17.3 Conservative Vector Fields

This is an action-packed section in which several fundamental ideas come together. At the heart of the matter are two questions.

- When can a vector field be expressed as the gradient of a potential function? A vector field with this property will be defined as a *conservative* vector field.
- What special properties do conservative vector fields have?

After some preliminary definitions, we present a test to determine whether a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ is conservative. The test is followed by a procedure to find a potential function for a conservative field. We then develop several equivalent properties shared by all conservative vector fields.

Types of Curves and Regions »

Many of the results in the remainder of this text rely on special properties of regions and curves. It's best to collect these definitions in one place for easy reference.

DEFINITION **Simple and Closed Curves**

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same (**Figure 17.28**).

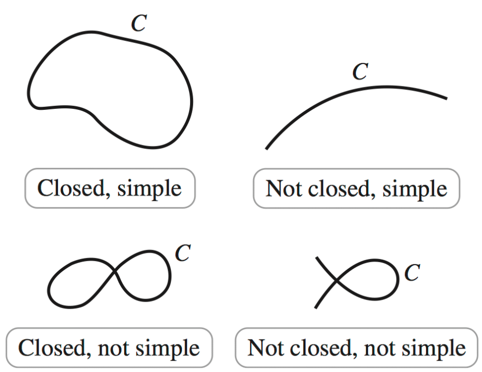


Figure 17.28

In all that follows, we generally assume that R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is an open region. Open regions are further classified according to whether they are *connected* and whether they are *simply connected*.

Note »

Recall that all points of an open set are interior points. An open set does not contain its boundary points.

DEFINITION **Connected and Simply Connected Regions**

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R (**Figure 17.29**).

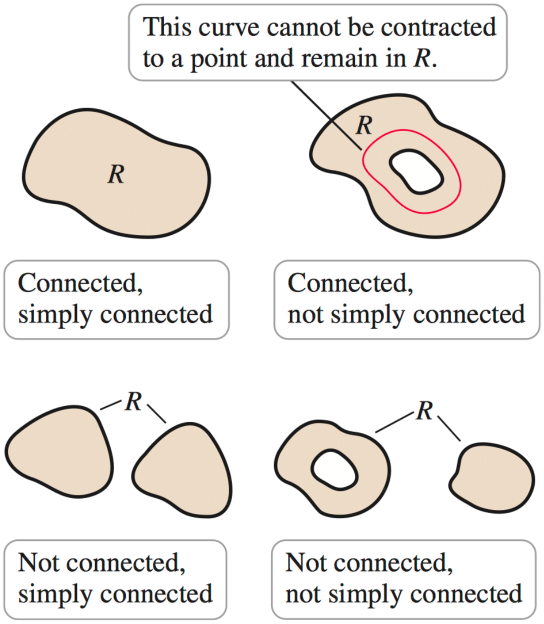


Figure 17.29

Note »

Roughly speaking, connected means that R is all in one piece and simply connected in $\mathbb{R}^2$ means that R has no holes. $\mathbb{R}^2$ and $\mathbb{R}^3$ are themselves connected and simply connected.

Quick Check 1 Is a figure-8 curve simple? Closed? Is a torus connected? Simply connected? ◆

Answer »

A figure-8 is closed but not simple; a torus is connected, but not simply connected.

Test for Conservative Vector Fields »

We begin with the central definition of this section.

DEFINITION **Conservative Vector Field**

A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Note »

The term *conservative* refers to conservation of energy. See Exercise 66 for an example of conservation of energy in a conservative force field.

Suppose the components of $\mathbf{F} = \langle f, g, h\rangle$ have continuous first partial derivatives on a region D in $\mathbb{R}^3$. Also assume $\mathbf{F}$ is conservative, which means by definition that there is a potential function φ such that $\mathbf{F} = \nabla\varphi$. Matching the components of $\mathbf{F}$ and $\nabla\varphi$, we see that $f = \varphi_x$, $g = \varphi_y$, and $h = \varphi_z$. Recall from Theorem 15.4 that if a function has continuous second partial derivatives, the order of differentiation in the second partial derivatives does not matter. Under these conditions on φ, we conclude the following:

- $\varphi_{xy} = \varphi_{yx}$, which implies that $f_y = g_x$,

- $\varphi_{xz} = \varphi_{zx}$, which implies that $f_z = h_x$, and
- $\varphi_{yz} = \varphi_{zy}$, which implies that $g_z = h_y$.

Note »

> Depending on the context and the interpretation of the vector field, the potential may be defined such that $\mathbf{F} = -\nabla\varphi$ (with a negative sign).

These observations comprise half of the proof of the following theorem. The remainder of the proof is given in Section 17.4.

THEOREM 17.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then, $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

EXAMPLE 1 Testing for conservative fields

Determine whether the following vector fields are conservative on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

SOLUTION »

a. Letting $f(x, y) = e^x \cos y$ and $g(x, y) = -e^x \sin y$, we see that

$$\frac{\partial f}{\partial y} = -e^x \sin y = \frac{\partial g}{\partial x}.$$

The conditions of Theorem 17.3 are met and $\mathbf{F}$ is conservative.

b. Letting $f(x, y, z) = 2xy - z^2$, $g(x, y, z) = x^2 + 2z$, and $h(x, y, z) = 2y - 2xz$, we have

$$\frac{\partial f}{\partial y} = 2x = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = -2z = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = 2 = \frac{\partial h}{\partial y}.$$

By Theorem 17.3, $\mathbf{F}$ is conservative.

Related Exercises 13–14 ◆

Finding Potential Functions »

Like antiderivatives, potential functions are determined up to an arbitrary additive constant. Unless an additive constant in a potential function has some physical meaning, it is usually omitted. Given a conservative vector field, there are several methods for finding a potential function. One method is shown in the following example. Another approach is illustrated in Exercise 71.

Quick Check 2 Explain why a potential function for a conservative vector field is determined up to an additive constant. ◆

Answer »

> The vector field is obtained by differentiating the potential function. So additive constants in the potential give the same vector field: $\nabla(\varphi + C) = \nabla\varphi$, where C is a constant.

EXAMPLE 2 Finding potential functions

Find a potential function for the conservative vector fields in Example 1.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y\rangle$

b. $\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz\right\rangle$

SOLUTION »

a. A potential function φ for $\mathbf{F} = \langle f, g\rangle$ has the property that $\mathbf{F} = \nabla\varphi$ and satisfies the conditions

$$\varphi_x = f(x, y) = e^x \cos y \quad \text{and} \quad \varphi_y = g(x, y) = -e^x \sin y.$$

The first equation is integrated with respect to x (holding y fixed) to obtain

$$\int \varphi_x\, dx = \int e^x \cos y\, dx,$$

which implies that

$$\varphi(x, y) = e^x \cos y + c(y).$$

Note »

> This procedure may begin with either of the two conditions, $\varphi_x = f$ or $\varphi_y = g$.

In this case, the "constant of integration" $c(y)$ is an arbitrary function of y. You can check the preceding calculation by noting that

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y + c(y)) = e^x \cos y = f(x, y).$$

To find the arbitrary function $c(y)$, we differentiate $\varphi(x, y) = e^x \cos y + c(y)$ with respect to y and equate the result to g (recall that $\varphi_y = g$):

$$\varphi_y = -e^x \sin y + c'(y) \quad \text{and} \quad g = -e^x \sin y.$$

We conclude that $c'(y) = 0$, which implies that $c(y)$ is any real number, which we typically take to be zero. So a potential function is $\varphi(x, y) = e^x \cos y$, a result that may be checked by differentiation.

b. The method of part (a) is more elaborate with three variables. A potential function φ must now satisfy these conditions:

$$\varphi_x = f = 2xy - z^2 \quad \varphi_y = g = x^2 + 2z \quad \varphi_z = h = 2y - 2xz.$$

Note »

> This procedure may begin with any of the three conditions.

Integrating the first condition with respect to x (holding y and z fixed), we have

$$\varphi = \int \left(2xy - z^2\right) dx = x^2 y - x z^2 + c(y, z).$$

Because the integration is with respect to x, the arbitrary "constant" is a function of y and z. To find $c(y, z)$, we differentiate φ with respect to y, which results in

$$\varphi_y = x^2 + c_y(y, z).$$

Equating φ_y and $g = x^2 + 2z$, we see that $c_y(y, z) = 2z$. To obtain $c(y, z)$, we integrate $c_y(y, z) = 2z$ with respect to y (holding z fixed), which results in $c(y, z) = 2yz + d(z)$. The "constant" of integration is now a function of z, which we call $d(z)$. At this point, a potential function looks like

$$\varphi(x, y, z) = x^2 y - x z^2 + 2yz + d(z).$$

To determine $d(z)$, we differentiate φ with respect to z:

$$\varphi_z = -2xz + 2y + d'(z).$$

Equating φ_z and $h = 2y - 2xz$, we see that $d'(z) = 0$, or $d(z)$ is a real number, which we generally take to be zero. Putting it all together, a potential function is

$$\varphi = x^2 y - x z^2 + 2yz.$$

Related Exercises 19, 24 ◆

Quick Check 3 Verify by differentiation that the potential functions found in Example 2 produce the corresponding vector fields. ◆

Answer »

> Show that $\nabla(e^x \cos y) = \langle e^x \cos y, -e^x \sin y\rangle$, which is the original vector field. A similar calculation may be done for part (b).

PROCEDURE **Finding Potential Functions in $\mathbb{R}^3$**

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Fundamental Theorem for Line Integrals and Path Independence »

Knowing how to find potential functions, we now investigate their properties. The first property is one of several beautiful parallels to the Fundamental Theorem of Calculus.

THEOREM 17.4 Fundamental Theorem for Line Integrals

Let R be a region of $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Note »

Compare the two versions of the Fundamental Theorem.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_C \nabla\varphi\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

Proof: Let the curve C in $\mathbb{R}^3$ be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$, where $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are the position vectors for the points A and B, respectively. By the Chain Rule, the rate of change of φ with respect to t along C is

$$\begin{aligned}
\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} && \text{Chain Rule}\\
&= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\rangle\cdot\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle && \text{Identify the dot product.}\\
&= \nabla\varphi\cdot\mathbf{r}'(t) && \mathbf{r} = \langle x, y, z\rangle\\
&= \mathbf{F}\cdot\mathbf{r}'(t). && \mathbf{F} = \nabla\varphi
\end{aligned}$$

Evaluating the line integral and using the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}
\int_C \mathbf{F}\cdot dr &= \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt\\
&= \int_a^b \frac{d\varphi}{dt}\,dt && \mathbf{F}\cdot\mathbf{r}'(t) = \frac{d\varphi}{dt}\\
&= \varphi(B) - \varphi(A). && \text{Fundamental Theorem of Calculus; } t = b \text{ corresponds to } B \text{ and } t = a \text{ corresponds to } A.
\end{aligned}$$

◆

Here is the meaning of Theorem 17.4: If $\mathbf{F}$ is a conservative vector field, then the value of a line integral of $\mathbf{F}$ depends only on the endpoints of the path. For this reason, we say the line integral is *independent of path*, which means that to evaluate line integrals of conservative vector fields, we do not need a parameterization of the path.

If we think of φ as an antiderivative of the vector field $\mathbf{F}$, then the parallel to the Fundamental Theorem of Calculus is clear. The line integral of $\mathbf{F}$ is the difference of the values of φ evaluated at the endpoints. Theorem 17.4 motivates the following definition.

DEFINITION **Independence of Path**

Let $\mathbf{F}$ be a continuous vector field with domain R. If $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

An important question concerns the converse of Theorem 17.4. With additional conditions on the domain R, the converse turns out to be true.

THEOREM 17.5

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

Note »

We state and prove Theorem 17.5 in two variables. It is easily extended to three or more variables.

Proof: Let $P(a, b)$ and $Q(x, y)$ be interior points of R and define $\varphi(x, y) = \int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a piecewise-smooth path from P to Q, and $\mathbf{F} = \langle f, g\rangle$. Because the integral defining φ is independent of path, any piecewise-smooth path in R from P to Q can be used. The goal is to compute the directional derivative $D_{\mathbf{u}}\,\varphi(x, y)$, where $\mathbf{u} = \langle u_1, u_2\rangle$ is an arbitrary unit vector, and then show that $\mathbf{F} = \nabla\varphi$. We let $S(x + t\,u_1, y + t\,u_2)$ be a point in R near Q and then apply the definition of the directional derivative at Q:

$$\begin{aligned} D_{\mathbf{u}}\,\varphi(x, y) &= \lim_{t\to 0} \frac{\varphi(x + t\,u_1, y + t\,u_2) - \varphi(x, y)}{t} \\ &= \lim_{t\to 0} \frac{1}{t}\left(\int_P^S \mathbf{F}\cdot d\mathbf{r} - \int_P^Q \mathbf{F}\cdot d\mathbf{r}\right) \\ &= \lim_{t\to 0} \frac{1}{t}\int_Q^S \mathbf{F}\cdot d\mathbf{r}. \end{aligned}$$

Using path independence, we choose the path from Q to S to be a line parametrized by $\mathbf{r}(s) = \langle x + s\,u_1, y + s\,u_2\rangle$, for $0 \le s \le t$. Noting that $\mathbf{r}'(s) = \mathbf{u}$, the directional derivative is

$$x, y) = \lim_{t\to 0} \frac{1}{t} \int_Q^S \mathbf{F} \cdot d\mathbf{r}$$

$$= \lim_{t\to 0} \frac{1}{t} \int_0^t \mathbf{F}(x + s\,u_1,\, y + s\,u_2) \cdot \mathbf{r}'(s)\, ds \qquad \text{Change line integral to ordinary integral.}$$

$$= \lim_{t\to 0} \frac{\int_0^t \mathbf{F}(x + s\,u_1,\, y + s\,u_2) \cdot \mathbf{r}'(s)\, ds - \int_0^0 \mathbf{F}(x + s\,u_1,\, y + s\,u_2) \cdot \mathbf{r}'(s)\, ds}{t} \qquad \text{Second integral equals 0.}$$

$$= \frac{d}{dt} \int_0^t \mathbf{F}(x + s\,u_1,\, y + s\,u_2) \cdot \mathbf{u}\, ds \Big|_{t=0} \qquad \text{Identify difference quotient; } \mathbf{r}'(s) = \mathbf{u}$$

$$= \mathbf{F}(x, y) \cdot \mathbf{u}. \qquad \text{Fundamental Theorem of Cal}$$

Choosing $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, we see that $D_{\mathbf{i}}\,\varphi(x, y) = \varphi_x(x, y) = \mathbf{F}(x, y) \cdot \mathbf{i} = f(x, y)$. Similarly, choosing $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, we have $D_{\mathbf{j}}\,\varphi(x, y) = \varphi_y(x, y) = \mathbf{F}(x, y) \cdot \mathbf{j} = g(x, y)$. Therefore, $\mathbf{F} = \langle f, g \rangle = \langle \varphi_x, \varphi_y \rangle = \nabla\varphi$, and $\mathbf{F}$ is conservative. ◆

EXAMPLE 3 Verifying path independence

Consider the potential function $\varphi(x, y) = \dfrac{x^2 - y^2}{2}$ and its gradient field $\mathbf{F} = \langle x, -y \rangle$.

- Let C_1 be the quarter circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le \dfrac{\pi}{2}$, from $A(1, 0)$ to $B(0, 1)$.
- Let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \le t \le 1$, also from A to B.

Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

SOLUTION »

On C_1 we have $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{F} = \langle x, -y \rangle = \langle \cos t, -\sin t \rangle$. The line integral on C_1 is

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{\pi/2} \underbrace{\langle \cos t, -\sin t \rangle}_{\mathbf{F}} \cdot \underbrace{\langle -\sin t, \cos t \rangle\, dt}_{\mathbf{r}'(t)\, dt} \qquad \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.$$

$$= \int_0^{\pi/2} (-\sin 2t)\, dt \qquad 2 \sin t \cos t = \sin 2t$$

$$= \left(\frac{1}{2} \cos 2t \right) \Bigg|_0^{\pi/2} = -1. \qquad \text{Evaluate integral.}$$

On C_2 we have $\mathbf{r}'(t) = \langle -1, 1 \rangle$ and $\mathbf{F} = \langle x, -y \rangle = \langle 1 - t, -t \rangle$; therefore,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \underbrace{\langle 1 - t, -t \rangle}_{\mathbf{F}} \cdot \underbrace{\langle -1, 1 \rangle\, dt}_{d\mathbf{r}} \qquad \text{Substitute for } \mathbf{F} \text{ and } d\mathbf{r}.$$

$$= \int_0^1 (-1)\, dt = -1. \qquad \text{Simplify.}$$

The two line integrals have the same value, which is

$$\varphi(B) - \varphi(A) = \varphi(0,\ 1) - \varphi(1,\ 0) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Related Exercises 31–32 ◆

EXAMPLE 4 **Line integral of a conservative vector field**

Evaluate

$$\int_C \left(\left(2\,x\,y - z^2\right)\mathbf{i} + \left(x^2 + 2\,z\right)\mathbf{j} + (2\,y - 2\,x\,z)\,\mathbf{k}\right)\cdot d\mathbf{r},$$

where C is a simple curve from $A(-3,\ -2,\ -1)$ to $B(1,\ 2,\ 3)$.

SOLUTION »

This vector field is conservative and has a potential function $\varphi = x^2\,y - x\,z^2 + 2\,y\,z$ (Example 2). By the Fundamental Theorem for line integrals,

$$\int_C \left(\left(2\,x\,y - z^2\right)\mathbf{i} + \left(x^2 + 2\,z\right)\mathbf{j} + (2\,y - 2\,x\,z)\,\mathbf{k}\right)\cdot d\mathbf{r} = \int_C \nabla\underbrace{\left(x^2\,y - x\,z^2 + 2\,y\,z\right)}_{\varphi}\cdot\, d\mathbf{r}$$
$$= \varphi(1,\ 2,\ 3) - \varphi(-3,\ -2,\ -1) = 16.$$

Related Exercise 34 ◆

Quick Check 4 Explain why the vector field $\nabla(x\,y + x\,z - y\,z)$ is a conservative field. ◆

Answer »

The vector field $\nabla(x\,y + x\,z - y\,z)$ is the gradient of $x\,y + x\,z - y\,z$, so the vector field is conservative.

Line Integrals on Closed Curves »

It is a short step to another characterization of conservative vector fields. Suppose C is a simple *closed* piecewise-smooth oriented curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. To distinguish line integrals on closed curves, we adopt the notation $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where the small circle on the integral sign indicates that C is a closed curve. Let A be any point on C and think of A as both the initial point and the final point of C. Assuming $\mathbf{F}$ is a conservative vector field on an open connected region R containing C, it follows by Theorem 17.4 that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \varphi(A) - \varphi(A) = 0.$$

Because A is an arbitrary point on C, we see that the line integral of a conservative vector field on a closed curve is zero.

Note »

An argument can be made in the opposite direction as well: Suppose $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves in a region R, and let A and B be distinct points in R. Let C_1 denote any curve from A to B, let C_2 be any curve from B to A (distinct from and not intersecting C_1) and let C be the closed curve consisting of C_1 followed by C_2 (**Figure 17.30**). Then

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

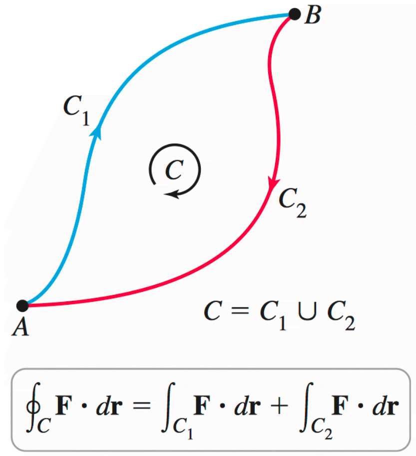


Figure 17.30

Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$, where $-C_2$ is the curve C_2 traversed in the opposite direction (from A to B). We see that the line integral has the same value on two arbitrary paths between A and B. It follows that the value of the line integral is independent of path, and by Theorem 17.5, $\mathbf{F}$ is conservative. This argument is a proof of the following theorem.

THEOREM 17.6 **Line Integrals on Closed Curves**

Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

EXAMPLE 5 A closed curve line integral in $\mathbb{R}^3$

Evaluate $\int_C \nabla(-xy + xz + yz) \cdot d\mathbf{r}$ on the curve $C: \mathbf{r}(t) = \langle \sin t, \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, without using Theorems 17.4 or 17.6.

SOLUTION »

The components of the vector field are

$$\mathbf{F} = \nabla(-xy + xz + yz) = \langle -y + z, -x + z, x + y \rangle.$$

Note that $\mathbf{r}'(t) = \langle \cos t, -\sin t, \cos t \rangle$ and $d\mathbf{r} = \mathbf{r}'(t)\,dt$. Substituting values of x, y, and z, the value of the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \langle -y+z,\ -x+z,\ x+y\rangle \cdot d\mathbf{r} \quad \text{Subsitute for } \mathbf{F}.$$

$$= \int_0^{2\pi} \sin 2\,t\,dt \quad \text{Substitute for } x,\ y,\ z,\ d\mathbf{r}.$$

$$= -\frac{1}{2}\cos 2\,t\Big|_0^{2\pi} = 0. \quad \text{Evaluate integral.}$$

The line integral of this conservative vector field on the closed curve C is zero. In fact, by Theorem 17.6, the line integral vanishes on any simple closed curve.

Related Exercise 50 ◆

Summary of the Properties of Conservative Vector Fields »

We have established three equivalent properties of conservative vector fields $\mathbf{F}$ defined on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$).

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).
- $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all piecewise-smooth oriented curves C from A to B (path independence).
- $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in R.

The connections between these properties were established by Theorems 17.4, 17.5, and 17.6 in the following way:

$$\text{Path–independence} \overset{\text{Theorem 17.4 and Theorem 17.5}}{\Longleftrightarrow} \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \overset{\text{Theorem 17.6}}{\Longleftrightarrow} \oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

Exercises »

Getting Started »

Practice Exercises »

9–16. Testing for conservative vector fields *Determine whether the following vector fields are conservative (in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*).*

9. $\mathbf{F} = \langle 1,\ 1\rangle$

10. $\mathbf{F} = \langle x,\ y\rangle$

11. $\mathbf{F} = \langle -y,\ x\rangle$

12. $\mathbf{F} = \langle -y,\ x+y\rangle$

13. $\mathbf{F} = \langle e^{-x}\cos y,\ e^{-x}\sin y\rangle$

14. $\mathbf{F} = \left\langle 2\,x^3 + x\,y^2,\ 2\,y^3 - x^2\,y\right\rangle$

15. $\mathbf{F} = \langle y\,z\cos x\,z,\ \sin x\,z,\ x\,y\cos x\,z\rangle$

16. $\mathbf{F} = \langle y\, e^{x-z},\, e^{x-z},\, y\, e^{x-z} \rangle$

17–30. Finding potential functions *Determine whether the following vector fields are conservative on the specified region. If so, determine a potential function. Let R^* and D^* be open regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, that do not include the origin.*

17. $\mathbf{F} = \langle x,\, y \rangle$ on $\mathbb{R}^2$

18. $\mathbf{F} = \langle -y,\, -x \rangle$ on $\mathbb{R}^2$

19. $\mathbf{F} = \left\langle x^3 - x\, y,\, \frac{x^2}{2} + y \right\rangle$ on $\mathbb{R}^2$

20. $\mathbf{F} = \frac{\langle x,\, y \rangle}{x^2 + y^2}$ on R^*

21. $\mathbf{F} = \frac{\langle x,\, y \rangle}{\sqrt{x^2 + y^2}}$ on R^*

22. $\mathbf{F} = \langle y,\, x,\, x - y \rangle$ on $\mathbb{R}^3$

23. $\mathbf{F} = \langle z,\, 1,\, x \rangle$ on $\mathbb{R}^3$

24. $\mathbf{F} = \langle y\, z,\, x\, z,\, x\, y \rangle$ on $\mathbb{R}^3$

25. $\mathbf{F} = \langle e^z,\, e^z,\, e^z\, (x - y) \rangle$ on $\mathbb{R}^3$

26. $\mathbf{F} = \langle 1,\, -z,\, y \rangle$ on $\mathbb{R}^3$

27. $\mathbf{F} = \langle y + z,\, x + z,\, x + y \rangle$ on $\mathbb{R}^3$

28. $\mathbf{F} = \frac{\langle x,\, y,\, z \rangle}{x^2 + y^2 + z^2}$ on D^*

29. $\mathbf{F} = \frac{\langle x,\, y,\, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$ on D^*

30. $\mathbf{F} = \left\langle x^3,\, 2\, y,\, -z^3 \right\rangle$ on $\mathbb{R}^3$

31–34. Evaluating line integrals *Use the given potential function φ of the gradient field* **F** *and the curve C to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.*

- ***a.*** *Use a parametric description of C and evaluate the integral directly.*
- ***b.*** *Use the Fundamental Theorem for line integrals.*

31. $\varphi(x,\, y) = x\, y;\ C : \mathbf{r}(t) = \langle \cos\, t,\, \sin\, t \rangle$, for $0 \le t \le \pi$

32. $\varphi(x,\, y) = x\ + 3\, y;\ C : \mathbf{r}(t) = \langle 2 - t,\, t \rangle$, for $0 \le t \le 2$

33. $\varphi(x, y, z) = \left(x^2 + y^2 + z^2\right)/2$; $C: \mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{t}{\pi} \right\rangle$, for $0 \le t \le 2\pi$

34. $\varphi(x, y, z) = xy + xz + yz$; $C: \mathbf{r}(t) = \langle t, 2t, 3t \rangle$, for $0 \le t \le 4$

35–38. Applying the Fundamental Theorem for line integrals *Suppose the vector field* $\mathbf{F}$ *is continuous on* $\mathbb{R}^2$, $\mathbf{F} = \langle f, g \rangle = \nabla\varphi$, $\varphi(1, 2) = 7$, $\varphi(3, 6) = 10$, *and* $\varphi(6, 4) = 20$. *Evaluate the following integrals for the given curve C, if possible.*

35. $\int_C \mathbf{F} \cdot d\mathbf{r}$; C: $\mathbf{r}(t) = \left\langle 2t - 1, t^2 + t \right\rangle$, for $1 \le t \le 2$

36. $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; C is a smooth curve from $(1, 2)$ to $(6, 4)$.

37. $\int_C f\, dx + g\, dy$; C is the path consisting of the line segment from $A(6, 4)$ to $B(1, 2)$ followed by the line segment from $B(1, 2)$ to $C(3, 6)$.

38. $\oint_C \mathbf{F} \cdot d\mathbf{r}$; C is a circle, oriented clockwise, starting and ending at the point $A(6, 4)$.

39–44. Using the Fundamental Theorem for line integrals *For each of the following, verify that that the Fundamental Theorem of line integrals can be used to evaluate the line integral and then evaluate the line integral using this theorem.*

39. $\int_C \langle 2x, 2y \rangle \cdot d\mathbf{r}$, where C is a smooth curve from $(0, 1)$ to $(3, 4)$

40. $\int_C \langle 1, 1, 1 \rangle \cdot d\mathbf{r}$, where C is a smooth curve from $(1, -1, 2)$ to $(3, 0, 7)$

41. $\int_C \nabla(e^{-x} \cos y) \cdot d\mathbf{r}$, where C is the line segment from $(0, 0)$ to $(\ln 2, 2\pi)$

42. $\int_C \nabla\left(1 + x^2 yz\right) \cdot d\mathbf{r}$, where C is the helix $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$, for $0 \le t \le 4\pi$

43. $\int_C \cos(2y - z)\, dx - 2x \sin(2y - z)\, dy + x \sin(2y - z)\, dz$, where C is the curve $\mathbf{r}(t) = \left\langle t^2, t, t \right\rangle$, for $0 \le t \le \pi$

44. $\int_C e^x y\, dx + e^x\, dy$, where C is the parabola $\mathbf{r}(t) = \left\langle t + 1, t^2 \right\rangle$, for $-1 \le t \le 3$

45–50. Line integrals of vector fields on closed curves *Evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *for the following vector fields and closed oriented curves C by parameterizing C. If the integral is not zero, give an explanation.*

45. $\mathbf{F} = \langle x, y \rangle$; C is the circle of radius 4 centered at the origin oriented counterclockwise.

46. $\mathbf{F} = \langle y, x \rangle$; C is the circle of radius 8 centered at the origin oriented counterclockwise.

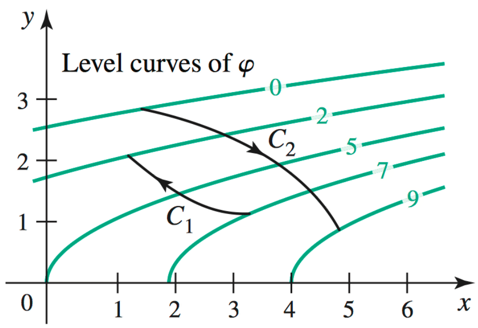
47. $\mathbf{F} = \langle x, y \rangle$; C is the triangle with vertices $(0, \pm 1)$ and $(1, 0)$ oriented counterclockwise.

48. $\mathbf{F} = \langle y, -x \rangle$; C is the circle of radius 3 centered at the origin oriented counterclockwise.

49. $\mathbf{F} = \langle x, y, z \rangle$; $C : \mathbf{r}(t) = \langle \cos t, \sin t, 2 \rangle$, for $0 \le t \le 2\pi$

50. $\mathbf{F} = \langle y - z, z - x, x - y \rangle$; $C : \mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \le t \le 2\pi$

51–52. Evaluating line integrals using level curves *Suppose the vector field* $\mathbf{F}$, *whose potential function is* φ, *is continuous on* $\mathbb{R}^2$. *Use the curves* C_1 *and* C_2 *and level curves of* φ (*see figure*) *to evaluate the following line integrals.*



51. $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$

52. $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

53–56. Line integrals *Evaluate the following line integrals using a method of your choice.*

53. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2xy + z^2, x^2, 2xz \rangle$ and C is the circle $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t \rangle$, for $0 \le t \le 2\pi$.

54. $\displaystyle\oint_C e^{-x}(\cos y\, dx + \sin y\, dy)$, where C is the square with vertices $(\pm 1, \pm 1)$ oriented counterclockwise

55. $\displaystyle\int_C \nabla(\sin xy) \cdot d\mathbf{r}$, where C is the line segment from $(0, 0)$ to $\left(2, \dfrac{\pi}{4}\right)$

56. $\displaystyle\int_C x^3\, dx + y^3\, dy$, where C is the curve $\mathbf{r}(t) = \langle 1 + \sin t, \cos^2 t \rangle$, for $0 \le t \le \pi/2$

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\mathbf{F} = \langle -y, x \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

b. If $\mathbf{F} = \langle x, -y \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

c. A constant vector field is conservative on $\mathbb{R}^2$.

d. The vector field $\mathbf{F} = \langle f(x), g(y) \rangle$ is conservative on $\mathbb{R}^2$ (assume f and g are defined for all real numbers).

e. Gradient fields are conservative.

58. Closed-curve integrals Evaluate $\oint_C ds$, $\oint_C dx$, and $\oint_C dy$, where C is the unit circle oriented counterclockwise.

59–62. Work in force fields *Find the work required to move an object in the following force fields along a line segment between the given points. Check to see if the force is conservative.*

59. $\mathbf{F} = \langle x, 2 \rangle$ from $A(0, 0)$ to $B(2, 4)$

60. $\mathbf{F} = \langle x, y \rangle$ from $A(1, 1)$ to $B(3, -6)$

61. $\mathbf{F} = \langle x, y, z \rangle$ from $A(1, 2, 1)$ to $B(2, 4, 6)$

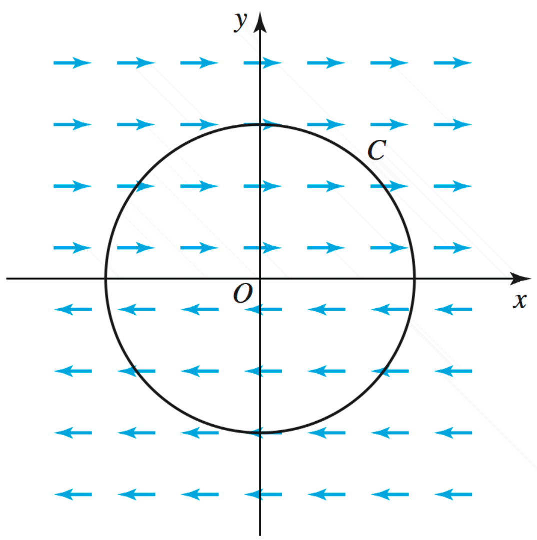
62. $\mathbf{F} = e^{x+y} \langle 1, 1, z \rangle$ from $A(0, 0, 0)$ to $B(-1, 2, -4)$

63. Suppose C is a circle centered at the origin in a vector field $\mathbf{F}$ (see figure).

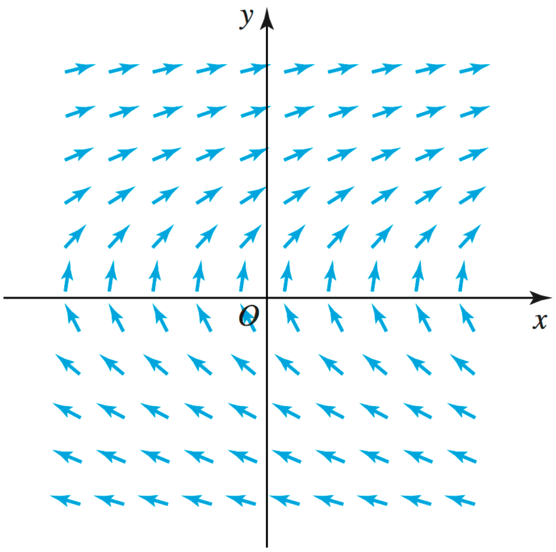
a. If C is oriented counterclockwise, is $\oint_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero?

b. If C is oriented clockwise, is $\oint_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero?

c. Is $\mathbf{F}$ conservative in $\mathbb{R}^2$? Explain.



64. A vector field that is continuous in $\mathbb{R}^2$ is given (see figure). Is it conservative?



65. **Work by a constant force** Evaluate a line integral to show that the work done in moving an object from point A to point B in the presence of a constant force $\mathbf{F} = \langle a, b, c \rangle$ is $\mathbf{F} \cdot \overrightarrow{AB}$.

Explorations and Challenges »

66. **Conservation of energy** Suppose an object with mass m moves in a region R in a conservative force field given by $\mathbf{F} = -\nabla \varphi$, where φ is a potential function in a region R. The motion of the object is governed by Newton's Second Law of Motion, $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{a}$ is the acceleration. Suppose the object moves from point A to point B in R.

a. Show that the equation of motion is $m \dfrac{d\mathbf{v}}{dt} = -\nabla \varphi$.

b. Show that $\dfrac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \dfrac{1}{2} \dfrac{d}{dt} (\mathbf{v} \cdot \mathbf{v})$.

c. Take the dot product of both sides of the equation in part (a) with $\mathbf{v}(t) = \mathbf{r}'(t)$ and integrate along a curve between A and B. Use part (b) and the fact that $\mathbf{F}$ is conservative to show that the total energy (kinetic plus potential) $\dfrac{1}{2} m |\mathbf{v}|^2 + \varphi$ is the same at A and B. Conclude that because A and B are arbitrary, energy is conserved in R.

67. **Gravitational potential** The gravitational force between two point masses M and m is

$$\mathbf{F} = G\,M\,m \frac{\mathbf{r}}{|\mathbf{r}|^3} = G\,M\,m \frac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}},$$

where G is the gravitational constant.

a. Verify that this force field is conservative on any region excluding the origin.

b. Find a potential function φ for this force field such that $\mathbf{F} = -\nabla \varphi$.

c. Suppose the object with mass m is moved from a point A to a point B, where A is a distance r_1 from M and B is a distance r_2 from M. Show that the work done in moving the object is $G\,M\,m \left(\dfrac{1}{r_2} - \dfrac{1}{r_1} \right)$.

d. Does the work depend on the path between A and B? Explain.

68. Radial fields in $\mathbb{R}^3$ are conservative Prove that the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number, is conservative on any region not containing the origin. For what values of p is $\mathbf{F}$ conservative on a region that contains the origin?

69. Rotation fields are usually not conservative

- **a.** Prove that the rotation field $\mathbf{F} = \dfrac{\langle -y, x \rangle}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y \rangle$ is not conservative for $p \neq 2$.
- **b.** For $p = 2$, show that $\mathbf{F}$ is conservative on any region not containing the origin.
- **c.** Find a potential function for $\mathbf{F}$ when $p = 2$.

70. Linear and quadratic vector fields

- **a.** For what values of a, b, c, and d is the field $\mathbf{F} = \langle a\,x + b\,y,\ c\,x + d\,y \rangle$ conservative?
- **b.** For what values of a, b, and c is the field $\mathbf{F} = \left\langle a\,x^2 - b\,y^2,\ c\,x\,y \right\rangle$ conservative?

71. Alternative construction of potential functions in $\mathbb{R}^2$ Assume that the vector field $\mathbf{F}$ is conservative on $\mathbb{R}^2$, so that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Use the following procedure to construct a potential function φ for the vector field $\mathbf{F} = \langle f, g \rangle = \langle 2\,x - y,\ -x + 2\,y \rangle$.

- **a.** Let A be $(0, 0)$ and let B be an arbitrary point (x, y). Define $\varphi(x, y)$ to be the work required to move an object from A to B, where $\varphi(A) = 0$. Let C_1 be the path from A to $(x, 0)$ to B and let C_2 be the path from A to $(0, y)$ to B. Draw a picture.
- **b.** Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} f\,dx + g\,dy$ and conclude that $\varphi(x, y) = x^2 - x\,y + y^2$.
- **c.** Verify that the same potential function is obtained by evaluating the line integral over C_2.

72–75. Alternative construction of potential functions *Use the procedure in Exercise 71 to construct potential functions for the following fields.*

72. $\mathbf{F} = \langle -y, -x \rangle$

73. $\mathbf{F} = \langle x, y \rangle$

74. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y \rangle$

75. $\mathbf{F} = \left\langle 2\,x^3 + x\,y^2,\ 2\,y^3 + x^2\,y \right\rangle$