17.5 Divergence and Curl

Green's Theorem sets the stage for the final act in our exploration of calculus. The last four sections of this chapter have the following goal: to lift both forms of Green's Theorem out of the plane ($\mathbb{R}^2$) and into space ($\mathbb{R}^3$). It is done as follows.

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. In an analogous manner, we will see that *Stokes' Theorem* (Section 17.7) relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is the same curve.
- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the *Divergence Theorem* (Section 17.8) relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by that surface.

In order to make these extensions, we need a few more tools.

- The two-dimensional divergence and two-dimensional curl must be extended to three dimensions (this section).
- The idea of a *surface integral* must be introduced (Section 17.6).

The Divergence »

Recall that in two dimensions the divergence of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. The extension to three dimensions is straightforward. If $\mathbf{F} = \langle f, g, h\rangle$ is a differentiable vector field defined on a region of $\mathbb{R}^3$, the divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$. The interpretation of the three-dimensional divergence is much the same as it is in two dimensions. It measures the expansion or contraction of the vector field at each point. If the divergence is zero at all points of a region, the vector field is *source free* on that region.

Note »

Recall the *del operator* ∇ that was introduced in Section 15.5 to define the gradient:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle.$$

This object is not really a vector; it is an operation that is applied to a function or a vector field. Applying it directly to a scalar function f results in the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \langle f_x, f_y, f_z\rangle.$$

However, if we form the *dot product* of ∇ and a vector field $\mathbf{F} = \langle f, g, h\rangle$, the result is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle f, g, h\rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial y},$$

which is the divergence of $\mathbf{F}$, also denoted div $\mathbf{F}$. Like all dot products, the divergence is a scalar; in this case, it is a scalar-valued function.

Note »

DEFINITION **Divergence of a Vector Field**

The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.

EXAMPLE 1 Computing the divergence

Compute the divergence of the following vector fields.

a. $\mathbf{F} = \langle x, y, z \rangle$ (a radial field)

b. $\mathbf{F} = \langle -y, x - z, y \rangle$ (a rotation field)

c. $\mathbf{F} = \langle -y, x, z \rangle$ (a spiral flow)

SOLUTION »

a. The divergence is $\nabla \cdot \mathbf{F} = \nabla \cdot \langle x, y, z \rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. Because the divergence is positive, the flow expands outward at all points (**Figure 17.38**).

b. The divergence is

$$\nabla \cdot \mathbf{F} = \nabla \cdot \langle -y, x - z, y \rangle = \frac{\partial(-y)}{\partial x} + \frac{\partial(x - z)}{\partial y} + \frac{\partial y}{\partial z} = 0 + 0 + 0 = 0,$$

so the field is source-free.

c. This field is a combination of the two-dimensional rotation field $\mathbf{F} = \langle -y, x \rangle$ and a vertical flow in the z-direction; the net effect is a field that spirals upward for $z > 0$ and spirals downward for $z < 0$. The divergence is

$$\nabla \cdot \mathbf{F} = \nabla \cdot \langle -y, x, z \rangle = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial z}{\partial z} = 0 + 0 + 1 = 1.$$

The rotational part of the field in x and y does not contribute to the divergence. However, the z-component of the field produces a nonzero divergence (Figure 17.38).

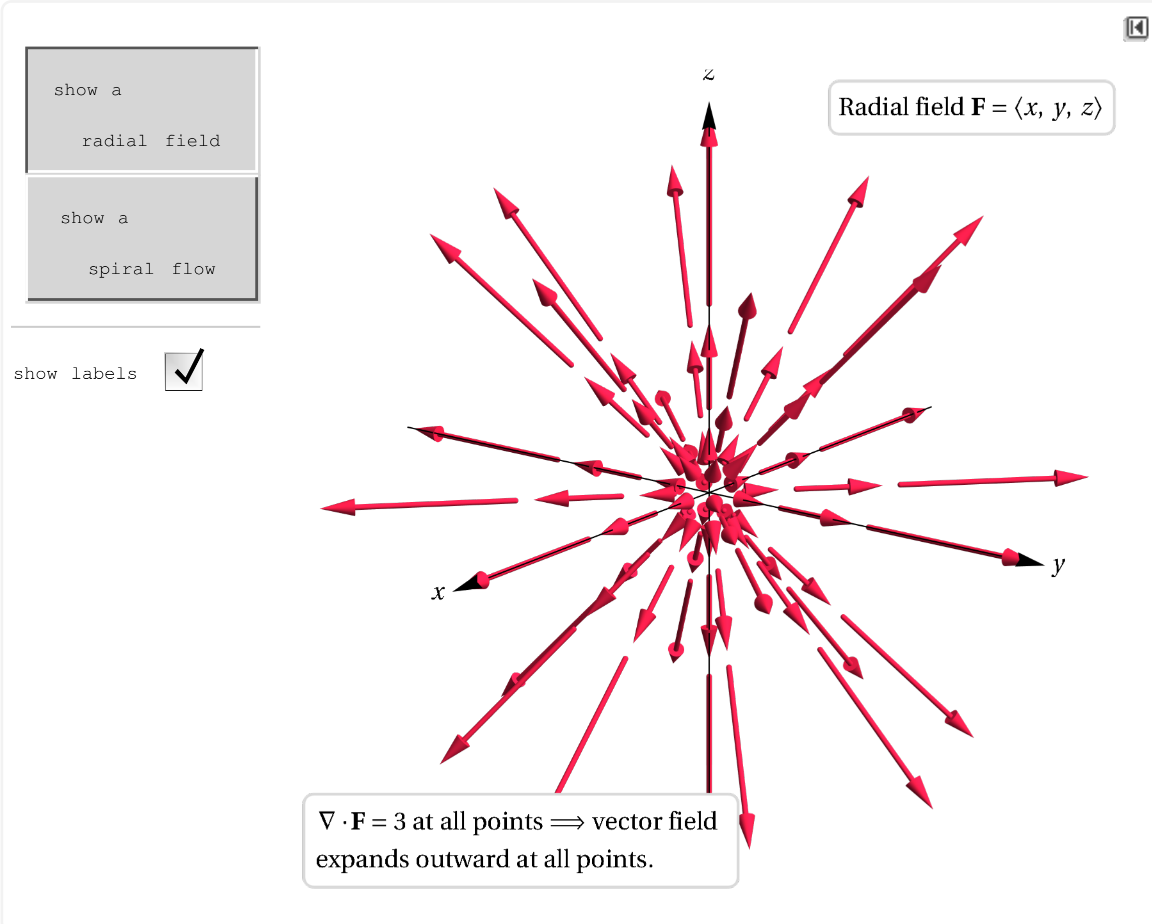


Figure 17.38

Related Exercises 10–11 ◆

Divergence of a Radial Vector Field

The vector field considered in Example 1a is just one of many radial fields that have important applications (for example, the inverse square laws of gravitation and electrostatics). The following example leads to a general result for the divergence of radial vector fields.

Quick Check 1 Show that if a vector field has the form $\mathbf{F} = \langle f(y, z), g(x, z), h(x, y) \rangle$, then div $\mathbf{F} = 0$. ◆

Answer »

EXAMPLE 2 Divergence of a radial field

Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

SOLUTION »

This radial field has the property that it is directed outward from the origin and all vectors have unit length ($|\mathbf{F}| = 1$). Let's compute one piece of the divergence; the others follow the same pattern. Using the Quotient Rule, the derivative with respect to x of the first component of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} \quad \text{Quotient Rule}$$

$$= \frac{|\mathbf{r}| - x^2\,|\mathbf{r}|^{-1}}{|\mathbf{r}|^2} \quad \sqrt{x^2+y^2+z^2} = |\mathbf{r}|$$

$$= \frac{|\mathbf{r}|^2 - x^2}{|\mathbf{r}|^3}. \quad \text{Simplify.}$$

A similar calculation of the y- and z-derivatives yields $\frac{|\mathbf{r}|^2 - y^2}{|\mathbf{r}|^3}$ and $\frac{|\mathbf{r}|^2 - z^2}{|\mathbf{r}|^3}$, respectively. Adding the three terms, we find that

$$\nabla \cdot \mathbf{F} = \frac{|\mathbf{r}|^2 - x^2}{|\mathbf{r}|^3} + \frac{|\mathbf{r}|^2 - y^2}{|\mathbf{r}|^3} + \frac{|\mathbf{r}|^2 - z^2}{|\mathbf{r}|^3}$$

$$= 3\,\frac{|\mathbf{r}|^2}{|\mathbf{r}|^3} - \frac{x^2+y^2+z^2}{|\mathbf{r}|^3} \quad \text{Collect terms.}$$

$$= \frac{2}{|\mathbf{r}|}. \quad x^2+y^2+z^2 = |\mathbf{r}|^2$$

Related Exercise 18 ◆

Examples 1a and 2 give two special cases of the following theorem about the divergence of radial vector fields (Exercise 73).

THEOREM 17.10 Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}.$$

EXAMPLE 3 Divergence from a graph

To gain some intuition about the divergence, consider the two-dimensional vector field $\mathbf{F} = \langle f, g\rangle = \left\langle x^2, y\right\rangle$ and a circle C of radius 2 centered at the origin (**Figure 17.39**).

a. Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1, 1)$. Why?

b. Confirm your conjecture in part (a) by computing the two-dimensional divergence at Q.

c. Based on part (b), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence negative?

d. By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

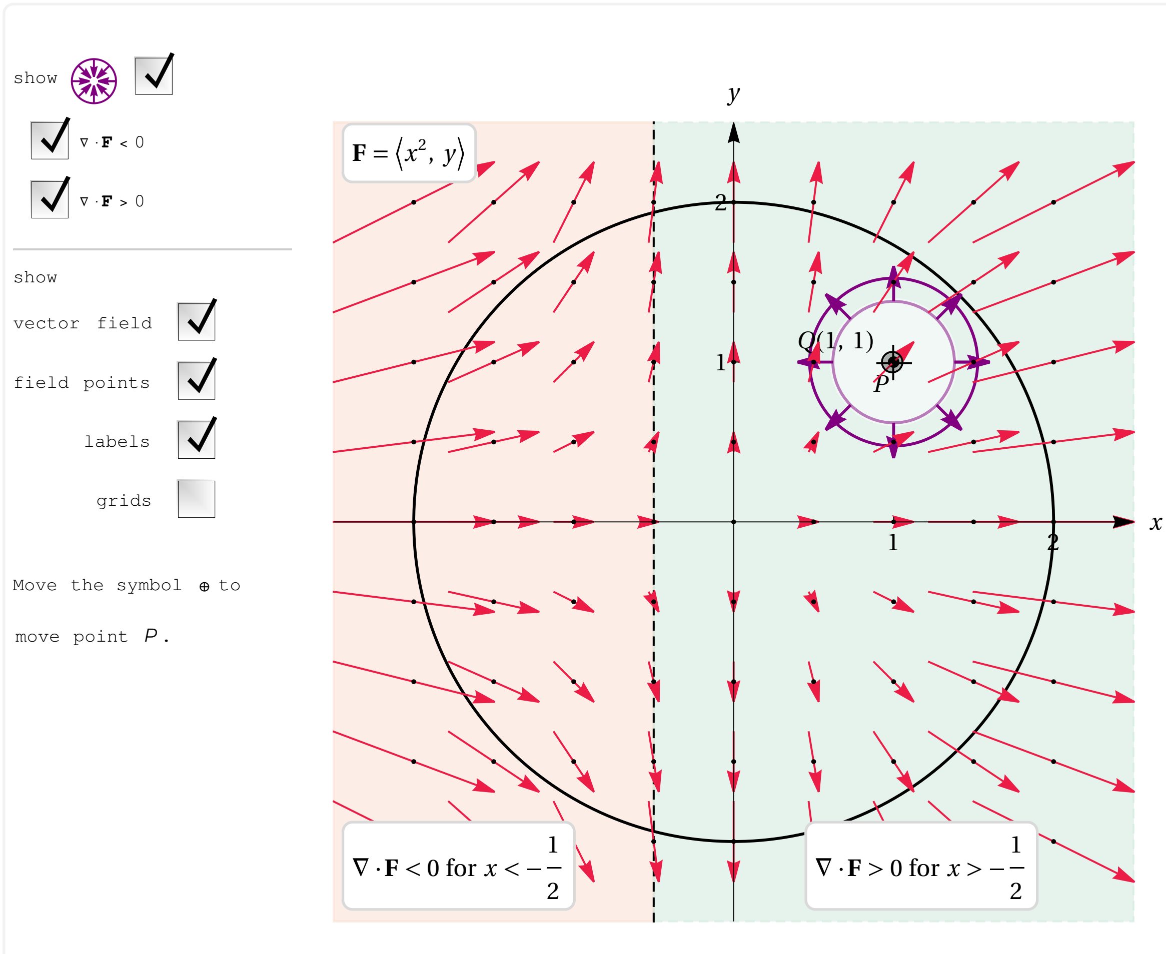


Figure 17.39

SOLUTION »

a. At $Q(1, 1)$ the x-component and the y-component of the field are increasing ($f_x > 0$ and $g_y > 0$), so the field is expanding at that point and the two-dimensional divergence is positive.

b. Calculating the two-dimensional divergence, we find that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(y) = 2x + 1.$$

At $Q(1, 1)$ the divergence is 3, confirming part (a).

Note »

To understand the conclusion reached in the solution to Example 3a, note that as you move through the point Q from left to right, the horizontal components of the vectors increase in length ($f_x > 0$). As you move through the point Q in the upward direction, the vertical components of the vectors also increase in length ($g_y > 0$).

c. From part (b) we see that $\nabla \cdot \mathbf{F} = 2x + 1 > 0$ for $x > -\frac{1}{2}$ and $\nabla \cdot \mathbf{F} < 0$ for $x < -\frac{1}{2}$. To the left of the line

$x = -\frac{1}{2}$ the field is contracting and to the right of the line the field is expanding.

d. Using Figure 17.39, it appears that the field is tangent to the circle at two points with $x \approx -1$. For points on the circle with $x < -1$, the flow is into the circle; for points on the circle with $x > -1$, the flow is out of the circle. It appears that the net outward flux across C is positive. The points where the field changes from inward to outward may be determined exactly (Exercise 46).

Related Exercises 21–22 ◆

Quick Check 2 Verify the claim made in part (d) of Example 3 by showing that the net outward flux of **F** across C is positive. (*Hint:* If you use Green's Theorem to evaluate the integral $\int_C f\,dy - g\,dx$, convert to polar coordinates.) ◆

Answer »

The net outward flux is 4π

The Curl »

Note »

Review: The *two-dimensional curl* $g_x - f_y$ measures the rotation of a vector field at a point. The circulation form of Green's Theorem implies that if the two-dimensional curl of a vector field is zero throughout a simply connected region, then the circulation on the boundary of the region is also zero. If the curl is nonzero, Green's Theorem gives the circulation along the curve.

Just as the divergence $\nabla \cdot \mathbf{F}$ is the dot product of the *del operator* and **F**, the three-dimensional curl is the cross product $\nabla \times \mathbf{F}$. If we formally use the notation for the cross product in terms of a 3×3 determinant, we obtain the definition of the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \begin{matrix} \longleftarrow \text{ Unit vectors} \\ \longleftarrow \text{ Components of } \nabla \\ \longleftarrow \text{ Components of } \mathbf{F} \end{matrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

The curl of a vector field, also denoted curl **F**, is a vector with three components. Notice that the **k**-component of the curl $(g_x - f_y)$ is the two-dimensional curl, which gives the rotation in the *xy*-plane at a point. The **i**- and **j**-components of the curl correspond to the rotation of the vector field in planes parallel to the *yz*-plane (orthogonal to **i**) and in planes parallel to the *xz*-plane (orthogonal to **j**) (**Figure 17.40**).

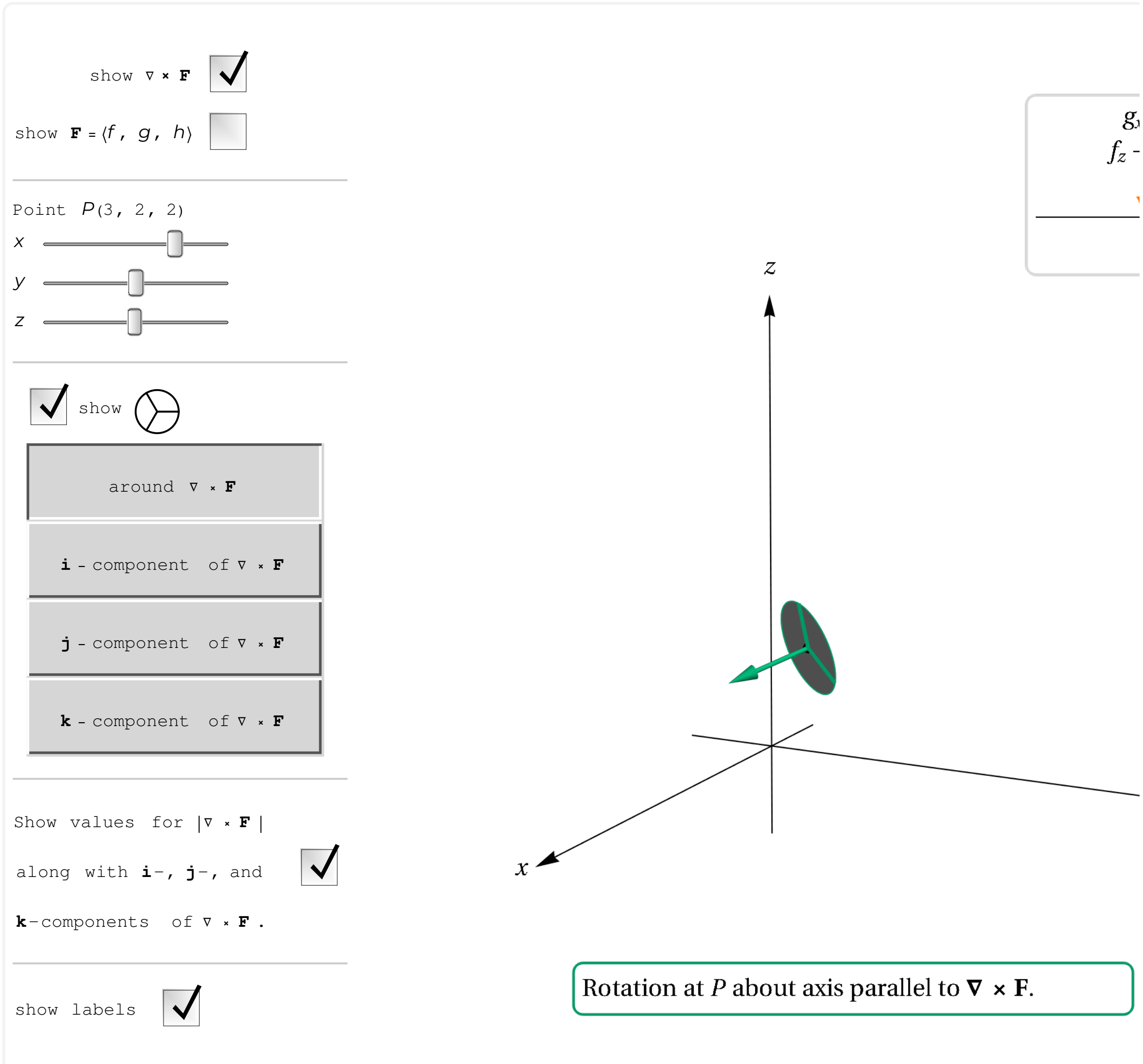


Figure 17.40

DEFINITION **Curl of a Vector Field**

The curl of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\begin{aligned}\nabla \times \mathbf{F} &= \text{curl } \mathbf{F}\\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\boldsymbol{k}.\end{aligned}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a General Rotation Vector Field

We can clarify the physical meaning of the curl by considering the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z \rangle$. Writing out its components, we see that

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2\, z - a_3\, y)\,\mathbf{i} + (a_3\, x - a_1\, z)\,\mathbf{j} + (a_1\, y - a_2\, x)\,\mathbf{k}.$$

This vector field is a *general rotation field* in three dimensions. With $a_1 = a_2 = 0$, and $a_3 = 1$, we have the familiar two-dimensional rotation field $\langle -y, x \rangle$ with its axis in the $\mathbf{k}$-direction. More generally, $\mathbf{F}$ is the superposition of three rotation fields with axes in the $\mathbf{i}$-, $\mathbf{j}$-, and $\mathbf{k}$-directions. The result is a single rotation field with an axis in the direction of $\mathbf{a}$ (**Figure 17.41**).

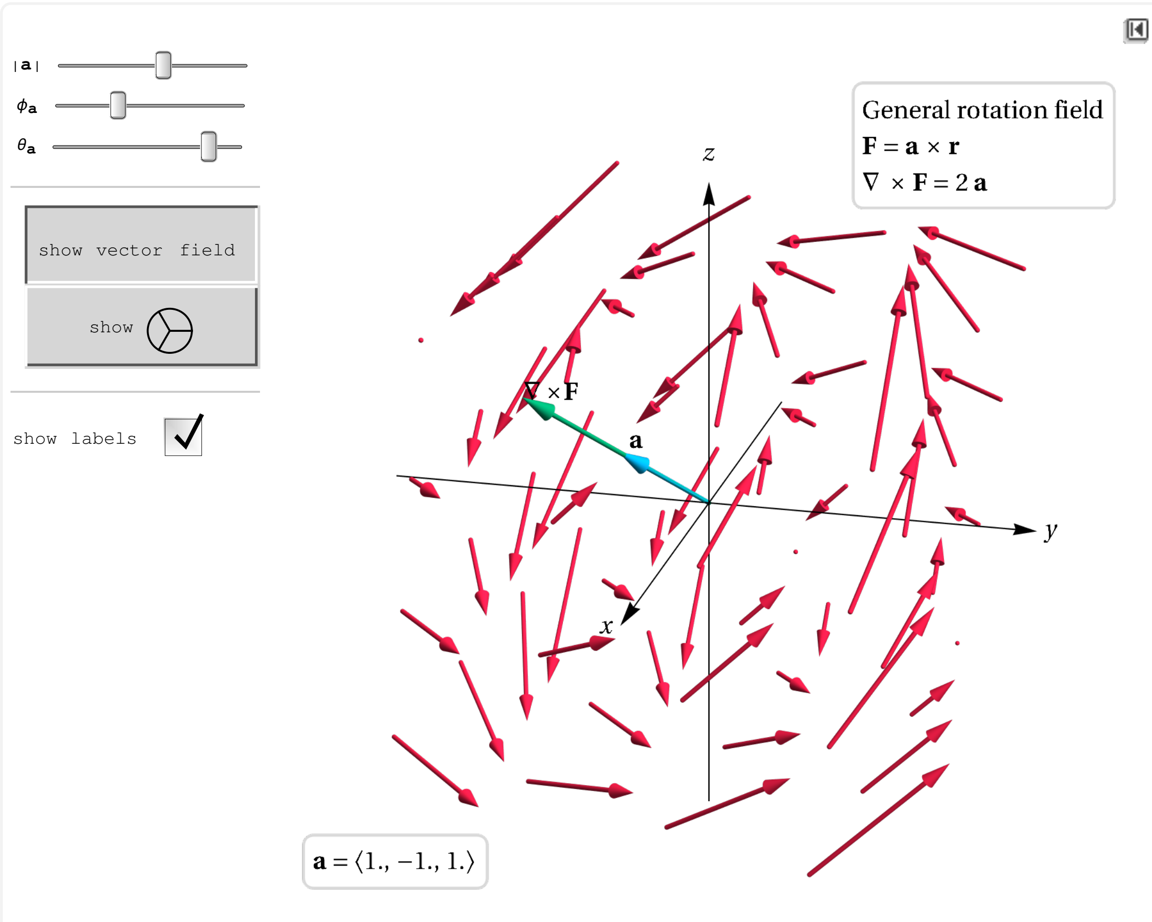


Figure 17.41

Three calculations tell us a lot about the general rotation field. The first calculation confirms that $\nabla \cdot \mathbf{F} = 0$ (Exercise 42). Just as with rotation fields in two dimensions, the divergence of a general rotation field is zero.

The second calculation (Exercises 43–44) uses the right-hand rule for cross products to show that the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ is indeed a rotation field that circles the vector $\mathbf{a}$ in a counterclockwise direction looking along the length of $\mathbf{a}$ from head to tail (Figure 17.41).

The third calculation (Exercise 45) says that $\nabla \times \mathbf{F} = 2\,\mathbf{a}$. Therefore, the curl of the general rotation field is in the direction of the axis of rotation $\mathbf{a}$ (Figure 17.41). The magnitude of the curl is $|\nabla \times \mathbf{F}| = 2\,|\mathbf{a}|$. It can be shown (Exercise 52) that if $\mathbf{F}$ is a velocity field, then $|\mathbf{a}|$ is the constant angular speed of rotation of the vector field, denoted ω. The angular speed is the rate (radians per unit time) at which a small particle in the vector field rotates about the axis of the field. Therefore the angular speed is half the magnitude of the curl, or

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

The rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ suggests a related question. Suppose a paddle wheel is placed in the vector field $\mathbf{F}$ at a point P with the axis of the wheel in the direction of a unit vector $\mathbf{n}$ (**Figure 17.42**). How should $\mathbf{n}$ be chosen so the paddle wheel spins fastest? The scalar component of $\nabla \times \mathbf{F}$ in the direction of $\mathbf{n}$ is

$$(\nabla \times \mathbf{F})\cdot\mathbf{n} = |\nabla \times \mathbf{F}| \cos\theta, \quad (|\mathbf{n}| = 1)$$

where θ is the angle between $\nabla \times \mathbf{F}$ and $\mathbf{n}$. The scalar component is greatest in magnitude and the paddle wheel spins fastest when $\theta = 0$ or $\theta = \pi$; that is, when $\mathbf{n}$ and $\nabla \times \mathbf{F}$ are parallel. If the axis of the paddle wheel is orthogonal to $\nabla \times \mathbf{F}$ $(\theta = \pm\frac{\pi}{2})$, the wheel doesn't spin.

Note »

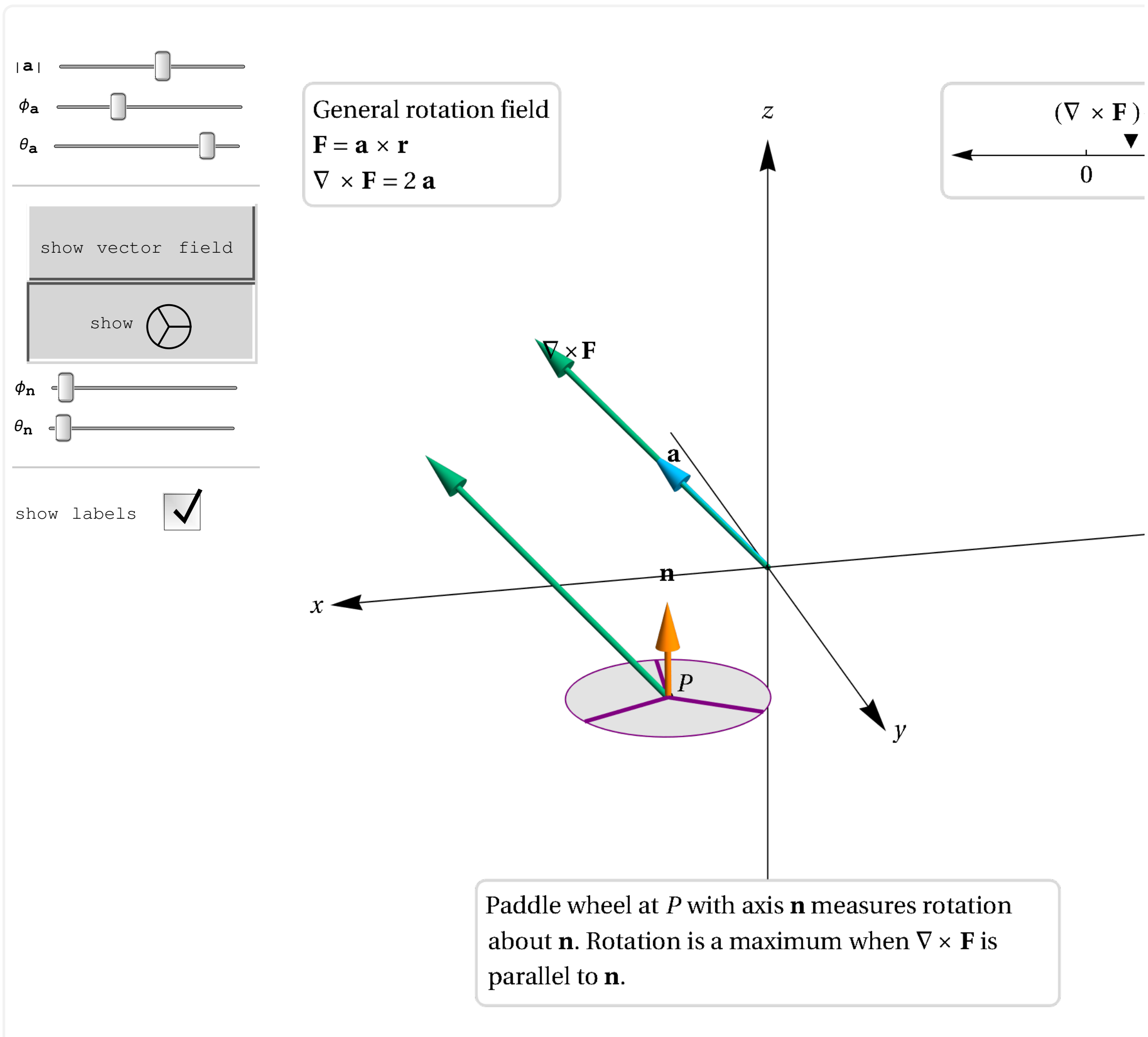


Figure 17.42

General Rotation Vector Field

The general rotation vector field is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\mathbf{r} = \langle x, y, z \rangle$. For all choices of $\mathbf{a}$, $|\nabla \times \mathbf{F}| = 2\,|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\mathbf{a}| = \frac{1}{2}\,|\nabla \times \mathbf{F}|.$$

Quick Check 3 Show that if a vector field has the form $\mathbf{F} = \langle f(x), g(y), h(z) \rangle$, then $\nabla \times \mathbf{F} = \mathbf{0}$. ◆

Answer »

EXAMPLE 4 Curl of a rotation field

Compute the curl of the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle 2, -1, 1 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$ (Figure 17.41). What is the direction and the magnitude of the curl?

SOLUTION »

A quick calculation shows that

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = (-y - z)\,\mathbf{i} + (x - 2\,z)\,\mathbf{j} + (x + 2\,y)\,\mathbf{k}.$$

The curl of the field is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y - z & x - 2\,z & x + 2\,y \end{vmatrix} = 4\,\boldsymbol{i} - 2\,\mathbf{j} + 2\,\mathbf{k} = 2\,\mathbf{a}.$$

We have confirmed that $\nabla \times \mathbf{F} = 2\,\mathbf{a}$ and that the direction of the curl is the direction of $\mathbf{a}$, which is the axis of rotation. The magnitude of $\nabla \times \mathbf{F}$ is $|2\,\mathbf{a}| = 2\sqrt{6}$, which is twice the angular speed of rotation.

Related Exercises 25–26 ◆

Working with Divergence and Curl »

The divergence and curl satisfy many of the same properties that ordinary derivatives satisfy. For example, given a real number c and differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$, we have the following properties.

Divergence Properties	Curl Properties
$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$
$\nabla \cdot (c\,\mathbf{F}) = c\,(\nabla \cdot \mathbf{F})$	$\nabla \times (c\,\mathbf{F}) = c\,(\nabla \times \mathbf{F})$

These and other properties are explored in Exercises 65–72.

Additional properties that have importance in theory and applications are presented in the following theorems and examples.

THEOREM 17.11 Curl of a Conservative Vector Field

Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$; that is, the curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Proof: We must calculate $\nabla \times \nabla\varphi$:

$$\nabla \times \nabla\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \underbrace{(\varphi_{zy} - \varphi_{yz})}_{0}\mathbf{i} + \underbrace{(\varphi_{xz} - \varphi_{zx})}_{0}\mathbf{j} + \underbrace{(\varphi_{yx} - \varphi_{xy})}_{0}\mathbf{k} = \mathbf{0}.$$

The mixed partial derivatives are equal by Clairaut's Theorem (Theorem 15.4).

The converse of this theorem (if $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative field) is handled in Section 17.7 by means of Stokes' Theorem. ◆

THEOREM 17.12 Divergence of the Curl

Suppose $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Note »

Proof: Again, a calculation is needed:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\ &= \underbrace{(h_{yx} - h_{xy})}_{0} + \underbrace{(g_{xz} - g_{zx})}_{0} + \underbrace{(f_{zy} - f_{yz})}_{0} = 0. \end{aligned}$$

Clairaut's Theorem (Theorem 15.4) ensures that the mixed partial derivatives are equal. ◆

The gradient, the divergence, and the curl may be combined in many ways—some of which are undefined. For example, the gradient of the curl ($\nabla(\nabla \times \mathbf{F})$) and the curl of the divergence ($\nabla \times (\nabla \cdot \mathbf{F})$) are undefined. However, a combination that *is* defined and is important is the divergence of the gradient $\nabla \cdot \nabla u$, where u is a scalar-valued function. This combination is denoted $\nabla^2 u$ and is called the **Laplacian** of u; it arises in many physical situations (Exercises 56–58, 62). Carrying out the calculation, we find that

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

We close with a result that is useful in its own right but also intriguing because it parallels the Product Rule from single-variable calculus.

THEOREM 17.13 **Product Rule for the Divergence**

Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\,\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\,(\nabla \cdot \mathbf{F}).$$

The rule says that the "derivative" of the product is the "derivative" of the first function multiplied by the second function plus the first function multiplied by the "derivative" of the second function. However, in each instance "derivative" must be interpreted correctly for the operations to make sense. The proof of the theorem requires a direct calculation (Exercise 67). Other similar vector calculus identities are presented in Exercises 68–72.

Quick Check 4 Is $\nabla \cdot (u\,\mathbf{F})$ a vector function or a scalar function? ◆

Answer »

EXAMPLE 5 **More properties of radial fields**

Let $\mathbf{r} = \langle x, y, z \rangle$ and let $\varphi = \dfrac{1}{|\mathbf{r}|} = \left(x^2 + y^2 + z^2\right)^{-1/2}$ be a potential function.

a. Find the associated gradient field $\mathbf{F} = \nabla \left(\dfrac{1}{|\mathbf{r}|}\right)$.

b. Compute $\nabla \cdot \mathbf{F}$.

SOLUTION »

a. The gradient has three components. Computing the first component reveals a pattern:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} 2\,x = -\frac{x}{|\mathbf{r}|^3}.$$

Making a similar calculation for the y- and z-derivatives, the gradient is

$$\mathbf{F} = \nabla \left(\frac{1}{|\mathbf{r}|}\right) = -\frac{\langle x, y, z \rangle}{|\mathbf{r}|^3} = -\frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

This result reveals that $\mathbf{F}$ is an inverse square vector field (for example, a gravitational or electric field), and its potential function is $\varphi = \dfrac{1}{|\mathbf{r}|}$.

b. The divergence $\nabla \cdot \mathbf{F} = \nabla \cdot \left(-\dfrac{\mathbf{r}}{|\mathbf{r}|^3}\right)$ involves a product of the vector function $\mathbf{r} = \langle x, y, z \rangle$ and the scalar function $|\mathbf{r}|^{-3}$. Applying Theorem 17.13, we find that

$$\nabla \cdot \mathbf{F} = \nabla \cdot \left(-\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = -\nabla \frac{1}{|\mathbf{r}|^3} \cdot \mathbf{r} - \frac{1}{|\mathbf{r}|^3}\,\nabla \cdot \mathbf{r}.$$

A calculation similar to part (a) shows that $\nabla \dfrac{1}{|\mathbf{r}|^3} = \dfrac{-3\,\mathbf{r}}{|\mathbf{r}|^5}$ (Exercise 35). Therefore,

$$\nabla\cdot\mathbf{F} = \nabla\cdot\left(-\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = -\underbrace{\nabla\frac{1}{|\mathbf{r}|^3}}_{-3\,\mathbf{r}/|\mathbf{r}|^5}\cdot\mathbf{r} - \frac{1}{|\mathbf{r}|^3}\underbrace{\nabla\cdot\mathbf{r}}_{3}$$

$$= \frac{3\,\mathbf{r}}{|\mathbf{r}|^5}\cdot\mathbf{r} - \frac{3}{|\mathbf{r}|^3} \qquad \text{Substitute for } \nabla\frac{1}{|\mathbf{r}|^3}.$$

$$= \frac{3\,|\mathbf{r}|^2}{|\mathbf{r}|^5} - \frac{3}{|\mathbf{r}|^3} \qquad \mathbf{r}\cdot\mathbf{r} = |\mathbf{r}|^2$$

$$= 0.$$

The result is consistent with Theorem 17.10 (with $p = 3$): The divergence of an inverse square vector field in $\mathbb{R}^3$ is zero. It does not happen for any other radial fields of this form.

Related Exercises 35–36 ◆

Summary of Properties of Conservative Vector Fields »

We can now extend the list of equivalent properties of conservative vector fields **F** defined on an open connected region. Theorem 17.11 is added to the list given at the end of Section 17.3.

Properties of a Conservative Vector Field

Let **F** be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then **F** has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).

2. $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all piecewise-smooth oriented curves C in D from A to B.

3. $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.

4. $\nabla\times\mathbf{F} = \mathbf{0}$ at all points of D.

Exercises »

Getting Started »

Practice Exercises »

9–16. Divergence of vectors fields *Find the divergence of the following vector fields.*

9. $\mathbf{F} = \langle 2x, 4y, -3z\rangle$

10. $\mathbf{F} = \langle -2y, 3x, z\rangle$

11. $\mathbf{F} = \langle 12x, -6y, -6z\rangle$

12. $\mathbf{F} = \left\langle x^2yz, -xy^2z, -xyz^2\right\rangle$

13. $\mathbf{F} = \left\langle x^2 - y^2, y^2 - z^2, z^2 - x^2\right\rangle$

14. $\mathbf{F} = \langle e^{-x+y}, e^{-y+z}, e^{-z+x}\rangle$

15. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{1 + x^2 + y^2}$

16. $\mathbf{F} = \langle yz \sin x,\ xz \cos y,\ xy \cos z \rangle$

17–20. Divergence of radial fields *Calculate the divergence of the following radial fields. Express the result in terms of the position vector* $\mathbf{r}$ *and its length* $|\mathbf{r}|$. *Check for agreement with Theorem 17.10.*

17. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

18. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

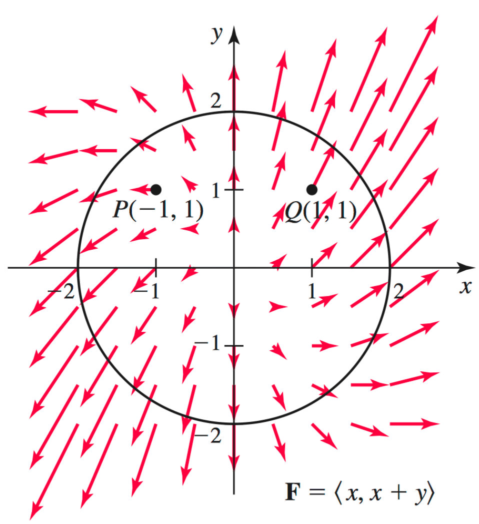
19. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^4}$

20. $\mathbf{F} = \langle x, y, z \rangle \left(x^2 + y^2 + z^2\right) = \mathbf{r}\,|\mathbf{r}|^2$

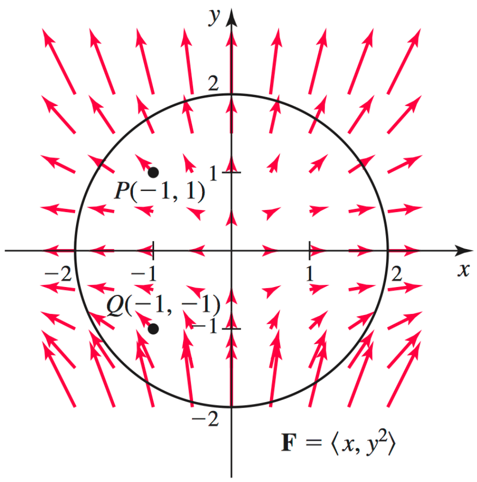
21–22. Divergence and flux from graphs *Consider the following vector fields, the circle C, and two points P and Q.*

- ***a.*** *Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.*
- ***b.*** *Compute the divergence and confirm your conjecture in part (a).*
- ***c.*** *On what part of C is the flux outward? Inward?*
- ***d.*** *Is the net outward flux across C positive or negative?*

21. $\mathbf{F} = \langle x, x + y \rangle$



22. $\mathbf{F} = \left\langle x, y^2 \right\rangle$



23–26. Curl of a rotational field *Consider the following vector fields, where* $\mathbf{r} = \langle x, y, z \rangle$.

a. *Compute the curl of the field and verify that it has the same direction as the axis of rotation.*

b. *Compute the magnitude of the curl of the field.*

23. $\mathbf{F} = \langle 1, 0, 0 \rangle \times \mathbf{r}$

24. $\mathbf{F} = \langle 1, -1, 0 \rangle \times \mathbf{r}$

25. $\mathbf{F} = \langle 1, -1, 1 \rangle \times \mathbf{r}$

26. $\mathbf{F} = \langle 1, -2, -3 \rangle \times \mathbf{r}$

27–34. Curl of a vector field *Compute the curl of the following vector fields.*

27. $\mathbf{F} = \left\langle x^2 - y^2, xy, z \right\rangle$

28. $\mathbf{F} = \left\langle 0, z^2 - y^2, -yz \right\rangle$

29. $\mathbf{F} = \left\langle x^2 - z^2, 1, 2xz \right\rangle$

30. $\mathbf{F} = \mathbf{r} = \langle x, y, z \rangle$

31. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

32. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$

33. $\mathbf{F} = \left\langle z^2 \sin y, x z^2 \cos y, 2xz \sin y \right\rangle$

34. $\mathbf{F} = \left\langle 3xz^3 e^{y^2}, 2xz^3 e^{y^2}, 3xz^2 e^{y^2} \right\rangle$

35–38. Derivative rules *Prove the following identities. Use Theorem 17.13 (Product Rule) whenever possible.*

35. $\nabla\left(\dfrac{1}{|\mathbf{r}|^3}\right) = \dfrac{-3\,\mathbf{r}}{|\mathbf{r}|^5}$ (used in Example 5)

36. $\nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = \dfrac{-2\,\mathbf{r}}{|\mathbf{r}|^4}$

37. $\nabla\cdot\nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = \dfrac{2}{|\mathbf{r}|^4}$ (*Hint:* Use Exercise 36.)

38. $\nabla(\ln|\mathbf{r}|) = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** For a function f of a single variable, if $f'(x) = 0$ for all x in the domain, then f is a constant function. If $\nabla\cdot\mathbf{F} = 0$ for all points in the domain, then $\mathbf{F}$ is constant.
- **b.** If $\nabla\times\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is constant.
- **c.** A vector field consisting of parallel vectors has zero curl.
- **d.** A vector field consisting of parallel vectors has zero divergence.
- **e.** curl $\mathbf{F}$ is orthogonal to $\mathbf{F}$.

40. Another derivative combination Let $\mathbf{F} = \langle f, g, h\rangle$ and let u be a differentiable scalar-valued function.

- **a.** Take the dot product of $\mathbf{F}$ and the del operator; then apply the result to u to show that

$$(\mathbf{F}\cdot\nabla)\,u = \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right)u$$
$$= f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} + h\frac{\partial u}{\partial z}.$$

- **b.** Evaluate $(\mathbf{F}\cdot\nabla)\left(x y^2 z^3\right)$ at $(1, 1, 1)$, where $\mathbf{F} = \langle 1, 1, 1\rangle$.

41. Does it make sense? Are the following expressions defined? If so, state whether the result is a scalar or a vector. Assume $\mathbf{F}$ is a sufficiently differentiable vector field and φ is a sufficiently differentiable scalar-valued function.

- **a.** $\nabla\cdot\varphi$
- **b.** $\nabla\mathbf{F}$
- **c.** $\nabla\cdot\nabla\varphi$
- **d.** $\nabla(\nabla\cdot\varphi)$
- **e.** $\nabla(\nabla\times\varphi)$
- **f.** $\nabla\cdot(\nabla\cdot\mathbf{F})$
- **g.** $\nabla\times\nabla\varphi$
- **h.** $\nabla\times(\nabla\cdot\mathbf{F})$
- **i.** $\nabla\times(\nabla\times\mathbf{F})$

42. Zero divergence of the rotation field Show that the general rotation field $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$, has zero divergence.

43. General rotation fields

a. Let $\mathbf{a} = \langle 0, 1, 0\rangle$, let $\mathbf{r} = \langle x, y, z\rangle$, and consider the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$. Use the right-hand rule for cross products to find the direction of $\mathbf{F}$ at the points $(0, 1, 1)$, $(1, 1, 0)$, $(0, 1, -1)$, and $(-1, 1, 0)$.

b. With $\mathbf{a} = \langle 0, 1, 0\rangle$, explain why the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ circles the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail (that is, in the negative y-direction).

44. General rotation fields Generalize Exercise 43 to show that the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ circles the vector $\mathbf{a}$ in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

45. Curl of the rotation field For the general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$, show that curl $\mathbf{F} = 2\,\mathbf{a}$.

46. Inward to outward Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\mathbf{F} = \langle f, g\rangle = \left\langle x^2, y\right\rangle$ switches from pointing inward to outward on the circle, or vice versa.

47. Maximum divergence Within the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$, where does div $\mathbf{F}$ have the greatest magnitude when $\mathbf{F} = \left\langle x^2 - y^2, x\,y^2\,z, 2\,xz\right\rangle$?

48. Maximum curl Let $\mathbf{F} = \langle z, 0, -y\rangle$.

a. Find the scalar component of curl $\mathbf{F}$ in the direction of the unit vector $\mathbf{n} = \langle 1, 0, 0\rangle$.

b. Find the scalar component of curl $\mathbf{F}$ in the direction of the unit vector $\mathbf{n} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$.

c. Find the unit vector $\mathbf{n}$ that maximizes $\text{scal}_{\mathbf{n}}\,\langle -1, 1, 0\rangle$ and state the value of $\text{scal}_{\mathbf{n}}\,\langle -1, 1, 0\rangle$ in this direction.

49. Zero component of the curl For what vectors $\mathbf{n}$ is $(\text{curl}\,\mathbf{F})\cdot\mathbf{n} = 0$ when $\mathbf{F} = \langle y, -2\,z, -x\rangle$?

50–51. Find a vector field *Find a vector field* $\mathbf{F}$ *with the given curl. In each case, is the vector field you found unique?*

50. curl $\mathbf{F} = \langle 0, 1, 0\rangle$

51. curl $\mathbf{F} = \langle 0, z, -y\rangle$

Explorations and Challenges »

52. Curl and angular speed Consider the rotational velocity field $\mathbf{v} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$. Use the fact that an object moving in a circular path of radius R with speed $|\mathbf{v}|$ has an angular speed of $\omega = \dfrac{|\mathbf{v}|}{R}$.

a. Sketch a position vector $\mathbf{a}$, which is the axis of rotation for the vector field, and a position vector $\mathbf{r}$ of a point P in $\mathbb{R}^3$. Let θ be the angle between the two vectors. Show that the perpendicular distance from P to the axis of rotation is $R = |\mathbf{r}|\sin\theta$.

b. Show that the speed of a particle in the velocity field is $|\mathbf{a} \times \mathbf{r}|$ and that the angular speed of the object is $|\mathbf{a}|$.

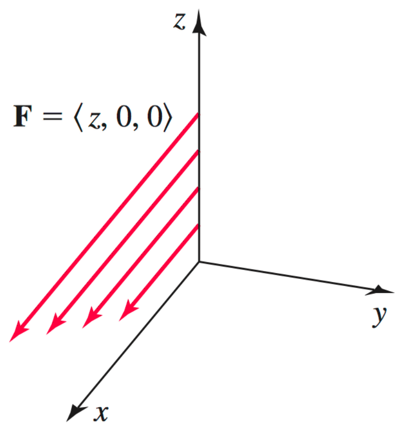
c. Conclude that $\omega = \dfrac{1}{2}\,|\nabla \times \mathbf{v}|$.

53. Paddle wheel in a vector field Let $\mathbf{F} = \langle z, 0, 0 \rangle$ and let $\mathbf{n}$ be a unit vector aligned with the axis of a paddle wheel located on the x-axis (see figure).

a. If the paddle wheel is oriented with $\mathbf{n} = \langle 1, 0, 0 \rangle$, in what direction (if any) does the wheel spin?

b. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 1, 0 \rangle$, in what direction (if any) does the wheel spin?

c. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 0, 1 \rangle$, in what direction (if any) does the wheel spin?



54. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle -2y, 2z, 0 \rangle$.

a. If a paddle wheel is placed in the xy-plane with its axis normal to this plane, what is its angular speed?

b. If a paddle wheel is placed in the xz-plane with its axis normal to this plane, what is its angular speed?

c. If a paddle wheel is placed in the yz-plane with its axis normal to this plane, what is its angular speed?

55. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

56–58. Heat flux *Suppose a solid object in* $\mathbb{R}^3$ *has a temperature distribution given by* $T(x, y, z)$. *The heat flow vector field in the object is* $\mathbf{F} = -k\nabla T$, *where the conductivity* $k > 0$ *is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is* $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ *(the Laplacian of* T*). Compute the heat flow vector field and its divergence for the following temperature distributions.*

56. $T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

57. $T(x, y, z) = 100\, e^{-x^2+y^2+z^2}$

58. $T(x, y, z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

59. Gravitational potential The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\varphi = \dfrac{GMm}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of the mass m and G is the gravitational constant.

a. Compute the gravitational force field $\mathbf{F} = -\nabla \varphi$.

b. Show that the field is irrotational; that is $\nabla \times \mathbf{F} = \mathbf{0}$.

60. Electric potential The potential function for the force field due to a charge q at the origin is $\varphi = \dfrac{1}{4\pi \epsilon_0} \dfrac{q}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y, z\rangle$ is the position vector of a point in the field and ϵ_0 is the permittivity of free space.

a. Compute the force field $\mathbf{F} = -\nabla \varphi$.

b. Show that the field is irrotational; that is $\nabla \times \mathbf{F} = \mathbf{0}$.

61. Navier-Stokes equation The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the flow in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right) = -\nabla p + \mu\,(\nabla\cdot\nabla)\,\mathbf{V}.$$

In this notation $\mathbf{V} = \langle u, v, w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation. (See Exercise 40 for an interpretation of the operations.)

T **62. Stream function and vorticity** The rotation of a three-dimensional velocity field $\mathbf{V} = \langle u, v, w\rangle$ is measured by the **vorticity** $\omega = \nabla \times \mathbf{V}$. If $\omega = \mathbf{0}$ at all points in the domain, the flow is irrotational.

a. Which of the following velocity fields is irrotational: $\mathbf{V} = \langle 2, -3y, 5z\rangle$ or $\mathbf{V} = \langle y, x - z, -y\rangle$?

b. Recall that for a two-dimensional source-free flow $\mathbf{V} = \langle u, v, 0\rangle$, a stream function $\psi(x, y)$ may be defined such that $u = \psi_y$ and $v = -\psi_x$. For such a two-dimensional flow, let $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}$ be the $\mathbf{k}$-component of the vorticity. Show that $\nabla^2\psi = \nabla\cdot\nabla\psi = -\zeta$.

c. Consider the stream function $\psi(x, y) = \sin x \sin y$ on the square region $R = \{(x, y) : 0 \le x \le \pi,\ 0 \le y \le \pi\}$. Find the velocity components u and v; then sketch the velocity field.

d. For the stream function in part (c), find the vorticity function ζ as defined in part (b). Plot several level curves of the vorticity function. Where on R is it a maximum? A minimum?

63. Ampere's Law One of Maxwell's equations for electromagnetic waves is $\nabla \times \mathbf{B} = C\dfrac{\partial \mathbf{E}}{\partial t}$, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and C is a constant.

a. Show that the fields

$$\mathbf{E}(z, t) = A\sin(kz - \omega t)\,\mathbf{i} \qquad \mathbf{B}(z, t) = A\sin(kz - \omega t)\,\mathbf{j}$$

satisfy the equation for constants A, k, and ω, provided $\omega = \dfrac{k}{C}$.

b. Make a rough sketch showing the directions of $\mathbf{E}$ and $\mathbf{B}$.

64. Splitting a vector field Express the vector field $\mathbf{F} = \langle xy, 0, 0\rangle$ in the form $\mathbf{V} + \mathbf{W}$, where $\nabla\cdot\mathbf{V} = 0$ and $\nabla \times \mathbf{W} = \mathbf{0}$.

65. Properties of div and curl Prove the following properties of the divergence and curl. Assume $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields and c is a real number.

a. $\nabla\cdot(\mathbf{F} + \mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$

b. $\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$

c. $\nabla \cdot (c\,\mathbf{F}) = c\,(\nabla \cdot \mathbf{F})$

d. $\nabla \times (c\,\mathbf{F}) = c\,(\nabla \times \mathbf{F})$

66. **Equal curls** If two functions of one variable, f and g, have the property that $f' = g'$, then f and g differ by a constant. Prove or disprove: If **F** and **G** are nonconstant vector fields in $\mathbb{R}^2$ with curl **F** = curl **G** and div **F** = div **G** at all points of $\mathbb{R}^2$, then **F** and **G** differ by a constant vector.

67–72. Identities *Prove the following identities. Assume φ is a differentiable scalar-valued function and* **F** *and* **G** *are differentiable vector fields, all defined on a region of* $\mathbb{R}^3$.

67. $\nabla \cdot (\varphi\,\mathbf{F}) = \nabla\varphi \cdot \mathbf{F} + \varphi\,\nabla \cdot \mathbf{F}$ (Product Rule)

68. $\nabla \times (\varphi\,\mathbf{F}) = (\nabla\varphi \times \mathbf{F}) + (\varphi\,\nabla \times \mathbf{F})$ (Product Rule)

69. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

70. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\,\mathbf{F} - \mathbf{G}\,(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\,\mathbf{G} + \mathbf{F}\,(\nabla \cdot \mathbf{G})$

71. $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\,\mathbf{F} + (\mathbf{F} \cdot \nabla)\,\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$

72. $\nabla \times (\nabla \times \mathbf{F}) = \nabla\,(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\,\mathbf{F}$

73. **Divergence of radial fields** Prove that for a real number p, with $\mathbf{r} = \langle x, y, z \rangle$, $\nabla \cdot \dfrac{\langle x, y, z \rangle}{|\mathbf{r}|^p} = \dfrac{3 - p}{|\mathbf{r}|^p}$.

74. **Gradients and radial fields** Prove that for a real number p, with $\mathbf{r} = \langle x, y, z \rangle$, $\nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{-p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}$.

75. **Divergence of gradient fields** Prove that for a real number p, with $r = \langle x, y, z \rangle$,
$\nabla \cdot \nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{p\,(p-1)}{|\mathbf{r}|^{p+2}}$.