17.4 Green's Theorem

The preceding section gave a version of the Fundamental Theorem of Calculus that applies to line integrals. In this and the remaining sections of the text, you will see additional extensions of the Fundamental Theorem that apply to regions in $\mathbb{R}^2$ and $\mathbb{R}^3$. All these fundamental theorems share a common feature. Part 2 of the Fundamental Theorem of Calculus (Chapter 5) says

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a),$$

which relates the integral of $\frac{df}{dx}$ on an interval $[a, b]$ to the values of f on the boundary of $[a, b]$. The Fundamental Theorem for line integrals says

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

which relates the integral of $\nabla\varphi$ on a piecewise-smooth oriented curve C to the boundary values of φ. (The boundary consists of the two endpoints A and B.)

The subject of this section is Green's Theorem, which is another step in this progression. It relates the double integral of derivatives of a function over a region in $\mathbb{R}^2$ to function values on the boundary of that region.

Circulation Form of Green's Theorem »

Throughout this section, unless otherwise stated, we assume curves in the plane are simple closed piecewise-smooth oriented curves. By a result called the *Jordan Curve Theorem*, such curves have a well-defined interior such that when the curve is traversed in the counterclockwise direction (viewed from above), the interior is on the left. With this orientation, there is a unique outward unit normal vector that points to the right (at points where the curve is smooth). We also assume curves in the plane lie in regions that are both connected and simply connected.

Suppose the vector field $\mathbf{F}$ is defined on a region R enclosed by a closed curve C. As we have seen, the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ (Section 17.2) measures the net component of $\mathbf{F}$ in the direction tangential to C. It is easiest to visualize the circulation if $\mathbf{F}$ represents the velocity of a fluid moving in two dimensions. For example, let C be the unit circle with a counterclockwise orientation. The vector field $\mathbf{F} = \langle -y, x\rangle$ has a positive circulation of 2π on C (Section 17.2) because the vector field is everywhere tangent to C (**Figure 17.31**). A nonzero circulation on a closed curve says that the vector field must have some property *inside* the curve that produces the circulation. You can think of this property as a *net rotation*.

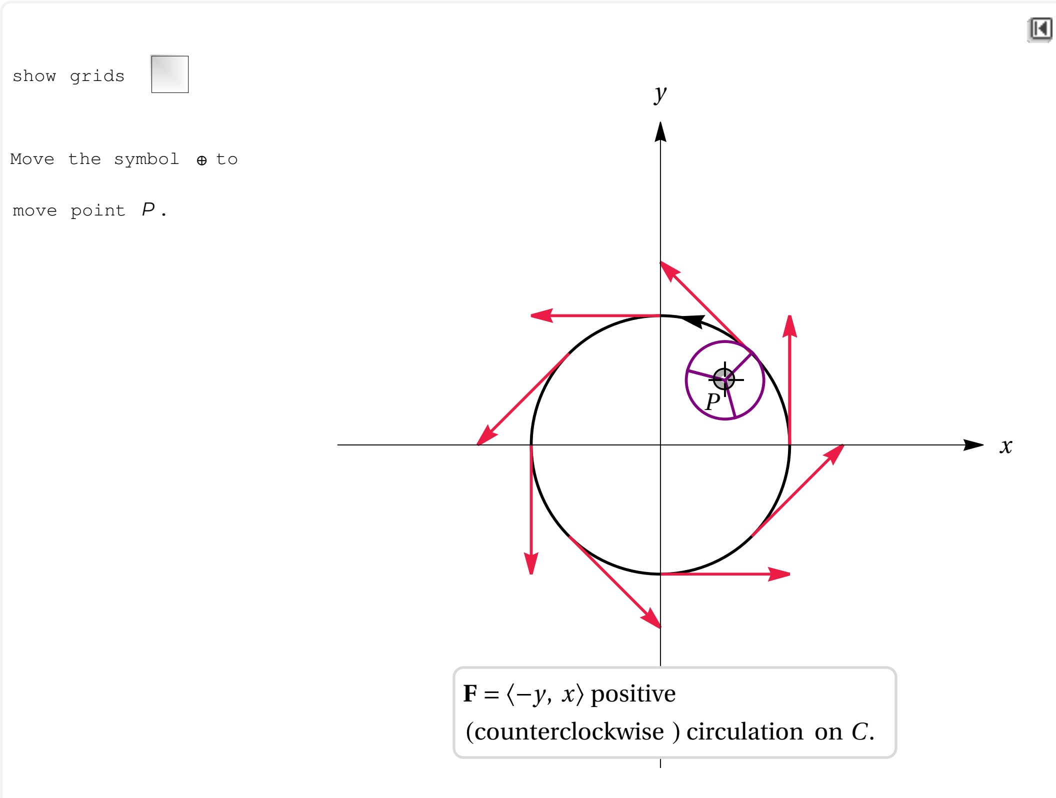


Figure 17.31

To visualize the rotation of a vector field, imagine a small paddle wheel, fixed at a point in the vector field, with its axis perpendicular to the *xy*-plane (Figure 17.31). The strength of the rotation at that point is seen in the speed at which the paddle wheel spins, and the direction of the rotation is the direction in which the paddle wheel spins. At a different point in the vector field, the paddle wheel will, in general, have a different speed and direction of rotation.

The first form of Green's Theorem relates the circulation on C to the double integral, over the region R, of a quantity that measures rotation at each point of R.

THEOREM 17.7 **Green's Theorem—Circulation Form**

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

Note »

The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

The proof of a special case of the theorem is given at the end of this section. Notice that the two line integrals on the left side of Green's Theorem give the circulation of the vector field on C. The double integral on the right side involves the factor $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$, which describes the rotation of the vector field *within* C that produces the circulation *on* C. This factor is called the *two-dimensional curl* of the vector field.

Figure 17.32 illustrates how the curl measures the rotation of one particular vector field at a point P. If the horizontal component of the field decreases in the y-direction at P ($f_y < 0$) and the vertical component increases in the x-direction at P ($g_x > 0$), then $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} > 0$, and the field has a counterclockwise rotation at P. The double integral in Green's Theorem computes the accumulated rotation of the field throughout R. The theorem says that the net rotation throughout R equals the circulation on the boundary of R.

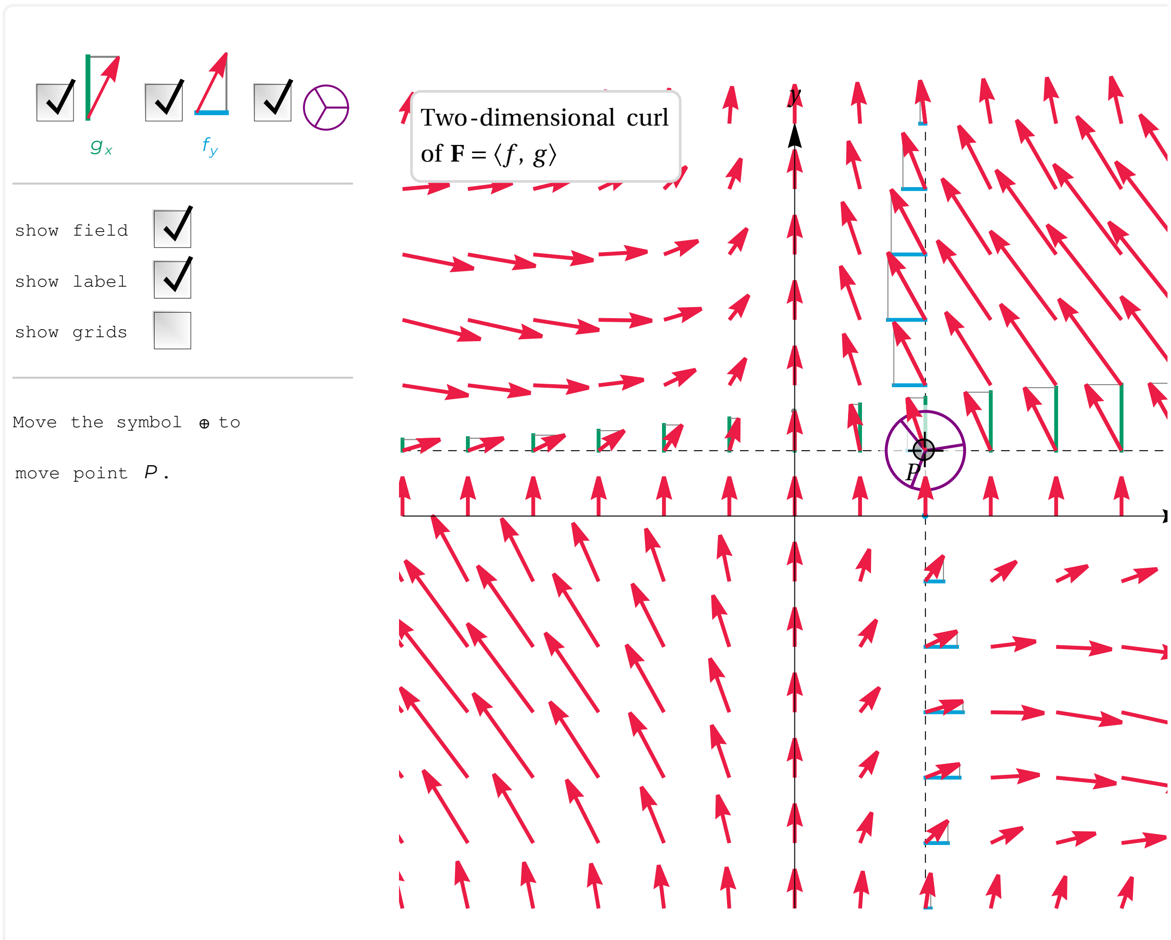


Figure 17.32

Quick Check 1 Compute $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$ for the radial vector field $\mathbf{F} = \langle x, y \rangle$. What does this tell you about the circulation on a simple closed curve? ◆

Answer »

$g_x - f_y = 0$, which implies zero circulation on a closed curve.

Green's Theorem has an important consequence when applied to a conservative vector field. Recall from Theorem 17.3 that if $\mathbf{F} = \langle f, g \rangle$ is conservative, then its components satisfy the condition $f_y = g_x$. If R is a region of $\mathbb{R}^2$ on which the conditions of Green's Theorem are satisfied, then for a conservative field we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{0} dA = 0.$$

Green's Theorem confirms the fact (Theorem 17.6) that if **F** is a conservative vector field in a region, then the circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is zero on any simple closed curve in the region. A two-dimensional vector field $\mathbf{F} = \langle f, g\rangle$ for which $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$ at all points of a region is said to be *irrotational*, because it produces zero circulation on closed curves in the region. Irrotational vector fields on simply connected regions in $\mathbb{R}^2$ are conservative.

Note »

In some cases, the rotation of a vector field may not be obvious. For example, the parallel flow in a channel $\mathbf{F} = \langle 0, 1 - x^2\rangle$ for $|x| \le 1$ has a nonzero curl for $x \ne 0$. See Exercise 72.

DEFINITION **Two-Dimensional Curl**

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is said to be **irrotational** on that region.

Evaluating circulation integrals of conservative vector fields on closed curves is easy. The integral is always zero. Green's Theorem provides a way to evaluate circulation integrals for nonconservative vector fields.

EXAMPLE 1 Circulation of a rotation field

Consider the rotation vector field $\mathbf{F} = \langle -y, x\rangle$ on the unit disk $R = \{(x, y) : x^2 + y^2 \le 1\}$ (Figure 17.31). In Example 6 of Section 17.2, we showed that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$, where C is the boundary of R oriented counterclockwise. Confirm this result using Green's Theorem.

SOLUTION »

Note that $f(x, y) = -y$ and $g(x, y) = x$; therefore, the curl of **F** is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2$. By Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{2} dA = \iint_R 2\, dA = 2\times(\text{area of } R) = 2\pi.$$

The curl $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$ is nonzero on R, which indicates a nonzero circulation on the boundary of R.

Related Exercises 18, 20 ◆

Calculating Area by Green's Theorem

A useful consequence of Green's Theorem arises with the vector fields $\mathbf{F} = \langle f, g\rangle = \langle 0, x\rangle$ and $\mathbf{F} = \langle f, g\rangle = \langle y, 0\rangle$.

In the first case, we have $g_x = 1$ and $f_y = 0$; therefore, by Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \underbrace{x\,dy}_{\mathbf{F}\cdot d\mathbf{r}} = \underbrace{\iint_R}_{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1} dA = \text{area of } R.$$

In the second case, $g_x = 0$ and $f_y = 1$, and Green's Theorem says

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C y\,dx = -\iint_R dA = -\text{area of } R.$$

These two results may be combined in one statement to give the following theorem.

THEOREM 17.8 **Area of a Plane Region by Line Integrals**

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

A remarkably simple calculation of the area of an ellipse follows from this result.

EXAMPLE 2 **Area of an ellipse**

Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION »

An ellipse with counterclockwise orientation is described parametrically by $\mathbf{r}(t) = \langle x, y\rangle = \langle a\cos t, b\sin t\rangle$, for $0 \le t \le 2\pi$. Noting that $dx = -a\sin t\,dt$ and $dy = b\cos t\,dt$, we have

$$\begin{aligned} x\,dy - y\,dx &= (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\ &= ab\left(\cos^2 t + \sin^2 t\right)dt \\ &= ab\,dt. \end{aligned}$$

Expressing the line integral as an ordinary integral with respect to t, the area of the ellipse is

$$\frac{1}{2}\oint_C \underbrace{(x\,dy - y\,dx)}_{ab\,dt} = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

Related Exercises 22–23 ◆

Flux Form of Green's Theorem »

Let C be a closed curve enclosing a region R in $\mathbb{R}^2$ and let $\mathbf{F}$ be a vector field defined on R. We assume C and R have the previously stated properties; specifically, C is oriented counterclockwise with an outward normal vector $\mathbf{n}$. Recall that the outward flux of $\mathbf{F}$ across C is $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ (Section 17.2). The second form of Green's Theorem relates the flux across C to a property of the vector field within R that produces the flux.

Note »

> The flux form of Green's Theorem is also called the *normal*, or *divergence*, form.

THEOREM 17.9 **Green's Theorem—Flux Form**

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\, ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

The two line integrals on the left side of Theorem 17.9 give the outward flux of the vector field across C. The double integral on the right side involves the quantity $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$, which is the property of the vector field that produces the flux across C. This factor is called the *two-dimensional divergence*.

Note »

> The two forms of Green's Theorem are related in the following way: Applying the circulation form of the theorem to $\mathbf{F} = \langle -g, f\rangle$ results in the flux form, and applying the flux form of the theorem to $\mathbf{F} = \langle g, -f\rangle$ results in the circulation form.

Figure 17.33 illustrates how the divergence measures the flux of one particular vector field at a point P. If $f_x > 0$ at P, it indicates an expansion of the vector field in the x-direction (if f_x is negative, it indicates a contraction). Similarly, if $g_y > 0$ at P, it indicates an expansion of the vector field in the y-direction. The combined effect of $f_x + g_y > 0$ at a point is a net outward flux across a small circle enclosing P.

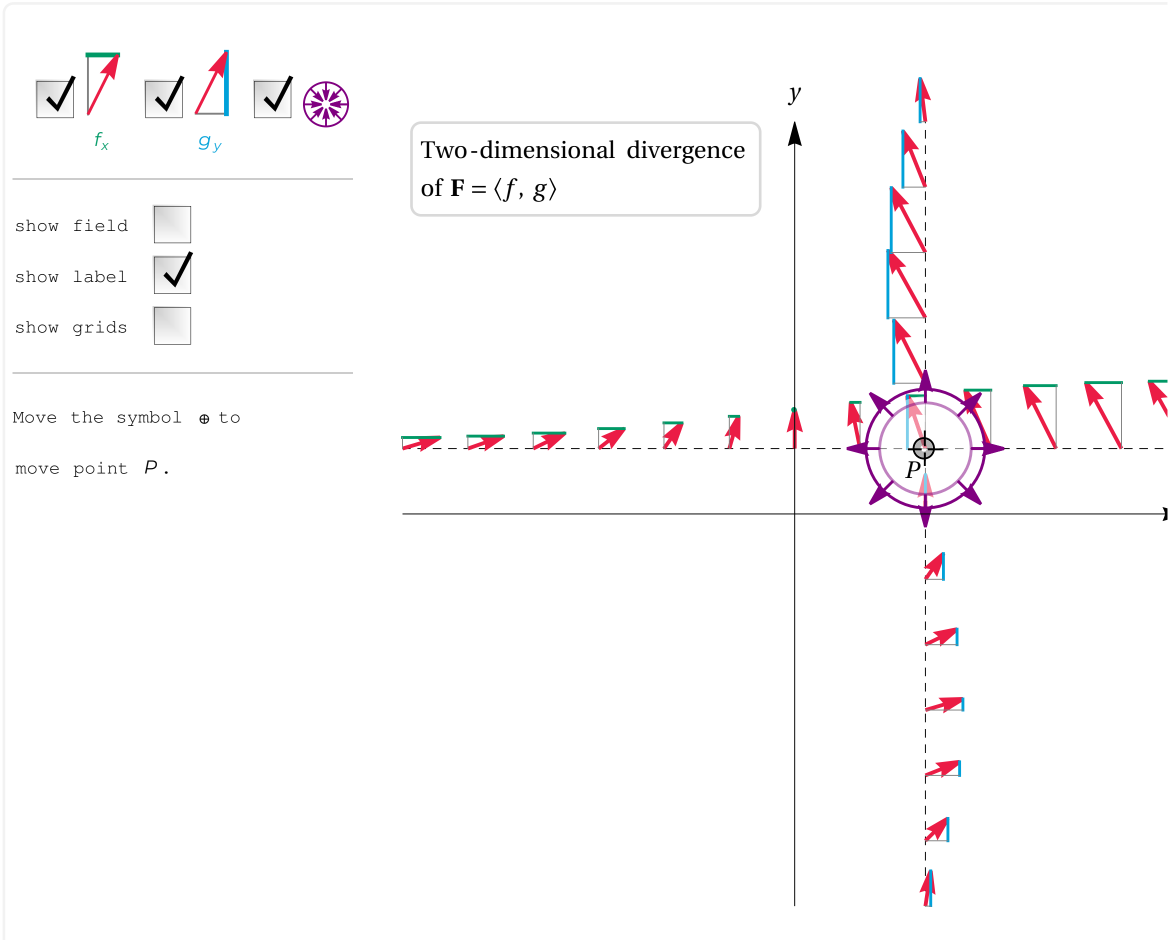


Figure 17.33

If the divergence of $\mathbf{F}$ is zero throughout a region on which $\mathbf{F}$ satisfies the conditions of Theorem 17.9, then the outward flux across the boundary is zero. Vector fields with a zero divergence are said to be *source free*. If the divergence is positive throughout R, the outward flux across C is positive, meaning that the vector field acts as a *source* in R. If the divergence is negative throughout R, the outward flux across C is negative, meaning that the vector field acts as a *sink* in R.

> **DEFINITION** **Two-Dimensional Divergence**
>
> The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is said to be **source free** on that region.

Quick Check 2 Compute $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ for the rotation field $\mathbf{F} = \langle -y, x \rangle$. What does this tell you about the outward flux of $\mathbf{F}$ across a simple closed curve? ◆

Answer »

> $f_x + g_y = 0$, which implies zero flux across a closed curve.

EXAMPLE 3 Outward flux of a radial field

Use Green's Theorem to compute the outward flux of the radial field $\mathbf{F} = \langle x, y \rangle$ across the unit circle $C = \{(x, y) : x^2 + y^2 = 1\}$ (**Figure 17.34**). Interpret the result.

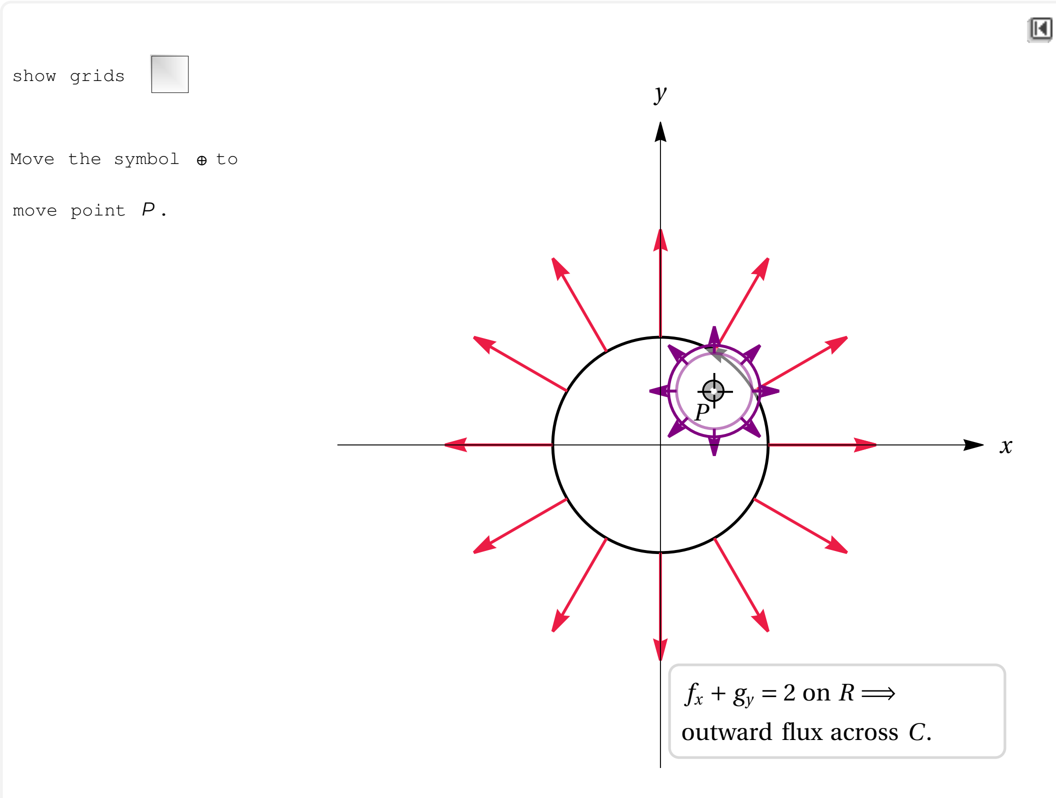


Figure 17.34

SOLUTION »

We have already calculated the outward flux of the radial field across C as a line integral and found it to be 2π (Example 8, Section 17.2). Computing the outward flux using Green's Theorem, note that $f(x, y) = x$ and $g(x, y) = y$; therefore, the divergence of $\mathbf{F}$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2$. By Green's Theorem, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}_{2} dA = \iint_R 2\, dA = 2 \times (\text{area of } R) = 2\pi.$$

The positive divergence on R results in an outward flux of the vector field across the boundary of R.

Related Exercise 27 ◆

As with the circulation form, the flux form of Green's Theorem can be used in either direction: to simplify line integrals or to simplify double integrals.

EXAMPLE 4 Line integral as a double integral

Evaluate $\oint_C \left(4x^3 + \sin y^2\right) dy - \left(4y^3 + \cos x^2\right) dx$, where C is the boundary of the disk $R = \{(x, y) : x^2 + y^2 \le 4\}$ oriented counterclockwise.

SOLUTION »

Letting $f(x, y) = 4x^3 + \sin y^2$ and $g(x, y) = 4y^3 + \cos x^2$, Green's Theorem takes the form

$$\oint_C \underbrace{\left(4x^3 + \sin y^2\right)}_{f} dy - \underbrace{\left(4y^3 + \cos x^2\right)}_{g} dx = \iint_R \Bigg(\underbrace{12x^2}_{f_x} + \underbrace{12y^2}_{g_y}\Bigg) dA \quad \text{Green's Theorem, flux form}$$

$$= 12\int_0^{2\pi}\int_0^2 r^2 \underbrace{r\,dr\,d\theta}_{dA} \quad \text{Polar coordinates; } x^2 + y^2 = r^2$$

$$= 12\int_0^{2\pi} \frac{r^4}{4}\Bigg|_0^2 d\theta \quad \text{Evaluate inner integral.}$$

$$= 48\int_0^{2\pi} d\theta = 96\pi. \quad \text{Evaluate outer integral.}$$

Related Exercises 35–36 ◆

Circulation and Flux on More General Regions »

Some ingenuity is required to extend both forms of Green's Theorem to more complicated regions. The next two examples illustrate Green's Theorem on two such regions: a half annulus and a full annulus.

EXAMPLE 5 Circulation on a half annulus

Consider the vector field $\mathbf{F} = \langle y^2, x^2\rangle$ on the half annulus $R = \{(x, y): 1 \le x^2 + y^2 \le 9,\ y \ge 0\}$, whose boundary is C. Find the circulation on C, assuming it has the orientation shown in **Figure 17.35**.

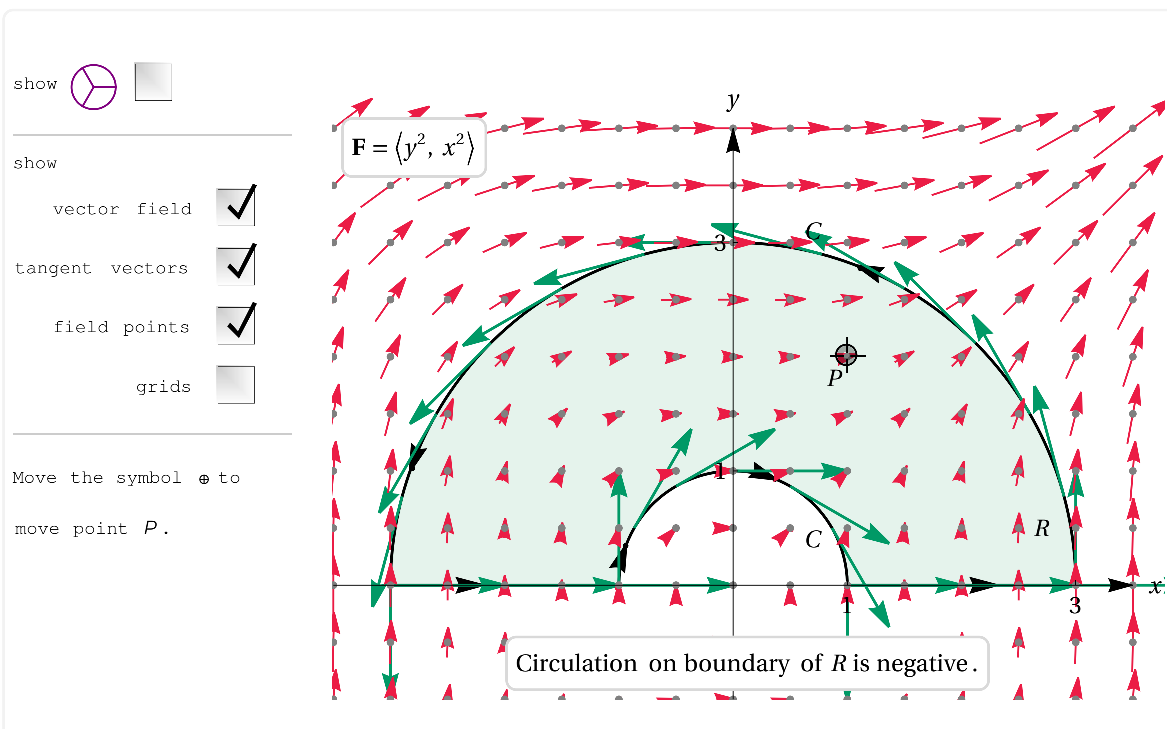


Figure 17.35

SOLUTION »

The circulation on C is

$$\oint_C f\,dx + g\,dy = \oint_C y^2\,dx + x^2\,dy.$$

With the given orientation, the curve runs counterclockwise on the outer semicircle and clockwise on the inner semicircle. Identifying $f(x, y) = y^2$ and $g(x, y) = x^2$, the circulation form of Green's Theorem converts the line integral into a double integral. The double integral is most easily evaluated in polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$:

$$\begin{aligned}
\oint_C y^2\,dx + x^2\,dy &= \iint_R \Big(\underbrace{2x}_{g_x} - \underbrace{2y}_{f_y}\Big)\,dA && \text{Green's Theorem, circulation form} \\
&= 2\int_0^{\pi}\int_1^3 (r\cos\theta - r\sin\theta)\,\underbrace{r\,dr\,d\theta}_{dA} && \text{Convert to polar coordinates.} \\
&= 2\int_0^{\pi} (\cos\theta - \sin\theta)\left(\frac{r^3}{3}\right)\Bigg|_1^3\,d\theta && \text{Evaluate inner integral.} \\
&= \frac{52}{3}\int_0^{\pi}(\cos\theta - \sin\theta)\,d\theta && \text{Simplify.} \\
&= -\frac{104}{3}. && \text{Evaluate outer integral.}
\end{aligned}$$

The vector field (Figure 17.35) suggests why the circulation is negative. The field is roughly *opposed* to the direction of C on the outer semicircle but roughly aligned with the direction of C on the inner semicircle. Because the outer semicircle is longer and the field has greater magnitudes on the outer curve than the inner curve, the greater contribution to the circulation is negative.

Related Exercise 41 ◆

EXAMPLE 6 Flux across the boundary of an annulus

Find the outward flux of the vector field $\mathbf{F} = \langle xy^2, x^2y\rangle$ across the boundary of the annulus $R = \{(x, y): 1 \le x^2 + y^2 \le 4\}$, which, when expressed in polar coordinates, is the set $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le 2\pi\}$ (**Figure 17.36**).

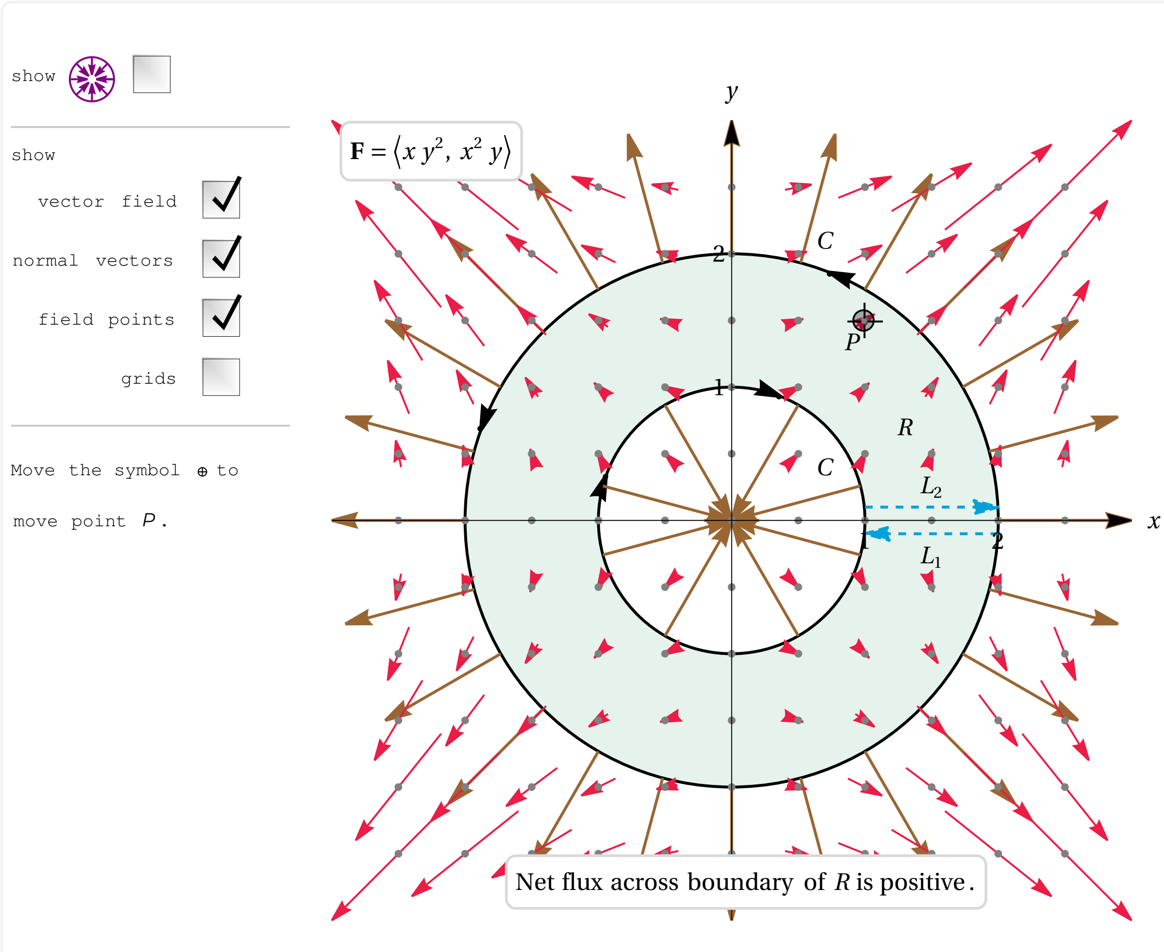


Figure 17.36

SOLUTION »

Because the annulus R is not simply connected, Green's Theorem does not apply as stated in Theorem 17.9. This difficulty is overcome by defining the curve C shown in Figure 17.36, which is simple, closed, and piecewise-smooth. The connecting links L_1 and L_2 along the x-axis are parallel and are traversed in opposite directions. Therefore, the contributions to the line integral cancel on L_1 and L_2. Because of this cancellation, we take C to be the curve that runs counterclockwise on the outer boundary and clockwise on the inner boundary.

Note »

> Another way to deal with the flux across the annulus is to apply Green's Theorem to the entire disk $|r| \le 2$ and compute the flux across the outer circle. Then apply Green's Theorem to the disk $|r| \le 1$ and compute the flux across the inner circle. Note that the flux *out* of the inner disk is a flux *into* the annulus. Therefore, the difference of the two fluxes gives the net flux for the annulus.

Using the flux form of Green's Theorem and converting to polar coordinates, we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \oint_C f\,dy - g\,dx \\
&= \oint_C x y^2\,dy - x^2 y\,dx && \text{Substitute for } f \text{ and } g. \\
&= \iint_R \Big(\underbrace{y^2}_{f_x} + \underbrace{x^2}_{g_y}\Big)\,dA && \text{Green's Theorem; flux form} \\
&= \int_0^{2\pi}\int_1^2 \left(r^2\right) r\,dr\,d\theta && \text{Polar coordinates; } x^2+y^2=r^2 \\
&= \int_0^{2\pi} \frac{r^4}{4}\Big|_1^2\,d\theta && \text{Evaluate inner integral.} \\
&= \frac{15}{4}\int_0^{2\pi} d\theta && \text{Simplify.} \\
&= \frac{15\pi}{2}. && \text{Evaluate outer integral.}
\end{aligned}$$

Figure 17.36 shows the vector field and explains why the flux across C is positive. Because the field increases in magnitude moving away from the origin, the outward flux across the outer boundary is greater than the inward flux across the inner boundary. Hence, the net outward flux across C is positive.

Note »

> Notice that the divergence of the vector field in Example 6 $\left(x^2+y^2\right)$ is positive on R, so we expect an outward flux across C.

Related Exercise 42 ◆

Stream Functions »

We can now see a wonderful parallel between circulation properties (and conservative vector fields) and flux properties (and source-free fields). We need one more piece to complete the picture; it is the *stream function*, which plays the same role for source-free fields that the potential function plays for conservative fields.

Consider a two-dimensional vector field $\mathbf{F} = \langle f, g\rangle$ that is differentiable on a region R. A **stream function** for the vector field—if it exists—is a function ψ (pronounced *psigh* or *psee*) that satisfies

$$\frac{\partial\psi}{\partial y} = f, \quad \frac{\partial\psi}{\partial x} = -g.$$

Note »

> Potential function:
>
> $$\varphi_x = f \quad \text{and} \quad \varphi_y = g$$
>
> Stream function:
>
> $$\psi_x = -g \quad \text{and} \quad \psi_y = f$$

If we compute the divergence of a vector field $\mathbf{F} = \langle f, g\rangle$ that has a stream function and use the fact that $\psi_{xy} = \psi_{yx}$, then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\psi_{yx} = \psi_{xy}} = 0.$$

We see that the existence of a stream function guarantees that the vector field has zero divergence or, equivalently, is source-free. The converse is also true on simply connected regions of $\mathbb{R}^2$.

As discussed in Section 17.1, the level curves of a stream function are called flow curves or streamlines—and for good reason. It can be shown (Exercise 70) that the vector field **F** is everywhere tangent to the streamlines, which means that a graph of the streamlines shows the flow of the vector field. Finally, just as circulation integrals of a conservative vector field are independent of path, flux integrals of a source-free field are also independent of path (Exercise 69).

Quick Check 3 Show that $\psi = \frac{1}{2}\left(y^2 - x^2\right)$ is a stream function for the vector field $\mathbf{F} = \langle y, x \rangle$. Show that **F** has zero divergence. ◆

Answer »

$\psi_y = y$ is the x-component of $\mathbf{F} = \langle y, x \rangle$, and $-\psi_x = x$ is the y-component of **F**. Also the divergence of **F** is $y_x + x_y = 0$.

Vector fields that are both conservative and source-free are quite interesting mathematically because they have both a potential function and a stream function. It can be shown that the level curves of the potential and stream functions form orthogonal families; that is, at each point of intersection, the line tangent to one level curve is orthogonal to the line tangent to the other level curve (equivalently, the gradient vector of one function is orthogonal to the gradient vector of the other function). Such vector fields have zero curl $(g_x - f_y = 0)$ and zero divergence $(f_x + g_y = 0)$. If we write the zero divergence condition in terms of the potential function φ, we find that

$$0 = f_x + g_y = \varphi_{xx} + \varphi_{yy}.$$

Writing the zero curl condition in terms of the stream function ψ, we find that

$$0 = g_x - f_y = -\psi_{xx} - \psi_{yy}.$$

Note »

In fluid dynamics, velocity fields that are both conservative and source-free are called *ideal flows*. They model fluids that are irrotational and incompressible.

We see that the potential function and the stream function both satisfy an important equation known as **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0 \text{ and } \psi_{xx} + \psi_{yy} = 0.$$

Any function satisfying Laplace's equation can be used as a potential function or stream function for a conservative, source free vector field. These vector fields are used in fluid dynamics, electrostatics, and other modeling applications.

Note »

Methods for finding solutions of Laplace's equation are discussed in advanced mathematics courses.

Table 17.1 shows the parallel properties of conservative and source-free vector fields in two dimensions. We assume C is a simple piecewise-smooth oriented curve and is either closed or has endpoints A and B.

Table 17.1

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	**Source-Free Fields $\mathbf{F} = \langle f, g \rangle$**
• curl $= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$	• divergence $= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$
• Potential function φ with $\mathbf{F} = \nabla\varphi$ or $f = \frac{\partial \varphi}{\partial x}, g = \frac{\partial \varphi}{\partial y}$	• Stream function ψ with $f = \frac{\partial \psi}{\partial y}$, $g = -\frac{\partial \psi}{\partial x}$
• Circulation $= \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all closed curves C.	• Flux $= \oint_C \mathbf{F}\cdot \mathbf{n}\, ds = 0$ on all closed curves C.
• Evaluation of line integral $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	• Evaluation of line integral $\int_C \mathbf{F}\cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

With Green's Theorem in the picture, we may also give a concise summary of the various cases that arise with line integrals of both the circulation and flux types (Table 17.2).

Table 17.2

Circulation /work integrals: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy$

	***C* closed**	***C* not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (f\,x' + g\,y')\,dt$

Flux integrals: $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx$

	***C* closed**	***C* not closed**
F source free ($f = \psi_y,\ g = -\psi_x$)	$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$	$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (f_x + g_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (f\,y' - g\,x')\,dt$

Proof of Green's Theorem on Special Regions »

The proof of Green's Theorem is straightforward when restricted to special regions. We consider regions R enclosed by a simple closed smooth curve C oriented in the counterclockwise direction, such that the region can be expressed in two ways (**Figure 17.37**):

- $R = \{(x, y) : a \le x \le b,\ G_1(x) \le y \le G_2(x)\}$ or
- $R = \{(x, y) : H_1(y) \le x \le H_2(y),\ c \le y \le d\}$

Note »

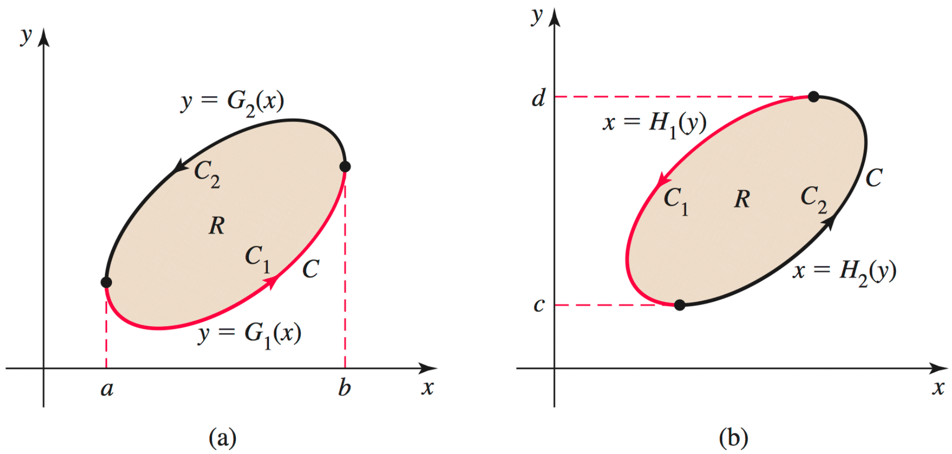


Figure 17.37

Under these conditions, we prove the circulation form of Green's Theorem:

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Beginning with the term $\iint_R \frac{\partial f}{\partial y}\,dA$, we write this double integral as an iterated integral, where $G_1(x) \le y \le G_2(x)$ in the inner integral and $a \le x \le b$ in the outer integral (Figure 17.37a). The upper curve is labeled C_2 and the lower curve is labeled C_1. Notice that the inner integral of $\frac{\partial f}{\partial y}$ with respect to y gives $f(x, y)$. Therefore, the first step of the double integration is

$$\iint_R \frac{\partial f}{\partial y}\,dA = \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\,dy\,dx \qquad \text{Convert to an iterated integral.}$$

$$= \int_a^b \Big(\underbrace{f(x, G_2(x))}_{\text{on } C_2} - \underbrace{f(x, G_1(x))}_{\text{on } C_1} \Big)\,dx.$$

Over the interval $a \le x \le b$, the points $(x, G_2(x))$ trace out the upper part of C (labeled C_2) in the *negative* (cloc-kwise) direction. Similarly, over the interval $a \le x \le b$, the points $(x, G_1(x))$ trace out the lower part of C (labeled C_1) in the *positive* (counterclockwise) direction.

Therefore,

$$\iint_R \frac{\partial f}{\partial y}\, dA = \int_a^b (f(x,\ G_2(x)) - f(x,\ G_1(x)))\, dx$$

$$= \int_{-C_2} f\, dx - \int_{C_1} f\, dx$$

$$= -\int_{C_2} f\, dx - \int_{C_1} f\, dx \qquad \int_{-C_2} f\, dx = -\int_{C_2} f\, dx$$

$$= -\oint_C f\, dx. \qquad \int_C f\, dx = \int_{C_1} f\, dx + \int_{C_2} f\, dx$$

A similar argument applies to the double integral of $\frac{\partial g}{\partial x}$, except we use the bounding curves $x = H_1(y)$ and $x = H_2(y)$, where C_1 is the left curve and C_2 is the right curve (Figure 17.37b). We have

$$\iint_R \frac{\partial g}{\partial x}\, dA = \int_c^d \int_{H_1(y)}^{H_2(y)} \frac{\partial g}{\partial x}\, dx\, dy \qquad \text{Convert to an iterated integral.}$$

$$= \int_c^d \left(\underbrace{g(H_2(y),\ y)}_{C_2} - \underbrace{g(H_1(y),\ y)}_{-C_1} \right) dy \qquad \int \frac{\partial g}{\partial x}\, dx = g$$

$$= \int_{C_2} g\, dy - \int_{-C_1} g\, dy$$

$$= \int_{C_2} g\, dy + \int_{C_1} g\, dy \qquad \int_{-C_1} g\, dy = -\int_{C_1} g\, dy$$

$$= \oint_C g\, dy. \qquad \int_C g\, dy = \int_{C_1} g\, dy + \int_{C_2} g\, dy$$

Combining these two calculations results in

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \oint_C f\, dx + g\, dy.$$

As mentioned earlier, with a change of notation (replace g by f and f by $-g$), the flux form of Green’s Theorem is obtained. This proof also completes the list of equivalent properties of conservative fields given in Section 17.3: From Green’s Theorem it follows that if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ on a simply connected region R, then the vector field $\mathbf{F} = \langle f,\ g \rangle$ is conservative on R.

Quick Check 4 Explain why Green’s Theorem proves that if $g_x = f_y$, then the vector field $\mathbf{F} = \langle f,\ g \rangle$ is conservative. ◆

Answer »

> If the curl is zero on a region, then all closed-path integrals are zero, which is a condition (Section 17.3) for a conservative field.

Exercises »

Getting Started »

Practice Exercises »

17–20. Green's Theorem, circulation form *Consider the following regions R and vector fields* **F**.

a. *Compute the two-dimensional curl of the vector field.*

b. *Evaluate both integrals in Green's Theorem and check for consistency.*

17. $\mathbf{F} = \langle 2y, -2x\rangle$; R is the region bounded by $y = \sin x$ and $y = 0$, for $0 \le x \le \pi$.

18. $\mathbf{F} = \langle -3y, 3x\rangle$; R is the triangle with vertices (0, 0), (1, 0), and (0, 2).

19. $\mathbf{F} = \langle -2xy, x^2\rangle$; R is the region bounded by $y = x(2 - x)$ and $y = 0$.

20. $\mathbf{F} = \langle 0, x^2 + y^2\rangle$; $R = \{(x, y) : x^2 + y^2 \le 1\}$.

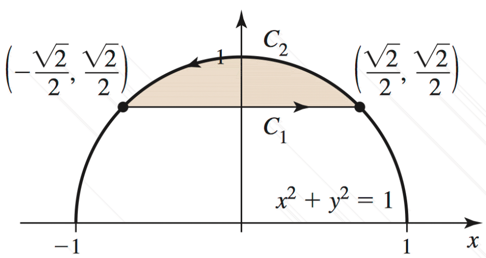
21–26. Area of regions *Use a line integral on the boundary to find the area of the following regions.*

21. A disk of radius 5

22. A region bounded by an ellipse with major and minor axes of length 12 and 8, respectively

23. $\{(x, y) : x^2 + y^2 \le 16\}$

24. The region shown in the figure.



25. The region bounded by the parabolas $\mathbf{r}(t) = \langle t, 2t^2\rangle$ and $\mathbf{r}(t) = \langle t, 12 - t^2\rangle$, for $-2 \le t \le 2$

26. The region bounded by the curve $\mathbf{r}(t) = \langle t(1 - t^2), 1 - t^2\rangle$, for $-1 \le t \le 1$ (*Hint:* Plot the curve.)

27–30. Green's Theorem, flux form *Consider the following regions R and vector fields* **F**.

a. *Compute the two-dimensional divergence of the vector field.*

b. *Evaluate both integrals in Green's Theorem and check for consistency.*

27. $\mathbf{F} = \langle x, y\rangle$; $R = \{(x, y) : x^2 + y^2 \le 4\}$

28. $\mathbf{F} = \langle x, -3y\rangle$; R is the triangle with vertices (0, 0), (1, 2), and (0, 2).

29. $\mathbf{F} = \langle 2xy, x^2\rangle$; $R = \{(x, y) :\ 0 \le y \le x(2 - x)\}$

30. $\mathbf{F} = \langle x^2 + y^2, 0\rangle$; $R = \{(x, y) : x^2 + y^2 \le 1\}$.

31–40. Line integrals *Use Green's Theorem to evaluate the following line integrals. Assume all curves are oriented counterclockwise. A sketch is helpful.*

31. $\oint_C \langle 3y+1, 4x^2+3\rangle \cdot d\mathbf{r}$, where C is the boundary of the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$

32. $\oint_C \langle \sin y, x\rangle \cdot d\mathbf{r}$, where C is the boundary of the triangle with vertices $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$, and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

33. $\oint_C x e^y\, dx + x\, dy$, where C is the boundary of the region bounded by the curves $y = x^2$, $x = 2$, and the x-axis

34. $\oint_C \frac{1}{1+y^2}\, dx + y\, dy$,where C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

35. $\oint_C \left(2x + e^{y^2}\right) dy - \left(4y^2 + e^{x^2}\right) dx$, where C is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

36. $\oint_C (2x - 3y)\, dy - (3x + 4y)\, dx$, where C is the unit circle

37. $\oint_C f\, dy - g\, dx$, where $\langle f, g\rangle = \langle 0, xy\rangle$ and C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$

38. $\oint_C f\, dy - g\, dx$, where $\langle f, g\rangle = \langle x^2, 2y^2\rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$ oriented clockwise

39. The circulation line integral of $\mathbf{F} = \langle x^2 + y^2, 4x + y^3\rangle$, where C is the boundary of $\{(x, y) : 0 \le y \le \sin x, 0 \le x \le \pi\}$

40. The flux line integral of $\mathbf{F} = \langle e^{x-y}, e^{y-x}\rangle$, where C is the boundary of $\{(x, y) : 0 \le y \le x, 0 \le x \le 1\}$

41–48. Circulation and flux *For the following vector fields, compute (a) the circulation on, and (b) the outward flux across, the boundary of the given region. Assume boundary curves are oriented counterclockwise.*

41. $\mathbf{F} = \langle x, y\rangle$; R is the half-annulus $\{(r, \theta); 1 \le r \le 2, 0 \le \theta \le \pi\}$.

42. $\mathbf{F} = \langle -y, x\rangle$; R is the annulus $\{(r, \theta) : 1 \le r \le 3, 0 \le \theta \le 2\pi\}$.

43. $\mathbf{F} = \langle 2x + y, x - 4y\rangle$; R is the quarter-annulus $\left\{(r, \theta) : 1 \le r \le 4, 0 \le \theta \le \frac{\pi}{2}\right\}$.

44. $\mathbf{F} = \langle x - y, -x + 2y\rangle$; R is the parallelogram $\{(x, y) : 1 - x \le y \le 3 - x, 0 \le x \le 1\}$.

45. $\mathbf{F} = \nabla\left(\sqrt{x^2 + y^2}\right)$; R is the half-annulus $\{(r, \theta) : 1 \le r \le 3, 0 \le \theta \le \pi\}$.

46. $\mathbf{F} = \left\langle \ln\left(x^2 + y^2\right), \tan^{-1} \frac{y}{x}\right\rangle$; R is the eigth-annulus $\left\{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}\right\}$.

47. $\mathbf{F} = \langle x + y^2, x^2 - y \rangle$; $R = \{(x, y) : y^2 \le x \le 2 - y^2\}$.

48. $\mathbf{F} = \langle y \cos x, -\sin x \rangle$; R is the square $\left\{(x, y) : 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right\}$.

49. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

- **a.** The work required to move an object around a closed curve C in the presence of a vector force field is the circulation of the force field on the curve.
- **b.** If a vector field has zero divergence throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is zero.
- **c.** If the two-dimensional curl of a vector field is positive throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is positive (assuming counterclockwise orientation).

50–51. Special line integrals *Prove the following identities, where C is a simple closed smooth oriented curve.*

50. $\oint_C dx = \oint_C dy = 0$

51. $\oint_C f(x)\, dx + g(y)\, dy = 0$, where f and g have continuous derivatives on the region enclosed by C

52. **Double integral to line integral** Use the flux form of Green's Theorem to evaluate $\int\int_R \left(2xy + 4y^3\right) dA$, where R is the triangle with vertices (0, 0), (1, 0), and (0, 1).

53. **Area line integral** Show that the value of

$$\oint_C xy^2\, dx + \left(x^2 y + 2x\right) dy$$

depends only on the area of the region enclosed by C.

54. **Area line integral** In terms of the parameters a and b, how is the value of $\oint_C ay\, dx + bx\, dy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

55–58. Stream function *Recall that if the vector field* $\mathbf{F} = \langle f, g \rangle$ *is source free (zero divergence), then a stream function* ψ *exists such that* $f = \psi_y$ *and* $g = -\psi_x$.

- ***a.*** *Verify that the given vector field has zero divergence.*
- ***b.*** *Integrate the relations* $f = \psi_y$ *and* $g = -\psi_x$ *to find a stream function for the field.*

55. $\mathbf{F} = \langle 4, 2 \rangle$

56. $\mathbf{F} = \langle y^2, x^2 \rangle$

57. $\mathbf{F} = \langle -e^{-x} \sin y, e^{-x} \cos y \rangle$

58. $\mathbf{F} = \langle x^2, -2xy \rangle$

Explorations and Challenges »

59–62. Ideal flow *A two-dimensional vector field describes* ***ideal flow*** *if it has both zero curl and zero divergence on a simply connected region (excluding the origin if necessary).*

- **a.** *Verify that the curl and divergence of the given field is zero.*
- **b.** *Find a potential function* φ *and a stream function* ψ *for the field.*
- **c.** *Verify that* φ *and* ψ *satisfy Laplace's equation* $\varphi_{xx} + \varphi_{yy} = \psi_{xx} + \psi_{yy} = 0$.

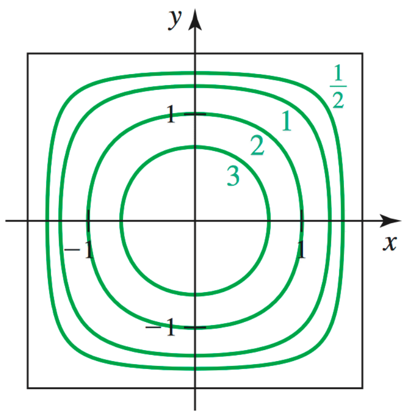
59. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

60. $\mathbf{F} = \left\langle x^3 - 3xy^2, y^3 - 3x^2 y \right\rangle$

61. $\mathbf{F} = \left\langle \tan^{-1} \frac{y}{x}, \frac{1}{2} \ln \left(x^2 + y^2\right) \right\rangle$, for $x > 0$

62. $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$, for $x > 0$, $y > 0$

63. Flow in an ocean basin An idealized two-dimensional ocean is modeled by the square region $R = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with boundary C. Consider the stream function $\psi(x, y) = 4 \cos x \cos y$ defined on R (see figure.)



- **a.** The horizontal (east-west) component of the velocity is $u = \psi_y$ and the vertical (north-south) component of the velocity is $v = -\psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.
- **b.** Is the velocity field source free? Explain.
- **c.** Is the velocity field irrotational? Explain.
- **d.** Let C be the boundary of R. Find the total outward flux across C.
- **e.** Find the circulation on C assuming counterclockwise orientation.

64. Green's Theorem as a Fundamental Theorem of Calculus Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0, \frac{f(x)}{c} \right\rangle$, where $c > 0$ and $R = \{(x, y): a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a).$$

65. Green's Theorem as a Fundamental Theorem of Calculus Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c}, 0 \right\rangle$, where $c > 0$ and $R = \{(x, y): a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a).$$

66. What's wrong? Consider the rotation field $\mathbf{F} = \frac{\langle -y, x \rangle}{x^2 + y^2}$.

- **a.** Verify that the two-dimensional curl of **F** is zero, which suggests that the double integral in the circulation form of Green's Theorem is zero.
- **b.** Use a line integral to verify that the circulation on the unit circle of the vector field is 2π.
- **c.** Explain why the results of parts (a) and (b) do not agree.

67. What's wrong? Consider the radial field $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$.

- **a.** Verify that the divergence of **F** is zero, which suggests that the double integral in the flux form of Green's Theorem is zero.
- **b.** Use a line integral to verify that the outward flux across the unit circle of the vector field is 2π.
- **c.** Explain why the results of parts (a) and (b) do not agree.

68. Conditions for Green's Theorem Consider the radial field $\mathbf{F} = \langle f, g \rangle = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

- **a.** Explain why the conditions of Green's Theorem do not apply to **F** on a region that includes the origin.
- **b.** Let R be the unit disk centered at the origin and compute $\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$.
- **c.** Evaluate the line integral in the flux form of Green's Theorem on the boundary of R.
- **d.** Do the results of parts (b) and (c) agree? Explain.

69. Flux integrals Assume the vector field $\mathbf{F} = \langle f, g \rangle$ is source free (zero divergence) with stream function ψ. Let C be any smooth simple curve from A to the distinct point B. Show that the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ is independent of path; that is, $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$.

70. Streamlines are tangent to the vector field Assume the vector field $\mathbf{F} = \langle f, g \rangle$ is related to the stream function ψ by $\psi_y = f$ and $\psi_x = -g$ on a region R. Prove that at all points of R, the vector field is tangent to the streamlines (the level curves of the stream function).

71. **Streamlines and equipotential lines** Assume that on $\mathbb{R}^2$, the vector field $\mathbf{F} = \langle f, g \rangle$ has a potential function φ such that $f = \varphi_x$ and $g = \varphi_y$, and it has a stream function ψ such that $f = \psi_y$ and $g = -\psi_x$. Show that the equipotential curves (level curves of φ) and the streamlines (level curves of ψ) are everywhere orthogonal.

72. **Channel flow** The flow in a long shallow channel is modeled by the velocity field $\mathbf{F} = \left\langle 0, 1 - x^2 \right\rangle$, where $R = \{(x, y) : |x| \le 1 \text{ and } |y| < 5\}$.

- **a.** Sketch R and several streamlines of $\mathbf{F}$.
- **b.** Evaluate the curl of $\mathbf{F}$ on the lines $x = 0$, $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = 1$.
- **c.** Compute the circulation on the boundary of the region R.
- **d.** How do you explain the fact that the curl of $\mathbf{F}$ is nonzero at points of R, but the circulation is zero?