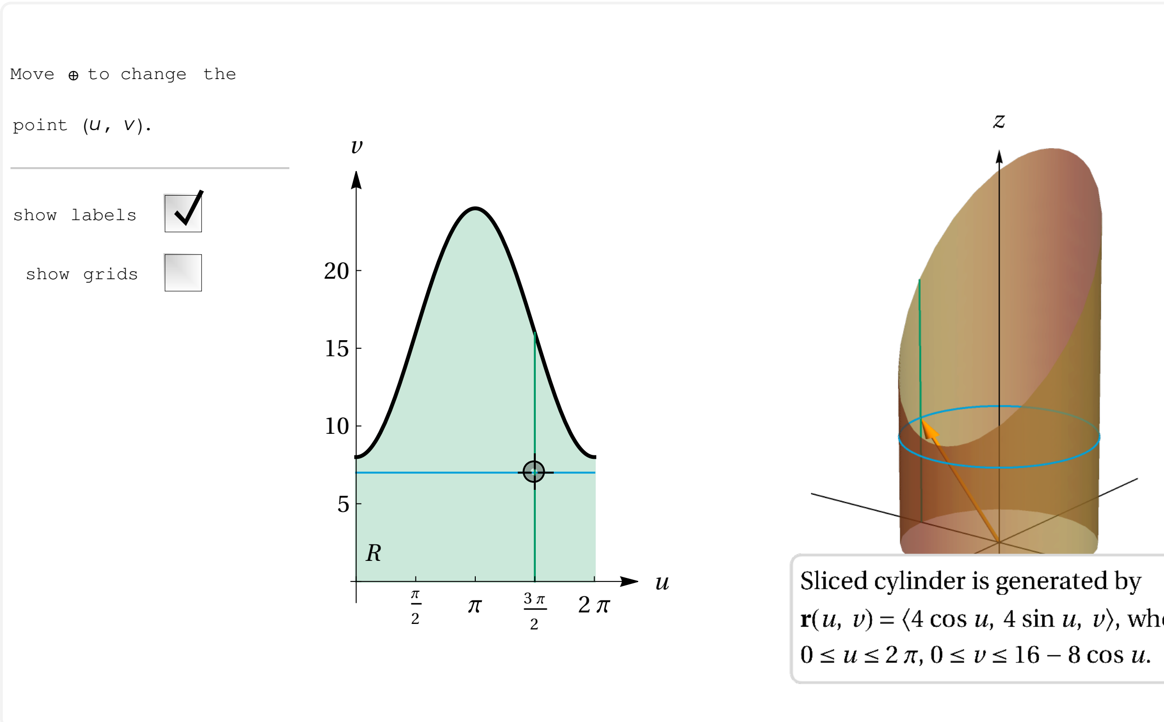


Figure 17.51

With $u = \theta$ and $v = z$, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \langle 4 \cos u, 4 \sin u, v\rangle.$$

The challenge is finding the limits on v, which is the z-coordinate. The plane $z = 16 - 2x$ intersects the cylinder in an ellipse; along this ellipse, as u varies between 0 and 2π, the parameter v also changes. To find the relationship between u and v along this intersection curve, notice that at any point on the cylinder, we have $x = 4 \cos u$ (remember that $u = \theta$). Making this substitution in the equation of the plane, we have

$$z = 16 - 2x = 16 - 2(4 \cos u) = 16 - 8 \cos u.$$

Substituting $v = z$, the relationship between u and v is $v = 16 - 8 \cos u$ (**Figure 17.52**).

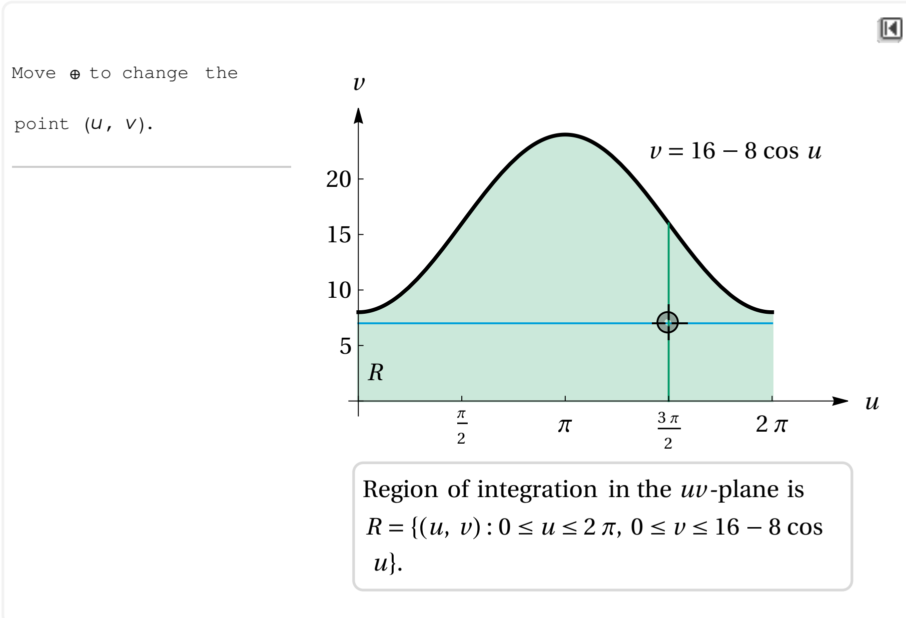


Figure 17.52

Therefore, the region of integration in the uv-plane is

$$R = \{(u, v) : 0 \le u \le 2\pi,\ 0 \le v \le 16 - 8\cos u\}.$$

Recall from Example 2a that for the cylinder, $|\mathbf{t}_u \times \mathbf{t}_v| = a = 4$. Setting $f(x, y, z) = 1$, the surface integral for the area is

$$\begin{aligned}
\iint_S 1\, dS &= \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{4}\, dA \\
&= \int_0^{2\pi}\int_0^{16-8\cos u} 4\, dv\, du \\
&= 4\int_0^{2\pi}(16 - 8\cos u)\, du && \text{Evaluate inner integral.} \\
&= 4\,(16u - 8\sin u)\Big|_0^{2\pi} && \text{Evaluate outer integral.} \\
&= 128\pi. && \text{Simplify.}
\end{aligned}$$

Related Exercise 24 ◆

EXAMPLE 4 Average temperature on a sphere

The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin\varphi$, for $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$ (φ and θ are spherical coordinates, so the temperature is 10° at the poles, increasing to 60° at the equator). Find the average temperature over the sphere.

SOLUTION »

We use the parametric description of a sphere. With $u = \varphi$ and $v = \theta$, the temperature function becomes $f(u, v) = 10 + 50\sin u$. Integrating the temperature over the sphere using the fact that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2\sin u$ (Example 2b), we have

$$\iint_S (10+50\sin u)\, dS = \iint_R (10+50\sin u)\, \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2 \sin u}\, dA$$

$$= \int_0^{\pi}\int_0^{2\pi} (10+50\sin u)\, a^2 \sin u\, dv\, du$$

$$= 2\pi a^2 \int_0^{\pi} (10+50\sin u)\sin u\, du \qquad \text{Evaluate inner integral.}$$

$$= 10\pi a^2(4+5\pi). \qquad \text{Evaluate outer integral.}$$

The average temperature is the integrated temperature $10\pi a^2(4+5\pi)$ divided by the surface area of the sphere $4\pi a^2$, so the average temperature is $\dfrac{20+25\pi}{2} \approx 49.3°$.

Related Exercise 42 ◆

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined not parametrically, but explicitly, in the form $z = g(x, y)$ over a region R in the xy-plane. Such a surface may be treated as a parameterized surface. We simply define parameters to be $u = x$ and $v = y$. Making these substitutions into the expression for $\mathbf{t}_u$ and $\mathbf{t}_v$, a short calculation (Exercise 71) reveals that $\mathbf{t}_u = \mathbf{t}_x = \langle 1, 0, z_x\rangle$, $\mathbf{t}_v = \mathbf{t}_y = \langle 0, 1, z_y\rangle$, and the required normal vector is

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1\rangle.$$

It follows that

$$|\mathbf{t}_x \times \mathbf{t}_y| = |\langle -z_x, -z_y, 1\rangle| = \sqrt{{z_x}^2 + {z_y}^2 + 1}.$$

With these observations, the surface integral over S can be expressed as a double integral over a region R in the xy-plane.

Note »

THEOREM 17.14 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{{z_x}^2 + {z_y}^2 + 1}\; dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Note »

EXAMPLE 5 Area of a roof over an ellipse

Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$ (**Figure 17.53**).

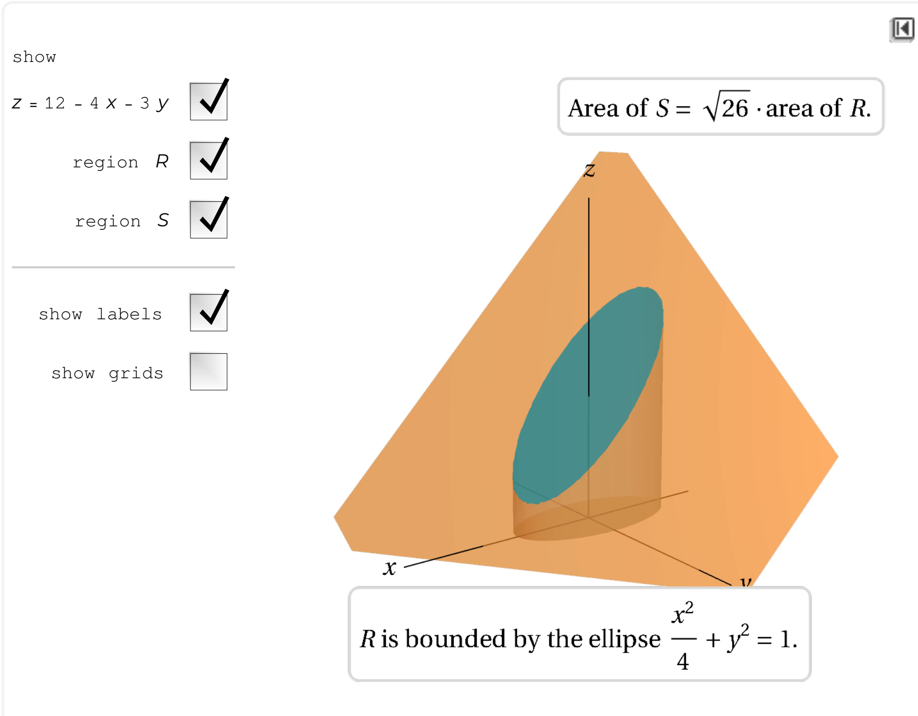


Figure 17.53

SOLUTION »

Because we are computing the area of the surface, we take $f(x, y, z) = 1$. Note that $z_x = -4$ and $z_y = -3$, so the factor $\sqrt{z_x^2 + z_y^2 + 1}$ has the value $\sqrt{(-4)^2 + (-3)^2 + 1} = \sqrt{26}$ (a constant because the surface is a plane). The relevant surface integral is

$$\iint_S 1\, dS = \iint_R \underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{\sqrt{26}}\, dA = \sqrt{26} \iint_R dA.$$

The double integral that remains is simply the area of the region R bounded by the ellipse. Because the ellipse has semiaxes of length $a = 2$ and $b = 1$, its area is $\pi a b = 2\pi$. Therefore, the area of S is $2\pi\sqrt{26}$.

This result has a useful interpretation. The plane surface S is not horizontal, so it has a greater area than the horizontal region R beneath it. The factor that converts the area of R to the area of S is $\sqrt{26}$. Notice that if the roof *were* horizontal, then the surface would be $z = c$, the area conversion factor would be 1, and the area of the roof would equal the area of the floor beneath it.

Related Exercises 29–30 ◆

Quick Check 4 The plane $z = y$ forms a $45°$ angle with the xy-plane. Suppose the plane is the roof of a room and the xy-plane is the floor of the room. Then $1\ \text{ft}^2$ on the floor becomes how many square feet when projected on the roof? ◆

Answer »

EXAMPLE 6 Mass of a conical sheet

A thin conical sheet is described by the surface $z = (x^2 + y^2)^{1/2}$, for $0 \le z \le 4$. The density of the sheet in g/cm^2 is

$\rho = f(x, y, z) = (8 - z)$ (decreasing from $8\text{ g}/\text{cm}^2$ at the vertex to $4\text{ g}/\text{cm}^2$ at the top of the cone; **Figure 17.54**). What is the mass of the cone?

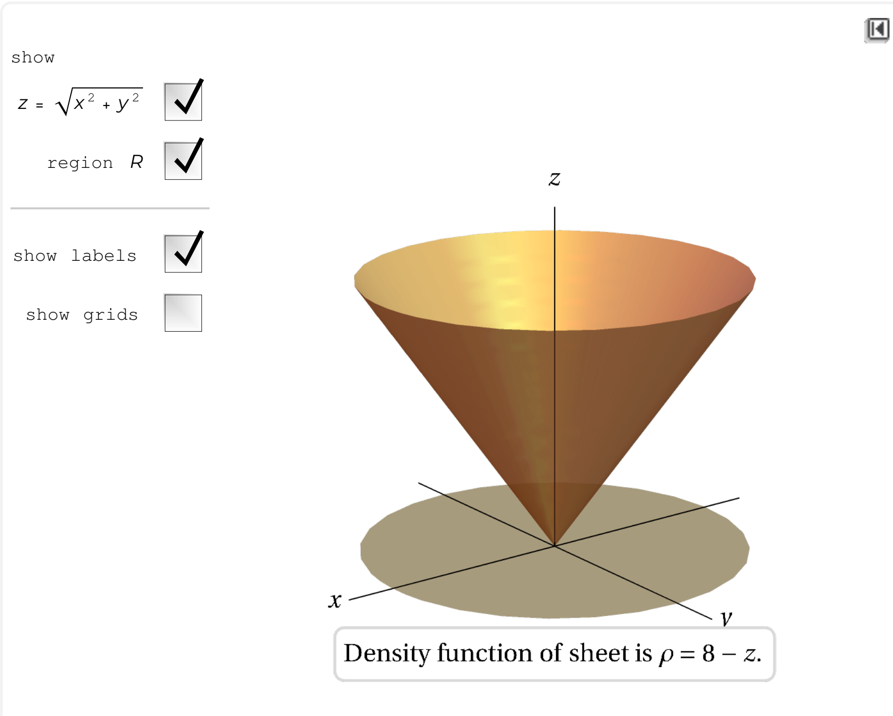


Figure 17.54

SOLUTION »

We find the mass by integrating the density function over the surface of the cone. The projection of the cone in the xy-plane is found by setting $z = 4$ (the top of the cone) in the equation of the cone. We find that $(x^2 + y^2)^{1/2} = 4$; therefore, the region of integration is the disk $R = \{(x, y) : x^2 + y^2 \le 16\}$. The next step is to compute z_x and z_y in order to evaluate $\sqrt{z_x{}^2 + z_y{}^2 + 1}$. Differentiating $z^2 = x^2 + y^2$ implicitly gives $2\, z\, z_x = 2\, x$, or $z_x = \dfrac{x}{z}$. Similarly, $z_y = \dfrac{y}{z}$. Using the fact that $z^2 = x^2 + y^2$, we have

$$\sqrt{z_x{}^2 + z_y{}^2 + 1} = \sqrt{\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + 1} = \sqrt{\underbrace{\frac{x^2 + y^2}{z^2}}_{1} + 1} = \sqrt{2}.$$

To integrate the density over the conical surface, we set $f(x, y, z) = 8 - z$. Replacing z in the integrand by $r = (x^2 + y^2)^{1/2}$ and using polar coordinates, the mass in grams is given by

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, z) \underbrace{\sqrt{z_x{}^2 + z_y{}^2 + 1}}_{\sqrt{2}}\, dA$$

$$= \sqrt{2} \iint_R (8 - z)\, dA \qquad \text{Substitute.}$$

$$= \sqrt{2} \iint_R \left(8 - \sqrt{x^2 + y^2}\right) dA \qquad z = \sqrt{x^2 + y^2}$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^4 (8 - r)\, r\, dr\, d\theta \qquad \text{Polar coordinates}$$

$$= \sqrt{2} \int_0^{2\pi} \left(4 r^2 - \frac{r^3}{3}\right)\Bigg|_0^4 d\theta \qquad \text{Evaluate inner integral.}$$

$$= \frac{128\sqrt{2}}{3} \int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{256\pi\sqrt{2}}{3} \approx 379. \qquad \text{Evaluate outer integral.}$$

As a check, note that the surface area of the cone is $\pi r \sqrt{r^2 + h^2} \approx 71\ \text{cm}^2$. If the entire cone had the maximum density $\rho = 8\ \text{g}/\text{cm}^2$, its mass would be approximately 568 g. If the entire cone had the minimum density $\rho = 4\ \text{g}/\text{cm}^2$, its mass would be approximately 284 g. The actual mass is between these extremes and closer to the low value because the cone is lighter at the top, where the surface area is greater.

Related Exercise 36 ◆

Table 17.3 summarizes the essential relationships for the explicit and parametric descriptions of cylinders, cones, spheres, and paraboloids. The listed normal vectors are chosen to point away from the z-axis.

Table 17.3

	Explicit Description $z = g(x, y)$		Parametric Description	
Surface	**Equation**	**Normal vector;** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x,$	**Equation**	**Normal vect magnitude** $\mathbf{t}_u \times \mathbf{t}_v; \lvert\mathbf{t}_u \times$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a$ si
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\left\langle \frac{x}{z}, \frac{y}{z}, -1\right\rangle; \sqrt{2}$	$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v$ si $\sqrt{2}\,v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle; \frac{a}{z}$	$\mathbf{r} = \langle a\sin u\cos v, a\sin u\sin v,$ $a\cos u\rangle, 0 \le u \le \pi,$ $0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos$ n$^2\, u\sin v,$ $a^2\sin u\cos$ $a^2\sin u$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4\left(x^2 + y^2\right)}$	$\mathbf{r} = \left\langle v\cos u, v\sin u, v^2\right\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2$ $-v\rangle; v\sqrt{1 +}$

Quick Check 5 Explain why the explicit description for a cylinder $x^2 + y^2 = a^2$ cannot be used for a surface integral over a cylinder and a parametric description must be used. ◆

Answer »

> The cylinder $x^2 + y^2 = a^2$ does not represent a function, so z_x and z_y cannot be computed.

Surface Integrals of Vector Fields »

Before beginning a discussion of surface integrals of vector fields, two technical issues about surfaces and normal vectors must be addressed.

The surfaces we consider in this text are called **two-sided**, or **orientable**, surfaces. To be orientable, a surface must have the property that the normal vectors vary continuously over the surface. In other words, when you walk on any closed path on an orientable surface and return to your starting point, your head must point in the same direction it did when you started. The most famous example of a *nonorientable* surface is the Möbius strip (**Figure 17.55**). Suppose you start walking along the surface of the Möbius strip at a point P with your head pointing upward. When you return to P, your head points in the opposite direction, or downward. Therefore, the Möbius strip is not orientable.

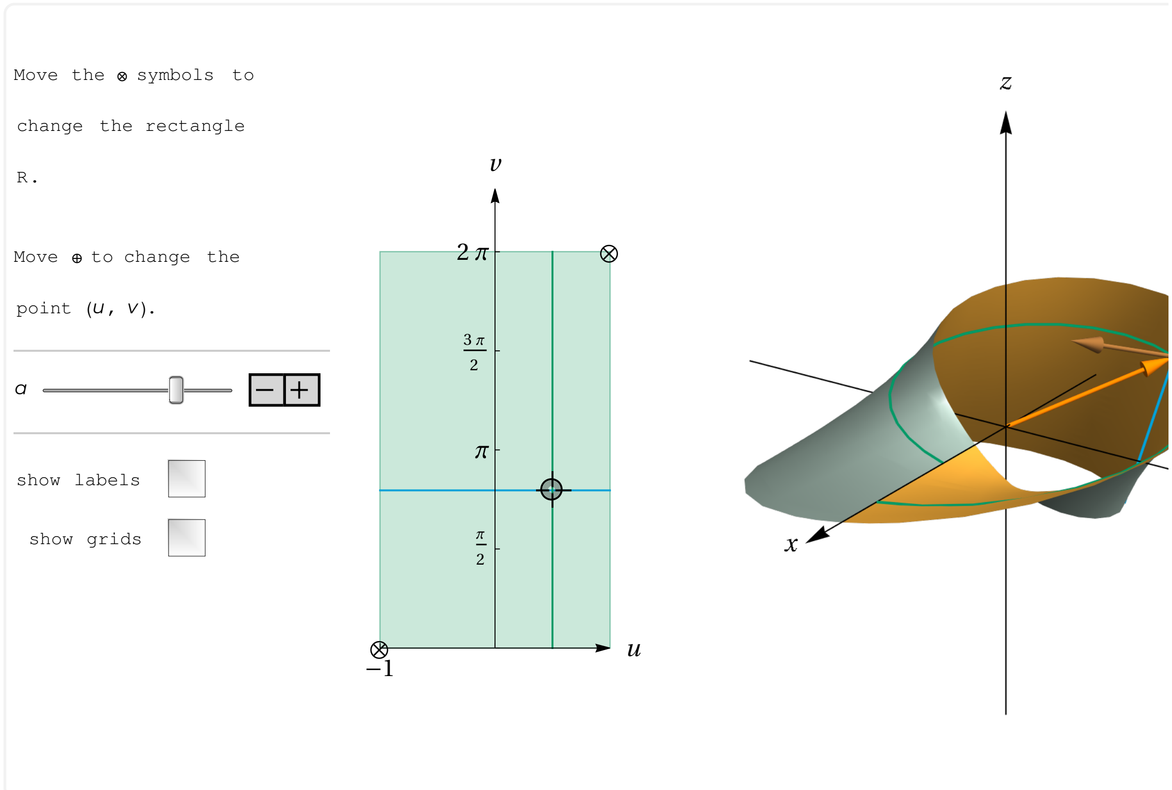


Figure 17.55

At any point of a parameterized orientable surface, there are two unit normal vectors. Therefore, the second point concerns the orientation of the surface or, equivalently, the choice of the direction of the normal vector. Once the direction of the normal vector is determined, the surface becomes **oriented**.

We make the common assumption that—unless specified otherwise—a closed orientable surface that fully encloses a region (such as a sphere) is oriented so that the normal vectors point in the *outward direction*. For a surface that is not closed, we assume that the orientation is specified in some way. For example, we might specify that the normal vectors for a particular surface point in the positive z-direction; that is, in an upward direction (**Figure 17.56**).

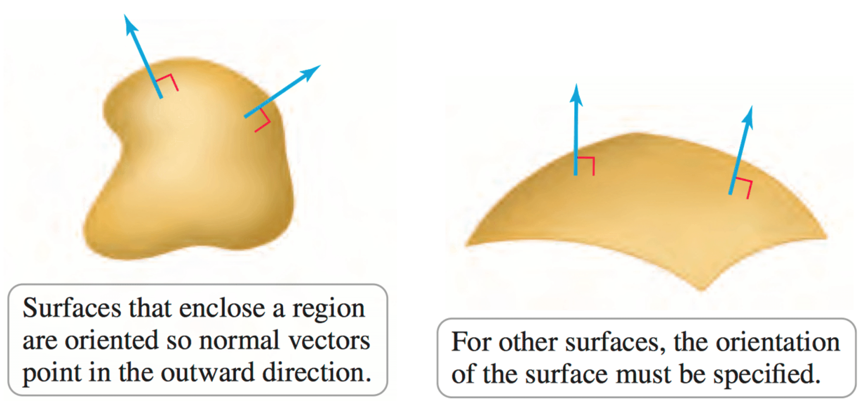


Figure 17.56

Now recall that the parameterization of a surface defines a normal vector $\mathbf{t}_u \times \mathbf{t}_v$ at each point. In many cases, the normal vectors are consistent with the specified orientation, in which case no adjustments need to be made. If the direction of $\mathbf{t}_u \times \mathbf{t}_v$ is not consistent with the specified orientation, then the sign of $\mathbf{t}_u \times \mathbf{t}_v$ must be reversed before doing calculations. This process is demonstrated in the following examples.

Flux Integrals

It turns out that the most common surface integral of a vector field is a *flux integral*. Consider a vector field $\mathbf{F} = \langle f, g, h \rangle$, continuous on a region in $\mathbb{R}^3$, that represents the flow of a fluid or the transport of a substance. Given a smooth oriented surface S, we aim to compute the net flux of the vector field across the surface. In a small region containing a point P, the flux across the surface is proportional to the component of $\mathbf{F}$ in the direction of the unit normal vector $\mathbf{n}$ at P. If θ is the angle between $\mathbf{F}$ and $\mathbf{n}$, then this component is $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}|\,|\mathbf{n}| \cos\theta = |\mathbf{F}| \cos\theta$ (because $|\mathbf{n}| = 1$; **Figure 17.57**). We have the following special cases.

- If $\mathbf{F}$ and the unit normal vector are aligned at P ($\theta = 0$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}|$; that is, all of $\mathbf{F}$ flows across the surface in the direction of $\mathbf{n}$.
- If $\mathbf{F}$ and the unit normal vector point in opposite directions at P ($\theta = \pi$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = -|\mathbf{F}|$; that is, all of $\mathbf{F}$ flows across the surface in the direction opposite to that of $\mathbf{n}$.
- If $\mathbf{F}$ and the unit normal vector are orthogonal at P ($\theta = \frac{\pi}{2}$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = 0$; that is, none of $\mathbf{F}$ flows across the surface at that point.

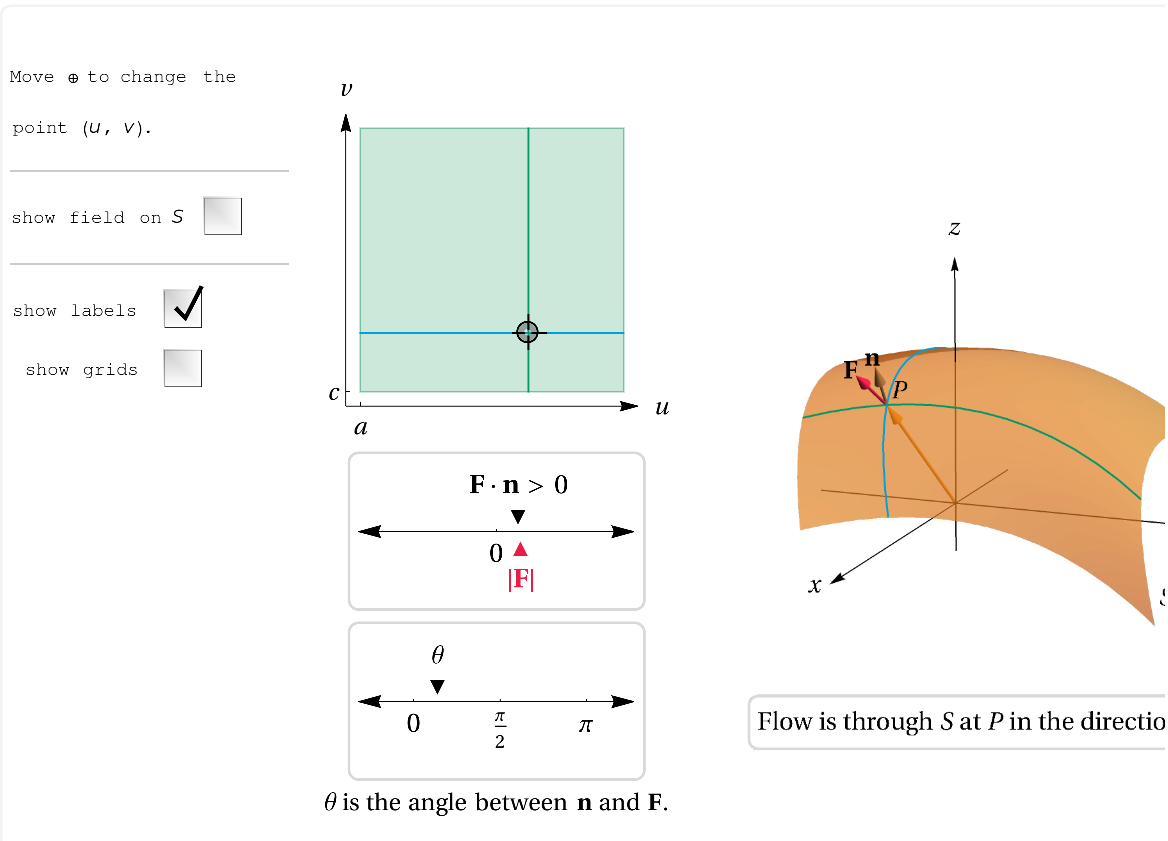


Figure 17.57

The flux integral, denoted $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ or $\iint_S \mathbf{F} \cdot d\mathbf{S}$, simply adds up the components of $\mathbf{F}$ normal to the surface at all points of the surface. Notice that $\mathbf{F} \cdot \mathbf{n}$ is a scalar-valued function. Here is how the flux integral is computed.

Suppose the smooth oriented surface S is parameterized in the form

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle,$$

where u and v vary over a region R in the uv-plane. The required vector normal to the surface at a point is $\mathbf{t}_u \times \mathbf{t}_v$, which we assume to be consistent with the orientation of S. Therefore, the *unit* normal vector consistent with the orientation is $\mathbf{n} = \dfrac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$. Appealing to the definition of the surface integral for parameterized surfaces, the flux integral is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R \mathbf{F}\cdot\mathbf{n}\, |\mathbf{t}_u \times \mathbf{t}_v|\, dA && \text{Definition of surface integral}\\ &= \iint_R \mathbf{F}\cdot\underbrace{\frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}}_{\mathbf{n}} |\mathbf{t}_u \times \mathbf{t}_v|\, dA && \text{Substitute for } \mathbf{n}.\\ &= \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\, dA. && \text{Convenient cancellation}\end{aligned}$$

The remarkable occurrence in the flux integral is the cancellation of the factor $|\mathbf{t}_u \times \mathbf{t}_v|$.

Note »

> If $\mathbf{t}_u \times \mathbf{t}_v$ is not consistent with the specified orientation, its sign must be reversed.

The special case in which the surface S is specified in the form $z = s(x, y)$ follows directly by recalling that the required normal vector to the surface is $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -z_x, -z_y, 1\right\rangle$. In this case, with $\mathbf{F} = \langle f, g, h\rangle$, the integrand of the surface integral is $\mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v) = -f\, z_x - g\, z_y + h$.

DEFINITION **Surface Integral of a Vector Field**

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

where $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right\rangle$ are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x, y)$, for (x, y) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left(-f\, z_x - g\, z_y + h\right) dA.$$

Note »

> The value of the surface integral is independent of the parameterization. However, in contrast to a surface integral of a scalar-valued function, the value of a surface integral of a vector field depends on the orientation of the surface. Changing the orientation changes the sign of the result.

EXAMPLE 7 Rain on a roof

Consider the vertical vector field $\mathbf{F} = \langle 0, 0, -1 \rangle$, corresponding to a constant downward flow. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.

SOLUTION »

In this case, the surface is given explicitly. With $z = 4 - 2x - y$, we have $z_x = -2$ and $z_y = -1$. Therefore, a vector normal to the plane is $\langle -z_x, -z_y, 1 \rangle = \langle 2, 1, 1 \rangle$, which points *upward* (the z-component of the vector is positive). Because we are interested in the *downward* flux of $\mathbf{F}$ across S, the surface must be oriented so the normal vectors point in the negative z-direction. So, we take the normal vector to be $\langle -2, -1, -1 \rangle$ (**Figure 17.58**). Letting R be the region in the xy-plane beneath S and noting that $\mathbf{F} = \langle f, g, h \rangle = \langle 0, 0, -1 \rangle$, the flux integral is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \langle 0, 0, -1 \rangle \cdot \langle -2, -1, -1 \rangle\, dA = \iint_R dA = \text{area } (R).$$

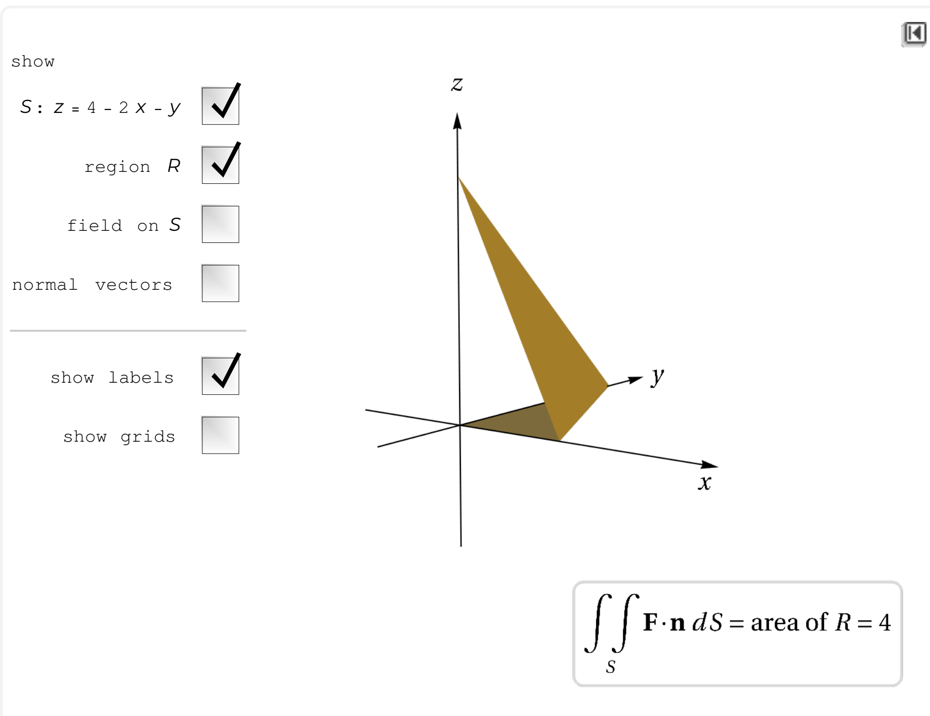


Figure 17.58

The base R is a triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, so its area is 4. Therefore, the *downward* flux across S is 4. This flux integral has an interesting interpretation. If the vector field $\mathbf{F}$ represents the rate of rainfall with units of, say, g/m^2 per unit time, then the flux integral gives the mass of rain (in grams) that falls on the surface in a unit of time. This result says that (because the vector field is vertical) the mass of rain that falls on the roof equals the mass that would fall on the floor beneath the roof if the roof were not there. This property is explored further in Exercise 73.

Related Exercises 43–44 ◆

EXAMPLE 8 **Flux of the radial field**

Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$. Is the upward flux of the field greater across the hemisphere $x^2 + y^2 + z^2 = 1$, for $z \ge 0$, or across the paraboloid $z = 1 - x^2 - y^2$, for $z \ge 0$? Note that the two surfaces have the same base in the xy-plane and the same high point $(0, 0, 1)$. Use the explicit description for the hemisphere and a parametric description for the paraboloid.

SOLUTION »

The base of both surfaces in the xy-plane is the unit disk $R = \left\{(x, y) : x^2 + y^2 \le 1\right\}$, which, when expressed in polar coordinates, is the set $\{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. To use the explicit description for the hemisphere, we must compute z_x and z_y. Differentiating $x^2 + y^2 + z^2 = 1$ implicitly, we find that $z_x = -\dfrac{x}{z}$ and $z_y = -\dfrac{y}{z}$. Therefore, the required normal vector is $\left\langle \dfrac{x}{z}, \dfrac{y}{z}, 1 \right\rangle$, which points upward on the surface.

Note »

The flux integral is evaluated by substituting for f, g, h, z_x, and z_y; eliminating z from the integrand; and converting the integral in x and y to an integral in polar coordinates:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \left(-f\, z_x - g\, z_y + h\right) dA \\
&= \iint_R \left(x \frac{x}{z} + y \frac{y}{z} + z\right) dA && \text{Substitute.} \\
&= \iint_R \left(\frac{x^2 + y^2 + z^2}{z}\right) dA && \text{Simplify.} \\
&= \iint_R \left(\frac{1}{z}\right) dA && x^2 + y^2 + z^2 = 1 \\
&= \iint_R \left(\frac{1}{\sqrt{1 - x^2 - y^2}}\right) dA && z = \sqrt{1 - x^2 - y^2} \\
&= \int_0^{2\pi} \int_0^1 \left(\frac{1}{\sqrt{1 - r^2}}\right) r\, dr\, d\theta && \text{Polar coordinates} \\
&= \int_0^{2\pi} \left(-\sqrt{1 - r^2}\right)\Big|_0^1 d\theta && \text{Evaluate inner integral as an improper integral.} \\
&= \int_0^{2\pi} d\theta = 2\pi. && \text{Evaluate outer integral.}
\end{aligned}$$

For the paraboloid $z = 1 - x^2 - y^2$, we use the parametric description (Example 1b or Table 17.3)

$$\mathbf{r}(u, v) = \langle x, y, z \rangle = \left\langle v \cos u, v \sin u, 1 - v^2 \right\rangle,$$

for $0 \le u \le 2\pi$ and $0 \le v \le 1$. The required vector normal to the surface is

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & -2\,v \end{vmatrix}$$
$$= \left\langle -2\,v^2 \cos u,\, -2\,v^2 \sin u,\, -v \right\rangle.$$

Notice that the normal vectors point downward on the surface (because the z-component is negative for $0 < v \le 1$). In order to find the upward flux, we negate the normal vector and use the upward normal vector

$$-(\mathbf{t}_u \times \mathbf{t}_v) = \left\langle 2\,v^2 \cos u,\, 2\,v^2 \sin u,\, v \right\rangle.$$

The flux integral is evaluated by substituting for $\mathbf{F} = \langle x, y, z \rangle$ and $-(\mathbf{t}_u \times \mathbf{t}_v)$, and then evaluating an iterated integral in u and v:

$$\int \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^{2\pi} \left\langle v \cos u,\, v \sin u,\, 1 - v^2 \right\rangle \cdot \left\langle 2\,v^2 \cos u,\, 2\,v^2 \sin u,\, v \right\rangle du\, dv \quad \text{Susbtitute for } \mathbf{F} \text{ and } -(\mathbf{t}_u \times \mathbf{t}$$

$$= \int_0^1 \int_0^{2\pi} \left(v^3 + v \right) du\, dv \quad \text{Simplify.}$$

$$= 2\pi \left(\frac{v^4}{4} + \frac{v^2}{2} \right) \Bigg|_0^1 = \frac{3\pi}{2}. \quad \text{Evaluate integrals.}$$

We see that the upward flux is greater for the hemisphere than for the paraboloid.

Related Exercises 45, 47 ◆

Quick Check 6 Explain why the upward flux for the radial field in Example 8 is greater for the hemisphere than for the paraboloid. ◆

Answer »

> The vector field is everywhere orthogonal to the hemisphere, so the hemisphere has maximum flux at every point.

Exercises »

Getting Started »

Practice Exercises »

9–14. Parametric descriptions *Give a parametric description of the form* $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ *for the following surfaces. The descriptions are not unique. Specify the required rectangle in the* uv*-plane.*

9. The plane $2x - 4y + 3z = 16$

10. The cap of the sphere $x^2 + y^2 + z^2 = 16$, for $2\sqrt{2} \le z \le 4$

11. The frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$

12. The cone $z^2 = 4\left(x^2 + y^2\right)$, for $0 \le z \le 4$

13. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

14. The cylinder $y^2 + z^2 = 36$, for $0 \le x \le 9$

15–18. Identify the surface *Describe the surface with the given parametric representation.*

15. $\mathbf{r}(u, v) = \langle u, v, 2u + 3v - 1 \rangle$, for $1 \le u \le 3, 2 \le v \le 4$

16. $\mathbf{r}(u, v) = \langle u, u + v, 2 - u - v \rangle$, for $0 \le u \le 2, 0 \le v \le 2$

17. $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, 4v \rangle$, for $0 \le u \le \pi, 0 \le v \le 3$

18. $\mathbf{r}(u, v) = \langle v, 6 \cos u, 6 \sin u \rangle$, for $0 \le u \le 2\pi, 0 \le v \le 2$

19–24. Surface area using a parametric description *Find the area of the following surfaces using a parametric description of the surface.*

19. The half cylinder $\{(r, \theta, z) : r = 4, 0 \le \theta \le \pi, 0 \le z \le 7\}$

20. The plane $z = 3 - x - 3y$ in the first octant

21. The plane $z = 10 - x - y$ above the square $|x| \le 2, |y| \le 2$

22. The hemisphere $x^2 + y^2 + z^2 = 100$, for $z \ge 0$

23. A cone with base radius r and height h, where r and h are positive constants

24. The cap of the sphere $x^2 + y^2 + z^2 = 4$, for $1 \le z \le 2$

25–28. Surface integrals using a parametric description *Evaluate the surface integral* $\iint_S f(x, y, z)\, dS$ *using a parametric description of the surface.*

25. $f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36$, for $z \ge 0$

26. $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9, 0 \le z \le 3$

27. $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1, 0 \le y \le 3$

28. $f(\rho, \varphi, \theta) = \cos \varphi$, where S is the part of the unit sphere in the first octant

29–34. Surface area using an explicit description *Find the area of the following surfaces using an explicit description of the surface.*

29. The part of the plane $z = 2x + 2y + 4$ over the region R bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$

30. The part of the plane $z = x + 3y + 5$ over the region $R = \{(x, y) : 1 \le x^2 + y^2 \le 4\}$

31. The cone $z^2 = 4\left(x^2 + y^2\right)$, for $0 \le z \le 4$

T **32.** The trough $z = \frac{1}{2} x^2$, for $-1 \le x \le 1, 0 \le y \le 4$

33. The paraboloid $z = 2\left(x^2 + y^2\right)$, for $0 \le z \le 8$

34. The part of the hyperbolic paraboloid $z = 3 + x^2 - y^2$ above the sector $R = \left\{(r, \theta) : 0 \le r \le \sqrt{2}, 0 \le \theta \le \frac{\pi}{2}\right\}$

35–38. Surface integrals using an explicit description *Evaluate the surface integral* $\iint_S f(x, y, z)\, dS$ *using an explicit representation of the surface.*

35. $f(x, y, z) = xy$; S is the plane $z = 2 - x - y$ in the first octant.

36. $f(x, y, z) = x^2 + y^2$; S is the paraboloid $z = x^2 + y^2$, for $0 \le z \le 1$.

37. $f(x, y, z) = 25 - x^2 - y^2$; S is the hemisphere centered at the origin with radius 5, for $z \ge 0$.

38. $f(x, y, z) = e^z$; S is the plane $z = 8 - x - 2y$ in the first octant.

39–42. Average values

39. Find the average temperature on that part of the plane $2x + 2y + z = 4$ over the square $0 \le x \le 1$, $0 \le y \le 1$, where the temperature is given by $T(x, y, z) = e^{2x+y+z-3}$.

T **40.** Find the average squared distance between the origin and the points on the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$.

41. Find the average value of the function $f(x, y, z) = xyz$ on the unit sphere in the first octant.

42. Find the average value of the temperature function $T(x, y, z) = 100 - 25z$ on the cone $z^2 = x^2 + y^2$, for $0 \le z \le 2$.

43–48. Surface integrals of vector fields *Find the flux of the following vector fields across the given surface with the specified orientation. You may use either an explicit or parametric description of the surface.*

43. $\mathbf{F} = \langle 0, 0, -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward.

44. $\mathbf{F} = \langle x, y, z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward.

45. $\mathbf{F} = \langle x, y, z \rangle$ across the slanted surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 1$; normal vectors point upward.

46. $\mathbf{F} = \langle e^{-y}, 2z, xy \rangle$ across the curved sides of the surface $S = \{(x, y, z) : z = \cos y, |y| \le \pi, 0 \le x \le 4\}$; normal vectors point upward.

47. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ across the sphere of radius a centered at the origin, where $\mathbf{r} = \langle x, y, z \rangle$; normal vectors point outward.

48. $\mathbf{F} = \langle -y, x, 1 \rangle$ across the cylinder $y = x^2$, for $0 \le x \le 1$, $0 \le z \le 4$; normal vectors point in the general direction of the positive y-axis.

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

b. If the surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

c. The surface $\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$, for $0 \le u \le \pi$, $0 \le v \le 2$, is the same as the surface $\mathbf{r} = \langle \sqrt{v} \cos 2u, \sqrt{v} \sin 2u, v \rangle$, for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 4$.

d. Given the standard parameterization of a sphere, the normal vectors $\mathbf{t}_u \times \mathbf{t}_v$ are outward normal vectors.

50–53. Miscellaneous surface integrals *Evaluate the following integrals using the method of your choice. Assume normal vectors point either outward or upward.*

50. $\iint_S \nabla \ln |\mathbf{r}| \cdot \mathbf{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\mathbf{r} = \langle x, y, z \rangle$

51. $\iint_S |\mathbf{r}|\, dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, where $\mathbf{r} = \langle x, y, z \rangle$

52. $\iint_S x\, y\, z\, dS$, where S is that part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 4$

53. $\iint_S \frac{\langle x, 0, z \rangle}{\sqrt{x^2 + z^2}} \cdot \mathbf{n}\, dS$, where S is the cylinder $x^2 + z^2 = a^2$, $|y| \le 2$

54. Cone and sphere The cone $z^2 = x^2 + y^2$, for $z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a. Find the surface area of the sphere below C, for $z \ge 0$.

b. Find the surface area of the sphere above C.

c. Find the surface area of the cone below C, for $z \ge 0$.

T **55. Cylinder and sphere** Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x - 1)^2 + y^2 = 1$, for $z \ge 0$. Find the surface area of the cylinder inside the sphere.

56. Flux on a tetrahedron Find the upward flux of the field $\mathbf{F} = \langle x, y, z \rangle$ across the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant. Show that the flux equals c times the area of the base of the region. Interpret the result physically.

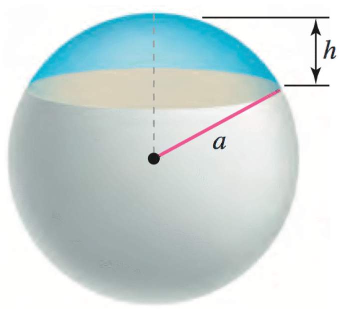
57. Flux across a cone Consider the field $\mathbf{F} = \langle x, y, z \rangle$ and the cone $z^2 = \frac{x^2 + y^2}{a^2}$, for $0 \le z \le 1$.

a. Show that when $a = 1$, the outward flux across the cone is zero. Interpret the result.

b. Find the outward flux (away from the z-axis), for any $a > 0$. Interpret the result.

58. Surface area formula for cones Find the general formula for the surface area of a cone with height h and base radius a (excluding the base).

59. Surface area formula for spherical cap A sphere of radius a is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane (see figure). Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h.



Explorations and Challenges »

60. Radial fields and spheres Consider the radial field $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number. Let S be the sphere of radius a centered at the origin. Show that the outward flux of $\mathbf{F}$ across the sphere is $\frac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

61–63. Heat flux *The heat flow vector field for conducting objects is* $\mathbf{F} = -k\nabla T$, *where* $T(x, y, z)$ *is the temperature in the object and* $k > 0$ *is a constant that depends on the material. Compute the outward flux of* $\mathbf{F}$ *across the following surfaces* S *for the given temperature distributions. Assume* $k = 1$.

61. $T(x, y, z) = 100\, e^{-x-y}$; S consists of the faces of the cube $|x| \le 1, |y| \le 1, |z| \le 1$.

62. $T(x, y, z) = 100\, e^{-x^2-y^2-z^2}$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

63. $T(x, y, z) = -\ln\left(x^2 + y^2 + z^2\right)$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

64. Flux across a cylinder Let S be the cylinder $x^2 + y^2 = a^2$, for $-L \le z \le L$.

a. Find the outward flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ across S.

b. Find the outward flux of the field $\mathbf{F} = \frac{\langle x, y, 0 \rangle}{\left(x^2 + y^2\right)^{p/2}} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$ across S, where $|\mathbf{r}|$ is the distance from the z-axis and p is a real number.

c. In part (b), for what values of p is the outward flux finite as $a \to \infty$ (with L fixed)?

d. In part (b), for what values of p is the outward flux finite as $L \to \infty$ (with a fixed)?

65. Flux across concentric spheres Consider the radial fields $\mathbf{F} = \frac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{p/2}} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number. Let S consist of the spheres A and B centered at the origin with radii $0 < a < b$, respectively. The total outward flux across S consists of the flux out of S across the outer sphere B minus the flux into S across the inner sphere A.

a. Find the total flux across S with $p = 0$. Interpret the result.

b. Show that for $p = 3$ (an inverse square law), the flux across S is independent of a and b.

66–69. Mass and center of mass *Let S be a surface that represents a thin shell with density ρ. The moments about the coordinate planes (see Section 16.6) are* $M_{yz} = \iint_S x\,\rho(x, y, z)\,dS$, $M_{xz} = \iint_S y\,\rho(x, y, z)\,dS$, *and* $M_{xy} = \iint_S z\,\rho(x, y, z)\,dS$. *The coordinates of the center of mass of the shell are* $\bar{x} = \frac{M_{yz}}{m}$, $\bar{y} = \frac{M_{xz}}{m}$, *and* $\bar{z} = \frac{M_{xy}}{m}$, *where m is the mass of the shell. Find the mass and center of mass of the following shells. Use symmetry whenever possible.*

66. The constant-density hemispherical shell $x^2 + y^2 + z^2 = a^2$, $z \ge 0$

67. The constant-density cone with radius a, height h, and base in the xy-plane

68. The constant-density half cylinder $x^2 + z^2 = a^2$, $-\frac{h}{2} \le y \le \frac{h}{2}$, $z \ge 0$

69. The cylinder $x^2 + y^2 = a^2$, $0 \le z \le 2$, with density $\rho(x, y, z) = 1 + z$

70. Outward normal to a sphere Show that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ for a sphere of radius a defined parametrically by $\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

71. Special case of surface integrals of scalar-valued functions Suppose a surface S is defined as $z = g(x, y)$ on a region R. Show that $\mathbf{t}_x \times \mathbf{t}_y = \left\langle -z_x, -z_y, 1 \right\rangle$ and that
$$\iint_S f(x, y, z)\,dS = \iint_R f(x, y, g(x, y))\,\sqrt{{z_x}^2 + {z_y}^2 + 1}\;dA.$$

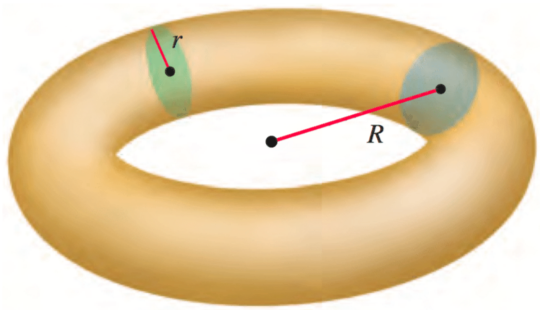
72. Surfaces of revolution Suppose $y = f(x)$ is a continuous and positive function on $[a, b]$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis.

- **a.** Show that S is described parametrically by $\mathbf{r}(u, v) = \langle u, f(u) \cos v, f(u) \sin v \rangle$, for $a \le u \le b$, $0 \le v \le 2\pi$.
- **b.** Find an integral that gives the surface area of S.
- **c.** Apply the result of part (b) to the surface generated with $f(x) = x^3$, for $1 \le x \le 2$.
- **d.** Apply the result of part (b) to the surface generated with $f(x) = \left(25 - x^2\right)^{1/2}$, for $3 \le x \le 4$.

73. Rain on roofs Let $z = s(x, y)$ define a surface over a region R in the xy-plane, where $z \ge 0$ on R. Show that the downward flux of the vertical vector field $\mathbf{F} = \langle 0, 0, -1 \rangle$ across S equals the area of R. Interpret the result physically.

74. Surface area of a torus

- **a.** Show that a torus with radii $R > r$ (see figure) may be described parametrically by $r(u, v) = \langle (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u \rangle$, for $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.
- **b.** Show that the surface area of the torus is $4\pi^2 R r$.



75. Surfaces of revolution—single variable Let f be differentiable and positive on the interval $[a, b]$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis. Use Theorem 17.14 to show that the area of S (as given in Section 6.6) is

$$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx.$$