17.2 Line Integrals

With integrals of a single variable, we integrate over intervals in $\mathbb{R}$ (the real line). With double and triple integrals, we integrate over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. *Line integrals* (which really should be called *curve integrals*) are another class of integrals that play an important role in vector calculus. They are used to integrate either scalar-valued functions or vector fields along curves.

Suppose a thin, circular plate has a known temperature distribution and you must compute the average temperature along the edge of the plate. The required calculation involves integrating the temperature function over the *curved* boundary of the plate. Similarly, to calculate the amount of work needed to put a satellite into orbit, we integrate the gravitational force (a vector field) along the curved path of the satellite. Both these calculations require line integrals. As you will see, line integrals take several different forms. It is the goal of this section to distinguish these various forms and show how and when each form should be used.

Scalar Line Integrals in the Plane »

We focus first on line integrals of scalar-valued functions over curves in the xy-plane. Assume C is a smooth curve of finite length given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. We divide $[a, b]$ into n subintervals using the grid points

$$a = t_o < t_1 < \cdots < t_{n-1} < t_n = b.$$

This partition of $[a, b]$ divides C into n subarcs (**Figure 17.16**), where the arc length of the kth subarc is denoted Δs_k. Let t_k^* be a point in the kth subinterval $[t_{k-1}, t_k]$, which corresponds to a point $(x(t_k^*), y(t_k^*))$ on the kth subarc of C, for $k = 1, 2, \ldots, n$.

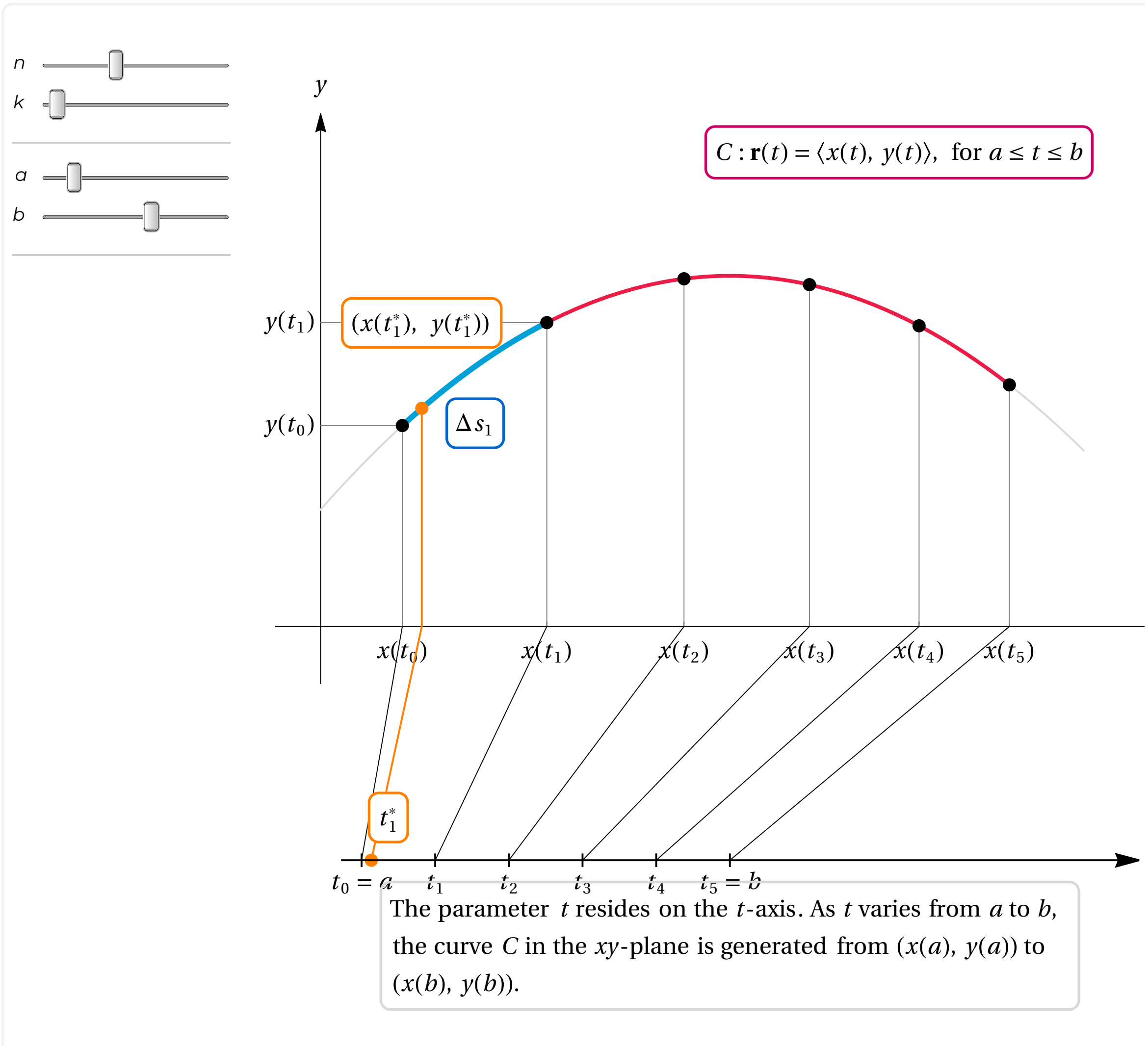


Figure 17.16

Now consider a scalar-valued function $z = f(x,\ y)$ defined on a region containing C. Evaluating f at $(x(t_k^*),\ y(t_k^*))$ and multiplying this value by Δs_k, we form the sum

$$S_n = \sum_{k=1}^{n} f(x(t_k^*),\ y(t_k^*))\,\Delta s_k,$$

which is similar to a Riemann sum. We now let Δ be the maximum value of $\{\Delta s_1,\ \ldots,\ \Delta s_n\}$. If the limit of the sum as $n \to \infty$ and $\Delta \to 0$ exists over all partitions, then the limit is called *the line integral of f over C*.

DEFINITION **Scalar Line Integral in the Plane**

Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. The **line integral of *f* over *C*** is

$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\, \Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.

The more compact notation $\int_C f(\mathbf{r}(t))\, ds$, $\int_C f(x, y)\, ds$, and $\int_C f\, ds$ are also used for the line integral of f over C. It can be shown that if f is continuous on a region containing C, then f is integrable over C.

There are several useful interpretations of the line integral of a scalar function. If $f(x, y) = 1$, the line integral $\int_C ds$ gives the length of the curve C, just as the ordinary integral $\int_a^b dx$ gives the length of the interval $[a, b]$, which is $b - a$. If $f(x, y) \ge 0$ on C, then $\int_C f(x, y)\, ds$ can be viewed as the area of one side of the vertical, curtain-like surface that lies between the graphs of f and C (**Figure 17.17**). This interpretation results from regarding the product $f(x(t_k^*), y(t_k^*))\, \Delta s_k$ as an approximation to the area of the kth panel of the curtain. Similarly, if f is a density function for a thin wire represented by the curve C, then $\int_C f(x, y)\, ds$ gives the mass of the wire—the product $f(x(t_k^*), y(t_k^*))\, \Delta s_k$ is an approximation to the mass of the kth piece of the wire (Exercises 35–36).

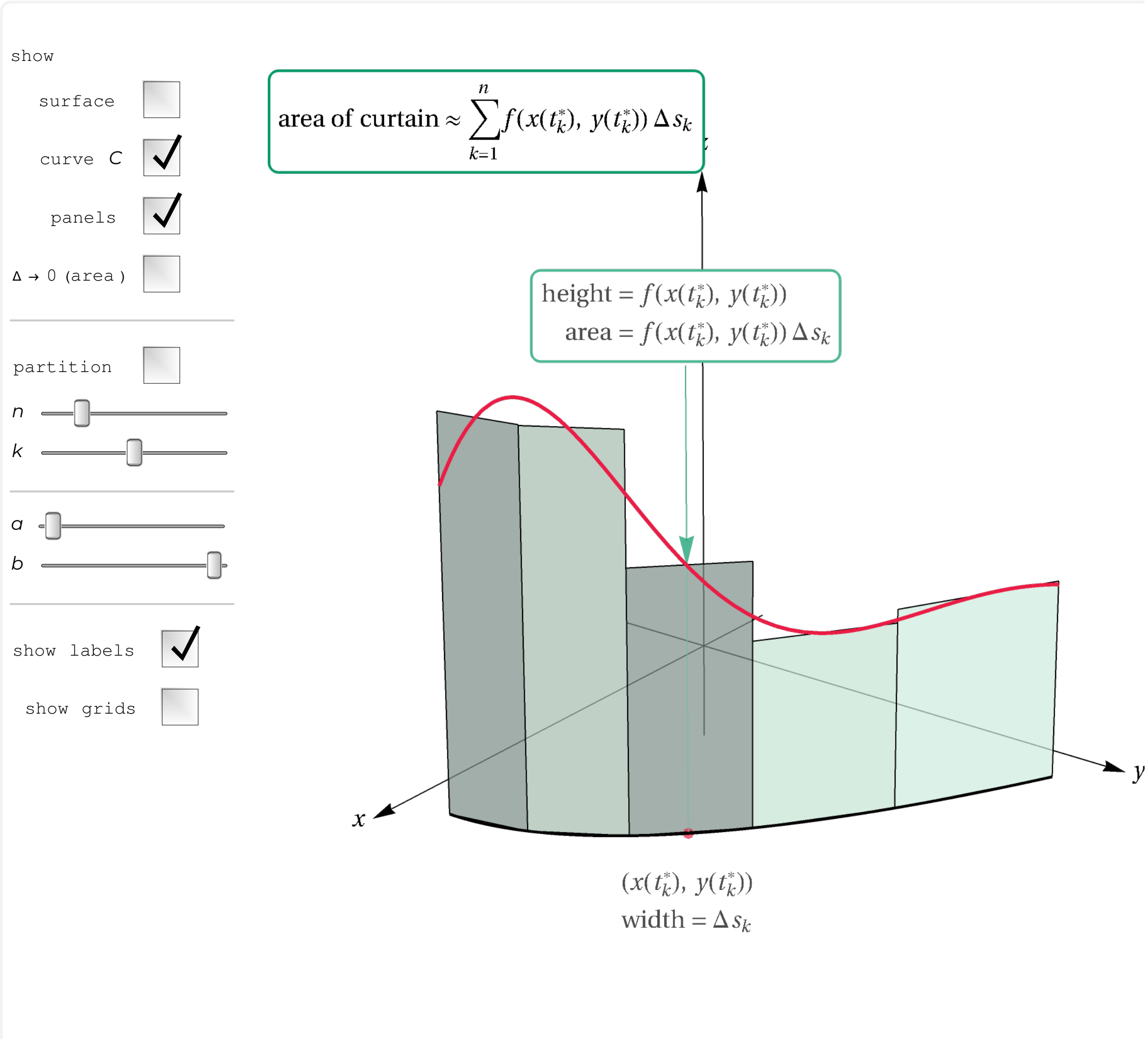


Figure 17.17

Evaluating Line Integrals »

The line integral of f over C given in the definition is not an ordinary Riemann integral, because the integrand is expressed as a function of t while the variable of integration is the arc length parameter s. We need a practical way to evaluate such integrals; the key is to use a change of variables to convert a line integral into an ordinary integral. Let C be given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. Recall from Section 14.4 that the length of C over the interval $[a, t]$ is

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du.$$

Differentiating both sides of this equation and using the Fundamental Theorem of Calculus yields $s'(t) = |\mathbf{r}'(t)|$. We now make a standard change of variables using the relationship

$$ds = s'(t)\,dt = |\mathbf{r}'(t)|\,dt.$$

Quick Check 1 Explain mathematically why differentiating the arc length integral leads to $s'(t) = |\mathbf{r}'(t)|$.

◆

Answer »

> The Fundamental Theorem of Calculus says that $\dfrac{d}{dt}\displaystyle\int_a^t f(u)\,du = f(t)$, which applies to differentiating the arc length integral.

Relying on a result from advanced calculus, the original line integral with respect to s can be converted into an ordinary integral with respect to t:

$$\int_C f\,ds = \int_a^b f(x(t),\,y(t))\,\underbrace{|\mathbf{r}'(t)|\,dt}_{ds}.$$

Note »

> If t represents time, then the relationship $ds = |\mathbf{r}'(t)|\,dt$ is a generalization of the familiar formula *distance* = (*speed*) (*time*).

THEOREM 17.1 **Evaluating Scalar Line Integrals in $\mathbb{R}^2$**

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t),\,y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(x(t),\,y(t))\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b f(x(t),\,y(t))\,\sqrt{x'(t)^2 + y'(t)^2}\;dt.$$

Note »

If t represents time and C is the path of a moving object, then $|\mathbf{r}'(t)|$ is the speed of the object. The *speed factor* $|\mathbf{r}'(t)|$ that appears in the integral relates distance traveled along the curve as measured by s to the elapsed time as measured by the parameter t.

Notice that if $f(x,\,y) = 1$, then the line integral is $\displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\;dt$, which is the arc length formula for C. Theorem 17.1 leads to the following procedure for evaluating line integrals.

PROCEDURE **Evaluating the Line Integral $\displaystyle\int_C f\,ds$**

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\,y(t)\rangle$, for $a \le t \le b$.

2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.

3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\,ds = \int_a^b f\,(x\,(t),\,y\,(t))\,|\mathbf{r}'(t)|\,dt.$$

EXAMPLE 1 Average temperature on a circle

The temperature of the circular plate $R = \{(x, y) : x^2 + y^2 \le 1\}$ is $T(x, y) = 100\left(x^2 + 2y^2\right)$. Find the average temperature along the edge of the plate.

Note »

When we compute the average value by an ordinary integral, we divide by the length of the interval of integration. Analogously, when we compute the average value by a line integral, we divide by the length of the curve:

$$\bar{f} = \frac{1}{L} \int_C f\, ds.$$

SOLUTION »

Calculating the average value requires integrating the temperature function over the boundary circle $C = \{(x, y) : x^2 + y^2 = 1\}$ and dividing by the length (circumference) of C. The first step is to find a parametric description for C. We use the standard parametrization for a unit circle centered at the origin, $\mathbf{r} = \langle x, y\rangle = \langle \cos t, \sin t\rangle$, for $0 \le t \le 2\pi$. Next, we compute the speed factor

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$$

We substitute $x = \cos t$ and $y = \sin t$ into the temperature function and express the line integral as an ordinary integral with respect to t:

$\displaystyle\int_C T(x, y)\, ds = \int_0^{2\pi} \underbrace{100\left(x(t)^2 + 2y(t)^2\right)}_{T(t)} \underbrace{|\mathbf{r}'(t)|}_{1}\, dt$	Write the line integral as an ordinary integral with respect to t; $ds = |\mathbf{r}'(t)|\, dt$.
$\displaystyle = 100 \int_0^{2\pi} \left(\cos^2 t + 2\sin^2 t\right) dt$	Substitute for x and y.
$\displaystyle = 100 \underbrace{\int_0^{2\pi} \left(1 + \sin^2 t\right) dt}_{3\pi}$	$\cos^2 t + \sin^2 t = 1$
$= 300\pi.$	Use $\sin^2 t = \dfrac{1 - \cos 2t}{2}$ and integrate.

The geometry of this line integral is shown in **Figure 17.18**. The temperature function on the boundary of C is a function of t. The line integral is an ordinary integral with respect to t over the interval $[0, 2\pi]$. To find the average value we divide the line integral of the temperature by the length of the curve, which is 2π. Therefore, the average temperature on the boundary of the plate is $\dfrac{300\pi}{2\pi} = 150$.

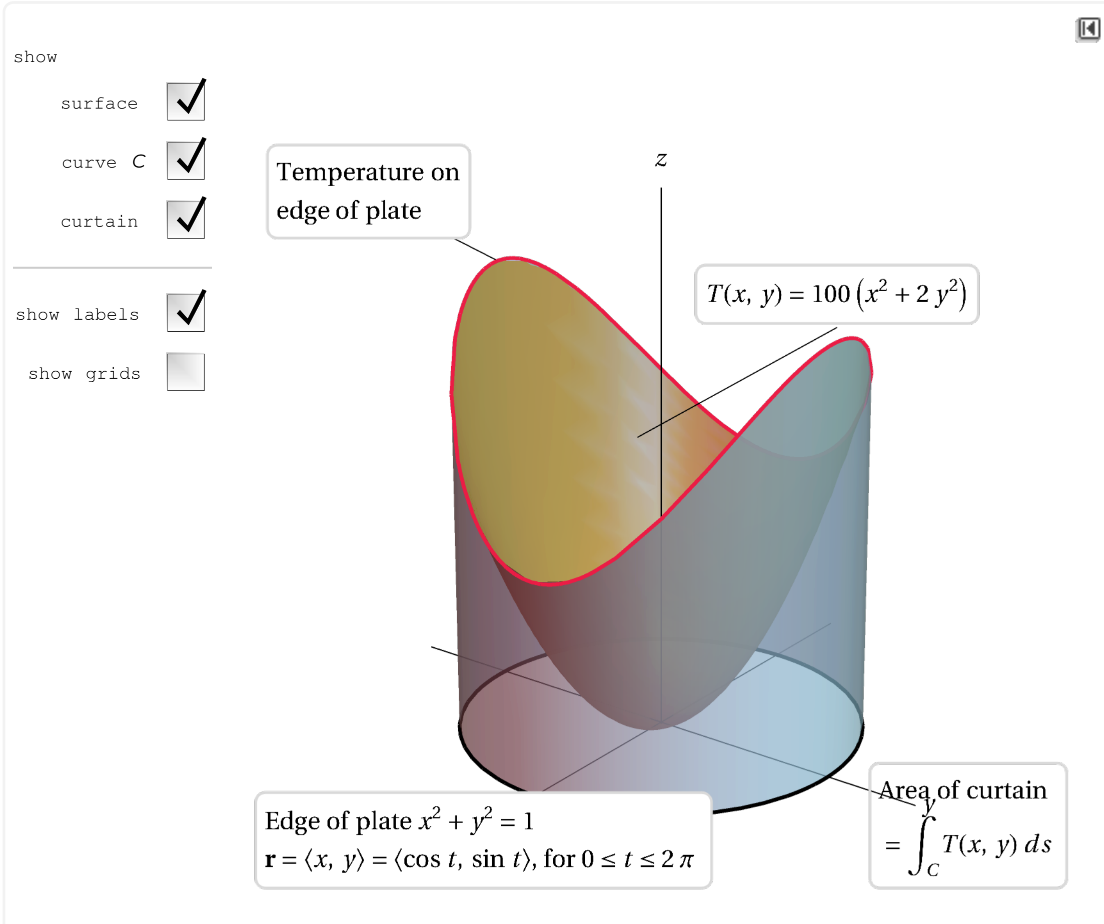


Figure 17.18

Note »

Related Exercises 17, 38 ◆

Quick Check 2 Suppose $\mathbf{r}(t) = \langle t, 0\rangle$, for $a \le t \le b$, is a parametric description of C; note that C is the interval $[a, b]$ on the x-axis. Show that $\int_C f(x, y)\, ds = \int_a^b f(t, 0)\, dt$, which is an ordinary, single-variable integral introduced in Chapter 5. ◆

Answer »

Line Integrals in $\mathbb{R}^3$ »

The argument that leads to line integrals on plane curves extends immediately to three or more dimensions. Here is the corresponding evaluation theorem for line integrals in $\mathbb{R}^3$.

THEOREM 17.2 **Evaluating Scalar Line Integrals in $\mathbb{R}^3$**

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f\, ds = \int_a^b f(x(t),\ y(t),\ z(t))\, |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b f(x(t),\ y(t),\ z(t))\, \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt.$$

Note »

EXAMPLE 2 Line integrals in $\mathbb{R}^3$

Evaluate $\int_C (x\,y + 2\,z)\, ds$ on the following line segments.

a. The line from $P(1,\ 0,\ 0)$ to $Q(0,\ 1,\ 1)$

b. The line from $Q(0,\ 1,\ 1)$ to $P(1,\ 0,\ 0)$

Note »

Recall that a parametric equation of a line is

$$\mathbf{r}(t) = \langle x_0,\ y_0,\ z_0\rangle + t\,\langle a,\ b,\ c\rangle,$$

where $\langle x_0,\ y_0,\ z_0\rangle$ is a position vector associated with a fixed point on the line and $\langle a,\ b,\ c\rangle$ is a vector parallel to the line.

SOLUTION »

a. A parametric description of the line from $P(1,\ 0,\ 0)$ to $Q(0,\ 1,\ 1)$ is

$$\mathbf{r}(t) = \langle 1,\ 0,\ 0\rangle + t\,\langle -1,\ 1,\ 1\rangle = \langle 1-t,\ t,\ t\rangle, \text{ for } 0 \le t \le 1.$$

The speed factor is

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}.$$

Substituting $x = 1 - t$, $y = t$, and $z = t$, the value of the line integral is

$$\begin{aligned}
\int_C (x\,y + 2\,z)\, ds &= \int_0^1 \left(\underbrace{(1-t)}_{x}\,\underbrace{t}_{y} + \underbrace{2\,t}_{2\,z}\right)\sqrt{3}\ dt && \text{Substitute for } x,\ y,\ z. \\
&= \sqrt{3}\int_0^1 \left(3\,t - t^2\right) dt && \text{Simplify.} \\
&= \sqrt{3}\left(\frac{3\,t^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^1 && \text{Integrate.} \\
&= \frac{7\sqrt{3}}{6}. && \text{Evaluate.}
\end{aligned}$$

b. The line from $Q(0,\ 1,\ 1)$ to $P(1,\ 0,\ 0)$ may be described parametrically by

$$\mathbf{r}(t) = \langle 0, 1, 1 \rangle + t\langle 1, -1, -1 \rangle = \langle t, 1-t, 1-t \rangle, \text{ for } 0 \le t \le 1.$$

The speed factor is

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

We substitute $x = t$, $y = 1 - t$, and $z = 1 - t$ and do a calculation similar to that in part (a). The value of the line integral is again $\frac{7\sqrt{3}}{6}$, emphasizing the fact that a scalar line integral is independent of the orientation and parameterization of the curve.

Related Exercises 32–33 ◆

EXAMPLE 3 Flight of an eagle

An eagle soars on the ascending spiral path

$$C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \left\langle 2400 \cos \frac{t}{2}, 2400 \sin \frac{t}{2}, 500\, t \right\rangle,$$

where x, y, and z are measured in feet and t is measured in minutes. How far does the eagle fly over the time interval $0 \le t \le 10$?

SOLUTION »

Note »

Because we are finding the length of a curve, the integrand in this line integral is $f(x, y, z) = 1$.

The distance traveled is found by integrating the element of arc length ds along C, that is, $L = \int_C ds$. We now make a change of variables to the parameter t using

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \\ &= \sqrt{\left(-1200 \sin \frac{t}{2}\right)^2 + \left(1200 \cos \frac{t}{2}\right)^2 + 500^2} && \text{Substitute derivatives.} \\ &= \sqrt{1200^2 + 500^2} = 1300. && \sin^2 \frac{t}{2} + \cos^2 \frac{t}{2} = 1 \end{aligned}$$

It follows that the distance traveled is

$$L = \int_C ds = \int_0^{10} |\mathbf{r}'(t)|\, dt = \int_0^{10} 1300\, dt = 13{,}000 \text{ ft}.$$

Related Exercise 39 ◆

Quick Check 3 What is the speed of the eagle in Example 3? ◆

Answer »

1300 ft/min

Line Integrals of Vector Fields »

Line integrals along curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ may also have integrands that involve vector fields. Such line integrals are different from scalar line integrals in two respects.

- Recall that an *oriented curve* is a parameterized curve for which a direction is specified. The *positive*, or *forward*, orientation is the direction in which the curve is generated as the parameter increases. For example, the positive direction of the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, is counterclockwise. As we will see, vector line integrals must be evaluated on oriented curves, and the value of a line integral depends on the orientation.
- The line integral of a vector field $\mathbf{F}$ along an oriented curve involves a specific component of $\mathbf{F}$ relative to the curve. We begin by defining vector line integrals for the *tangential* component of $\mathbf{F}$, a situation that has many physical applications.

Let $C : \mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$ be a smooth oriented curve in $\mathbb{R}^3$ parameterized by arc length and let $\mathbf{F}$ be a vector field that is continuous on a region containing C. At each point of C, the unit tangent vector $\mathbf{T}$ points in the positive direction on C (**Figure 17.19**). The component of $\mathbf{F}$ in the direction of $\mathbf{T}$ at a point of C is $|\mathbf{F}| \cos \theta$, where θ is the angle between $\mathbf{F}$ and $\mathbf{T}$. Because $\mathbf{T}$ is a unit vector,

$$|\mathbf{F}| \cos \theta = |\mathbf{F}|\,|\mathbf{T}| \cos \theta = \mathbf{F} \cdot \mathbf{T}.$$

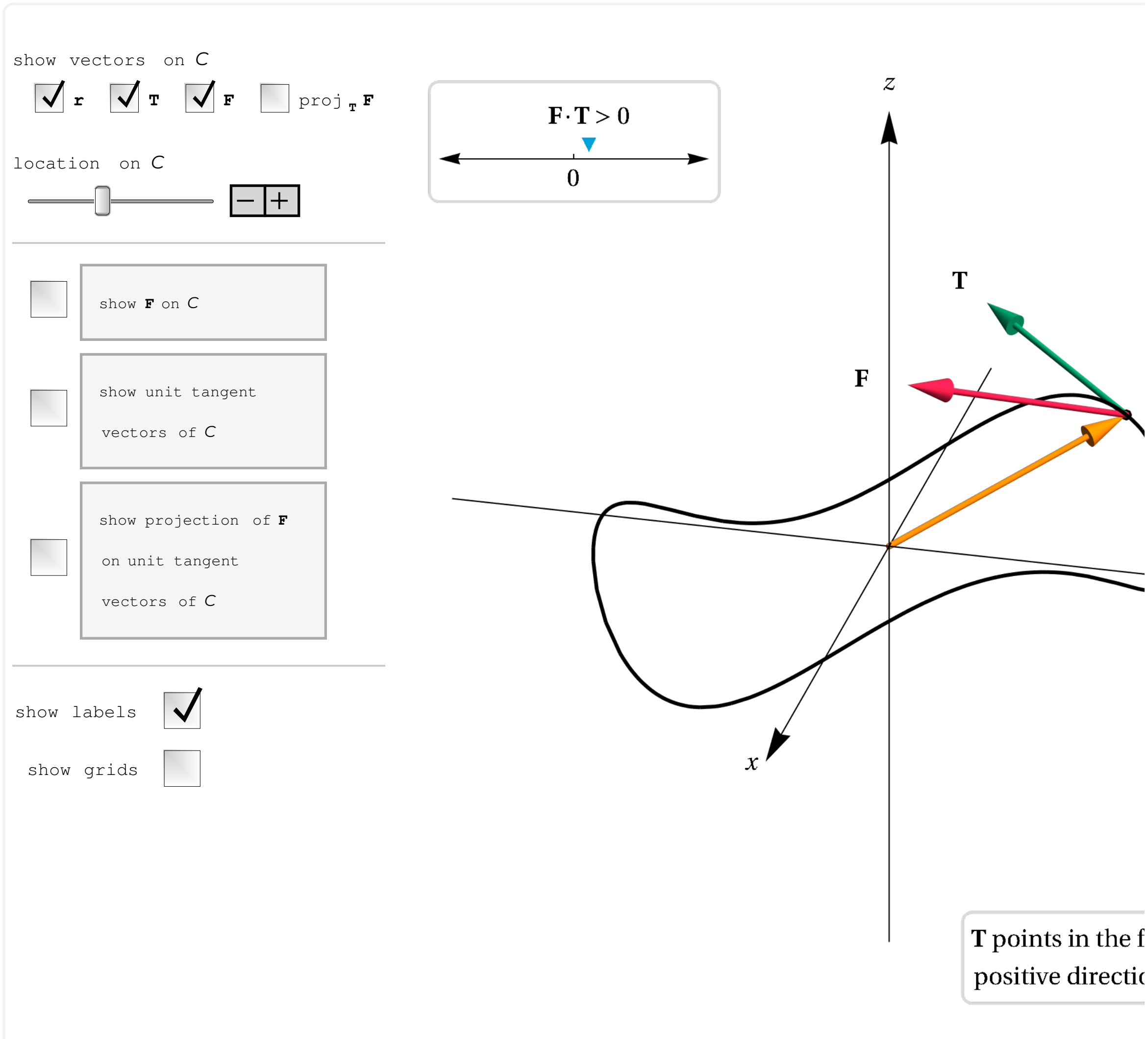


Figure 17.19

Note »

The first line integral of a vector field **F** that we introduce is the line integral of the scalar $\mathbf{F}\cdot\mathbf{T}$ along the curve C. When we integrate $\mathbf{F}\cdot\mathbf{T}$ along C, the effect is to add up the components of **F** in the direction of C at each point of C.

DEFINITION **Line Integral of a Vector Field**

Let **F** be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let **T** be the unit tangent vector at each point of C consistent with the orientation. The line integral of **F** over C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Note »

Some texts let $d\mathbf{s}$ stand for $\mathbf{T}\,ds$. Then the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is written $\int_C \mathbf{F}\cdot\,d\mathbf{s}$.

Just as we did for line integrals of scalar-valued functions, we need a method for evaluating vector line integrals when the parameter is not the arc length. Suppose that C has a parameterization $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ for $a \le t \le b$. Recall from Section 14.2 that the unit tangent vector at a point on the curve is $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Using the fact that $ds = |\mathbf{r}'(t)|\,dt$, the line integral becomes

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\underbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}_{\mathbf{T}}\underbrace{|\mathbf{r}'(t)|\,dt}_{ds} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

This integral may be written in several different forms. If $\mathbf{F} = \langle f,\, g,\, h\rangle$, then the line integral may be evaluated in component form as

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)\,x'(t) + g(t)\,y'(t) + h(t)\,z'(t))\,dt.$$

Note »

Keep in mind that $f(t)$ stands for $f(x(t),\, y(t),\, z(t))$ with analogous expressions for $g(t)$ and $h(t)$.

Another useful form is obtained by noting that

$$dx = x'(t)\,dt, \qquad dy = y'(t)\,dt, \qquad dz = z'(t)\,dt.$$

Making these replacements in the previous integral results in the form

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C f\,dx + g\,dy + h\,dz.$$

Finally, if we let $d\mathbf{r} = \langle dx,\, dy,\, dz\rangle$, then $f\,dx + g\,dy + h\,dz = \mathbf{F}\cdot d\mathbf{r}$, and we have

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

It is helpful to become familiar with these various forms of the line integral.

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$:

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\, dt &= \int_a^b (f(t)\, x'(t) + g(t)\, y'(t) + h(t)\, z'(t))\, dt\\ &= \int_C f\, dx + g\, dy + h\, dz\\ &= \int_C \mathbf{F}\cdot d\mathbf{r}\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_a^b (f(t)\, x'(t) + g(t)\, y'(t))\, dt = \int_C f\, dx + g\, dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

EXAMPLE 4 Different paths

Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\, ds$ with $\mathbf{F} = \langle y - x, x\rangle$ on the following oriented paths in $\mathbb{R}^2$ (**Figure 17.20**).

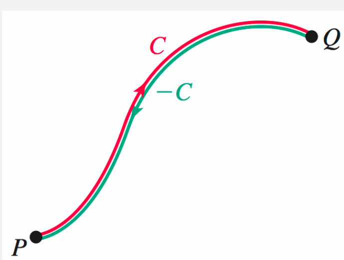
a. The quarter circle C_1 from $P(0, 1)$ to $Q(1, 0)$

b. The quarter circle $-C_1$ from $Q(1, 0)$ to $P(0, 1)$

c. The path C_2 from P to Q via two line segments through $O(0, 0)$

Note »

We use the convention that $-C$ is the curve C with the opposite orientation.



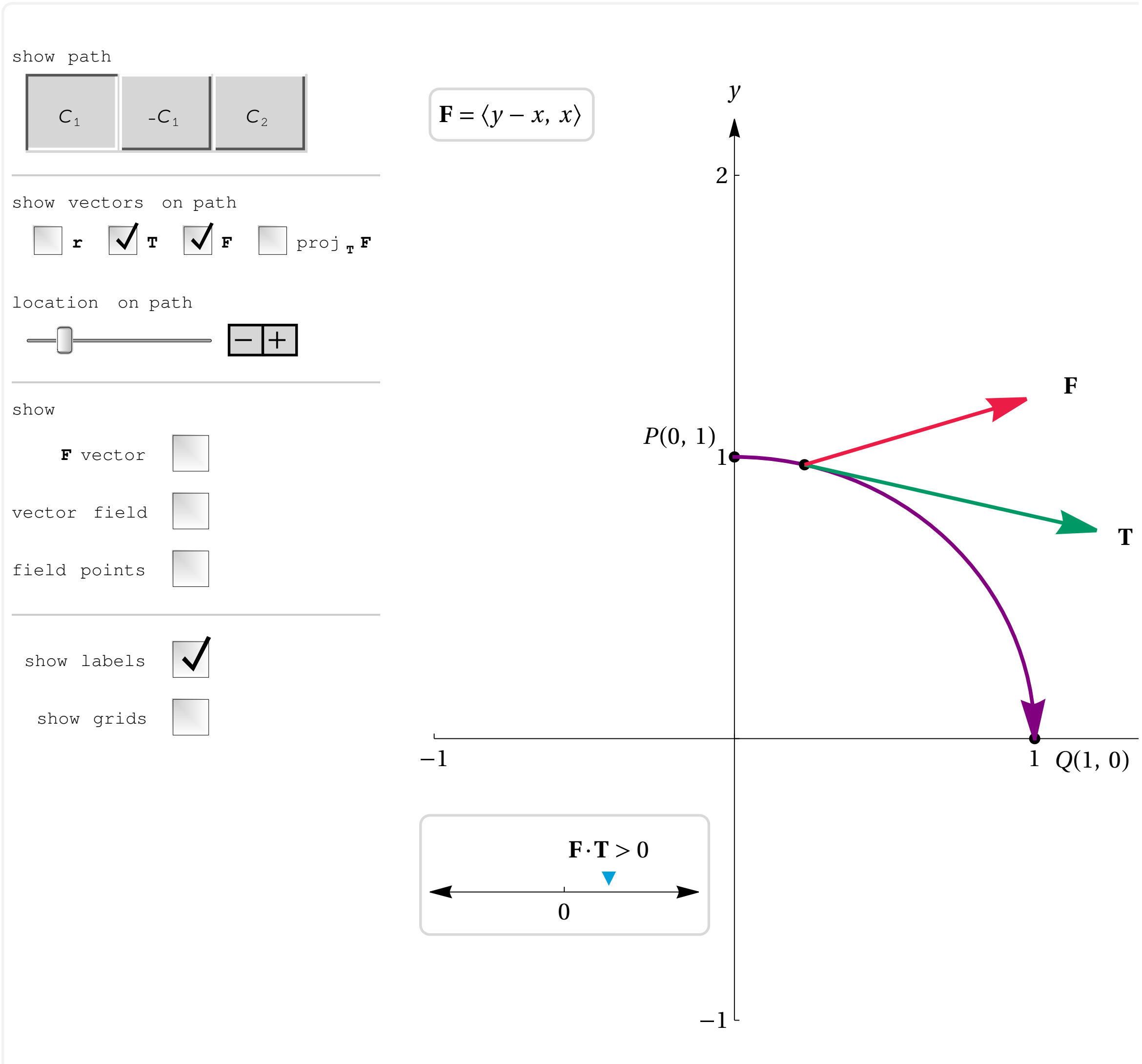


Figure 17.20

SOLUTION »

a. Working in $\mathbb{R}^2$, a parametric description of the curve C_1 with the required (clockwise) orientation is $\mathbf{r}(t) = \langle \sin t, \cos t \rangle$, for $0 \le t \le \dfrac{\pi}{2}$. Along C_1 the vector field is

$$\mathbf{F} = \langle y - x, x \rangle = \langle \cos t - \sin t, \sin t \rangle.$$

The velocity vector is $\mathbf{r}'(t) = \langle \cos t, -\sin t \rangle$, so the integrand of the line integral is

$$\mathbf{F} \cdot \mathbf{r}'(t) = \langle \cos t - \sin t, \sin t \rangle \cdot \langle \cos t, -\sin t \rangle = \underbrace{\cos^2 t - \sin^2 t}_{\cos 2t} - \underbrace{\sin t \cos t}_{\frac{1}{2} \sin 2t}.$$

The value of the line integral of $\mathbf{F}$ over C_1 is

$$\begin{aligned}\int_0^{\pi/2} \mathbf{F}\cdot\mathbf{r}'(t)\,dt &= \int_0^{\pi/2}\left(\cos 2t - \frac{1}{2}\sin 2t\right)dt && \text{Substitute for } \mathbf{F}\cdot\mathbf{r}'(t).\\ &= \left.\left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right)\right|_0^{\pi/2} && \text{Evaluate integral.}\\ &= -\frac{1}{2}. && \text{Simplify.}\end{aligned}$$

b. A parameterization of the curve $-C_1$ from Q to P is $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le \frac{\pi}{2}$. The vector field along the curve is

$$\mathbf{F} = \langle y - x, x\rangle = \langle \sin t - \cos t, \cos t\rangle,$$

and the velocity vector is $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$. A calculation very similar to that in part (a) results in

$$\int_{-C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2} \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \frac{1}{2}.$$

Comparing the results of parts (a) and (b), we see that reversing the orientation of C_1 reverses the sign of the line integral of a vector field.

c. The path C_2 consists of two line segments.

- The segment from P to O is parameterized by $\mathbf{r}(t) = \langle 0, 1 - t\rangle$, for $0 \le t \le 1$. Therefore, $\mathbf{r}'(t) = \langle 0, -1\rangle$ and $\mathbf{F} = \langle y - x, x\rangle = \langle 1 - t, 0\rangle$. On this segment, $\mathbf{T} = \langle 0, -1\rangle$.

The line integral is split into two parts and evaluated as follows:

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{PO} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{OQ} \mathbf{F}\cdot\mathbf{T}\,ds\\ &= \int_0^1 \langle 1 - t, 0\rangle\cdot\langle 0, -1\rangle\,dt + \int_0^1 \langle -t, t\rangle\cdot\langle 1, 0\rangle\,dt && \text{Substitute for } x, y, \mathbf{r}'.\\ &= \int_0^1 0\,dt + \int_0^1 (-t)\,dt && \text{Simplify.}\\ &= -\frac{1}{2}. && \text{Evaluate integrals.}\end{aligned}$$

The line integrals in parts (a) and (c) have the same value and run from P to Q, but along different paths. We might ask: For what vector fields are the values of a line integral independent of path? We return to this question in Section 17.3.

Note »

Line integrals of vector fields satisfy properties similar to those of ordinary integrals. If C is a smooth curve from A to B, C_1 is the curve from A to P, and C_2 is the curve from P to B, where P is a point on C between A and B, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Related Exercises 42–43 ◆

The solutions to parts (a) and (b) of Example 4 illustrate a general result that applies to line integrals of vector fields:

$$\int_{-C} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_{C} \mathbf{F}\cdot\mathbf{T}\,ds.$$

Figure 17.21 provides the justification of this fact: Reversing the orientation of C changes the sign of $\mathbf{F}\cdot\mathbf{T}$ at each point of C, which changes the sign of the line integral.

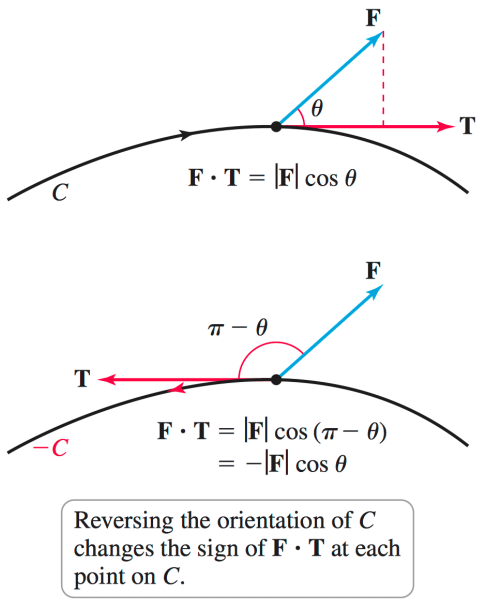


Figure 17.21

Work Integrals

A common application of line integrals of vector fields is computing the work done in moving an object in a force field (for example, a gravitational or electrical field). First recall (Section 6.7) that if $\mathbf{F}$ is a *constant* force field, the work done in moving an object a distance d along the x-axis is $W = F_x\,d$, where $F_x = |\mathbf{F}|\cos\theta$ is the component of the force along the x-axis (**Figure 17.22a**). Only the component of $\mathbf{F}$ in the direction of motion contributes to the work. More generally, if $\mathbf{F}$ is a *variable* force field, the work done in moving an object from $x = a$ to $x = b$ is $W = \int_a^b F_x(x)\,dx$, where again F_x is the component of the force in the direction of motion (parallel to the x-axis, **Figure 17.22b**).

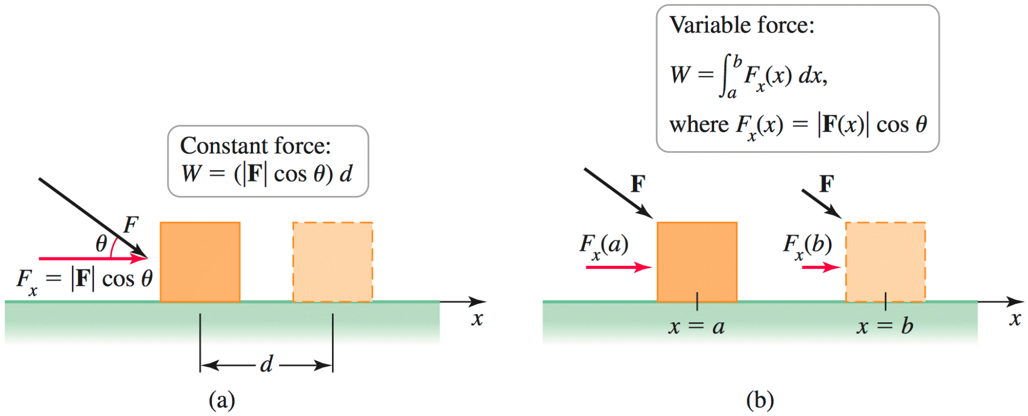


Figure 17.22

Quick Check 4 Suppose a two-dimensional force field is everywhere directed outward from the origin and C is a circle centered at the origin. What is the angle between the field and the unit vectors tangent to C? ◆

Answer »

$\frac{\pi}{2}$

We now take this progression one step further. Let **F** be a variable force field defined in a region D of $\mathbb{R}^3$, and suppose C is a smooth oriented curve in D along which an object moves. The direction of motion at each point of C is given by the unit tangent vector **T**. Therefore, the component of **F** in the direction of motion is $\mathbf{F} \cdot \mathbf{T}$, which is the tangential component of **F** along C. Summing the contributions to the work at each point of C, the work done in moving an object along C in the presence of the force is the line integral of $\mathbf{F} \cdot \mathbf{T}$ (**Figure 17.23**).

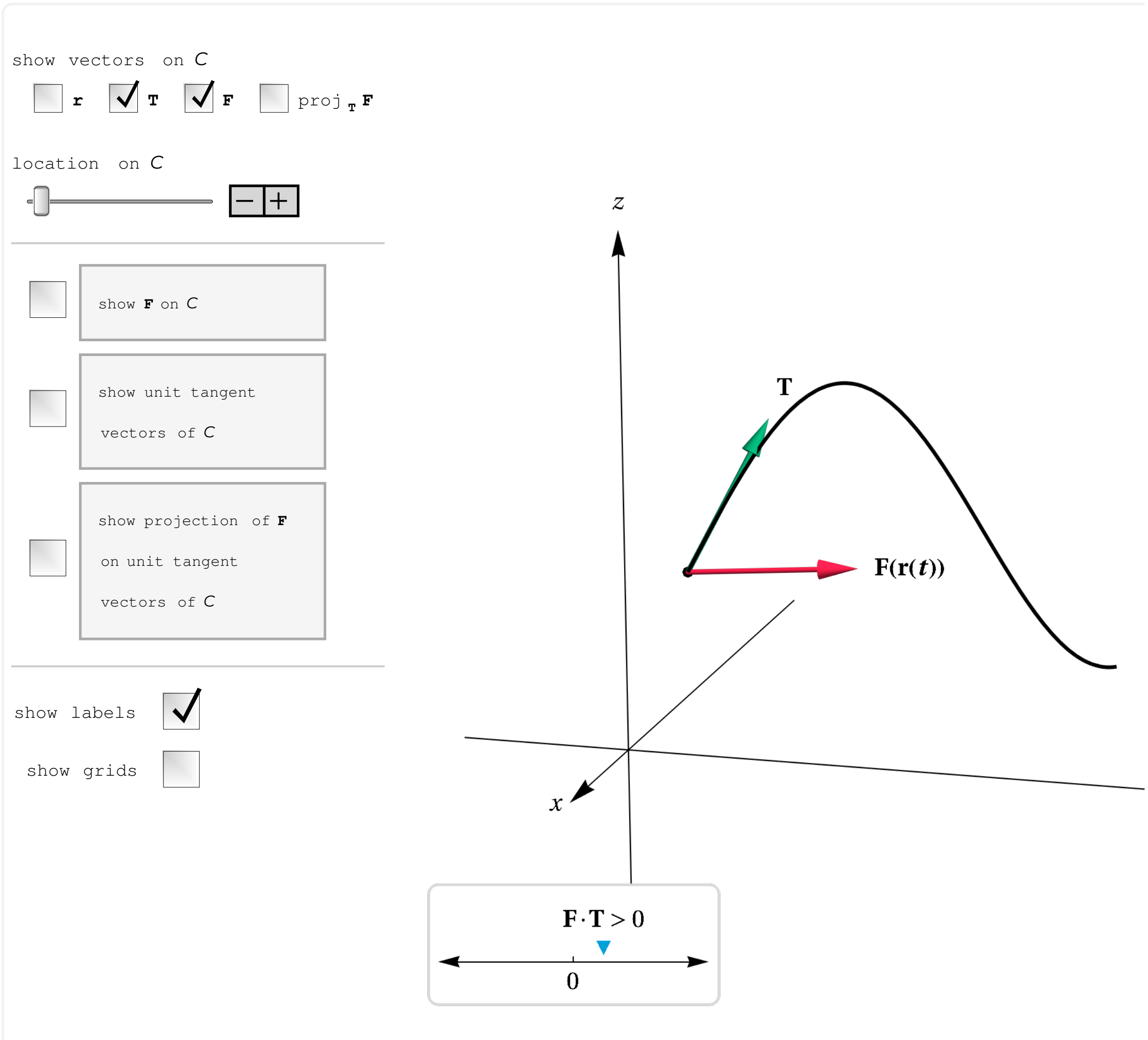


Figure 17.23

> **DEFINITION** **Work Done in a Force Field**
>
> Let **F** be a continuous force field in a region D of $\mathbb{R}^3$. Let
>
> $$C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \text{ for } a \leq t \leq b,$$
>
> be a smooth curve in D with a unit tangent vector **T** consistent with the orientation. The work done in moving an object along C in the positive direction is
>
> $$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Note »

> Just to be clear, a work integral is nothing more than a line integral of the tangential component of a force field.

EXAMPLE 5 An inverse square force

Gravitational and electrical forces between point masses and point charges obey inverse square laws: They act along the line joining the centers and they vary as $\frac{1}{r^2}$, where r is the distance between the centers. The force of attraction (or repulsion) of an inverse square force field is given by the vector field $\mathbf{F} = \dfrac{k\,\langle x, y, z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$, where k is a physical constant. Because $\mathbf{r} = \langle x, y, z\rangle$, this force may also be written $\mathbf{F} = \dfrac{k\,\mathbf{r}}{|\mathbf{r}|^3}$. Find the work done in moving an object along the following paths.

a. C_1 is the line segment from $(1, 1, 1)$ to (a, a, a), where $a > 1$.

b. C_2 is the extension of C_1 produced by letting $a \to \infty$.

SOLUTION »

a. A parametric description of C_1 consistent with the orientation is $\mathbf{r}(t) = \langle t, t, t\rangle$, for $1 \le t \le a$, with $\mathbf{r}'(t) = \langle 1, 1, 1\rangle$. In terms of the parameter t, the force field is

$$\mathbf{F} = \frac{k\,\langle x, y, z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{k\,\langle t, t, t\rangle}{\left(3\,t^2\right)^{3/2}}.$$

The dot product that appears in the work integral is

$$\mathbf{F}\cdot\mathbf{r}'(t) = \frac{k\,\langle t, t, t\rangle}{\left(3\,t^2\right)^{3/2}}\cdot\langle 1, 1, 1\rangle = \frac{3\,k\,t}{3\,\sqrt{3}\,t^3} = \frac{k}{\sqrt{3}\,t^2}.$$

Therefore, the work done is

$$W = \int_1^a \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \frac{k}{\sqrt{3}}\int_1^a t^{-2}\,dt = \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right).$$

b. The path C_2 is obtained by letting $a \to \infty$ in part (a). The required work is

$$W = \lim_{a\to\infty} \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right) = \frac{k}{\sqrt{3}}.$$

If $\mathbf{F}$ is a gravitational field, this result implies that the work required to escape Earth's gravitational field is finite (which makes space flight possible).

Related Exercise 55 ◆

Circulation and Flux of a Vector Field »

Line integrals are useful for investigating two important properties of vector fields: *circulation* and *flux*. These properties apply to any vector field, but they are particularly relevant and easy to visualize if you think of $\mathbf{F}$ as the velocity field for a moving fluid.

Circulation

We assume $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region D of $\mathbb{R}^3$, and we take C to be a *closed* smooth oriented curve in D. The *circulation* of $\mathbf{F}$ along C is a measure of how much of the vector field points in the direction of C. More simply, as you travel along C in the forward direction, how often is the vector field at your back and how often is it in your face? To determine the circulation, we simply "add up" the components of $\mathbf{F}$ in the direction of the unit tangent vector $\mathbf{T}$ at each point. Therefore, circulation integrals are another example of line integrals of vector fields.

Note »

In the definition of circulation, a *closed curve* is a curve whose initial and terminal points are the same, as defined formally in Section 17.3.

DEFINITION Circulation

Let **F** be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of **F** on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where **T** is the unit vector tangent to C consistent with the orientation.

Note »

Though we define circulation integrals for smooth curves, these integrals may be computed on piecewise-smooth curves. We adopt the convention that *piecewise* refers to a curve with finitely many pieces.

EXAMPLE 6 Circulation of two-dimensional flows

Let C be the unit circle with counterclockwise orientation. Find the circulation on C for the following vector fields.

a. The radial flow field $\mathbf{F} = \langle x, y \rangle$

b. The rotation flow field $\mathbf{F} = \langle -y, x \rangle$

SOLUTION »

a. The unit circle with the specified orientation is described parametrically by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$. Therefore, $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and the circulation of the radial field $\mathbf{F} = \langle x, y \rangle$ is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt && \text{Evaluation of a line integral} \\ &= \int_0^{2\pi} \underbrace{\langle \cos t, \sin t \rangle}_{\mathbf{F} = \langle x, y \rangle} \cdot \underbrace{\langle -\sin t, \cos t \rangle}_{\mathbf{r}'(t)}\, dt && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'. \\ &= \int_0^{2\pi} 0\, dt = 0. && \text{Simplify.} \end{aligned}$$

The tangential component of the radial vector field is zero everywhere on C, so the circulation is zero (**Figure 17.24**).

b. The circulation for the rotation field $\mathbf{F} = \langle -y, x \rangle$ is

$$
\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt && \text{Evaluation of a line integral}\\
&= \int_0^{2\pi} \underbrace{\langle -\sin t, \cos t\rangle}_{\mathbf{F}=\langle -y,x\rangle}\cdot\underbrace{\langle -\sin t, \cos t\rangle}_{\mathbf{r}'(t)}\,dt && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.\\
&= \int_0^{2\pi} (\underbrace{\sin^2 t + \cos^2 t}_{1})\,dt && \text{Simplify.}\\
&= 2\pi.
\end{aligned}
$$

In this case, at every point of C, the vector field is in the direction of the tangent vector; the result is a positive circulation (Figure 17.24).

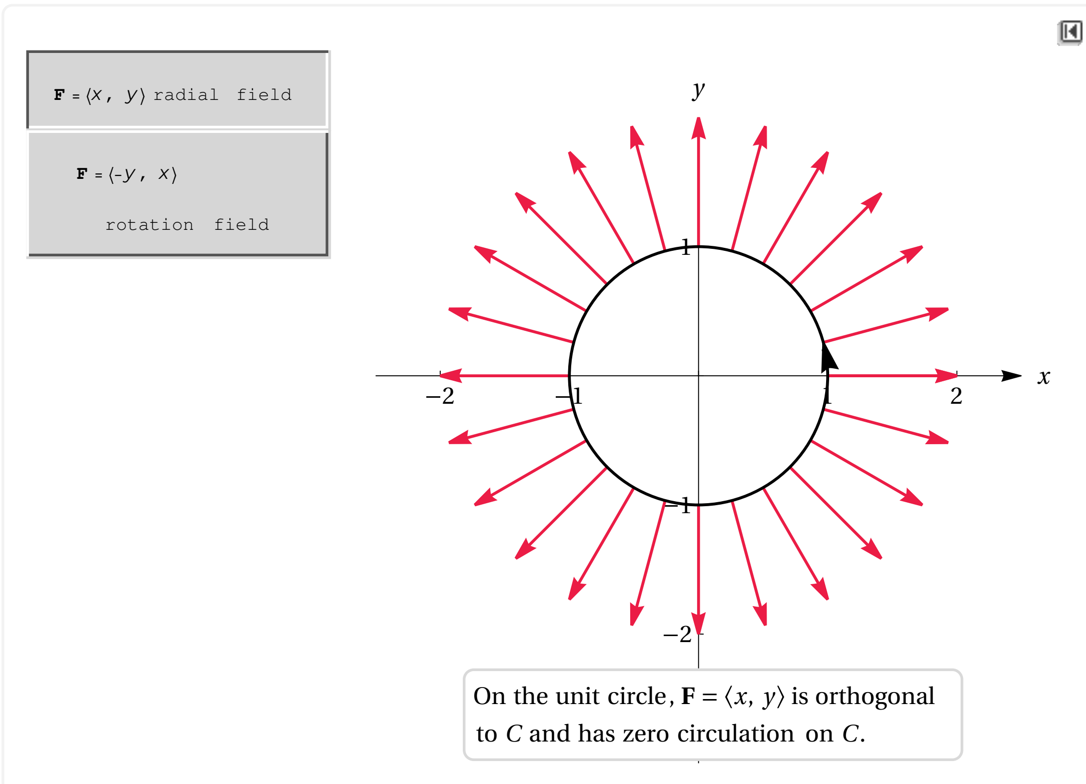


Figure 17.24

Related Exercise 57 ◆

EXAMPLE 7 Circulation of a three-dimensional flow

Find the circulation of the vector field $\mathbf{F} = \langle z, x, -y\rangle$ on the tilted ellipse $C : \mathbf{r}(t) = \langle \cos t, \sin t, \cos t\rangle$, for $0 \le t \le 2\pi$ (**Figure 17.25**).

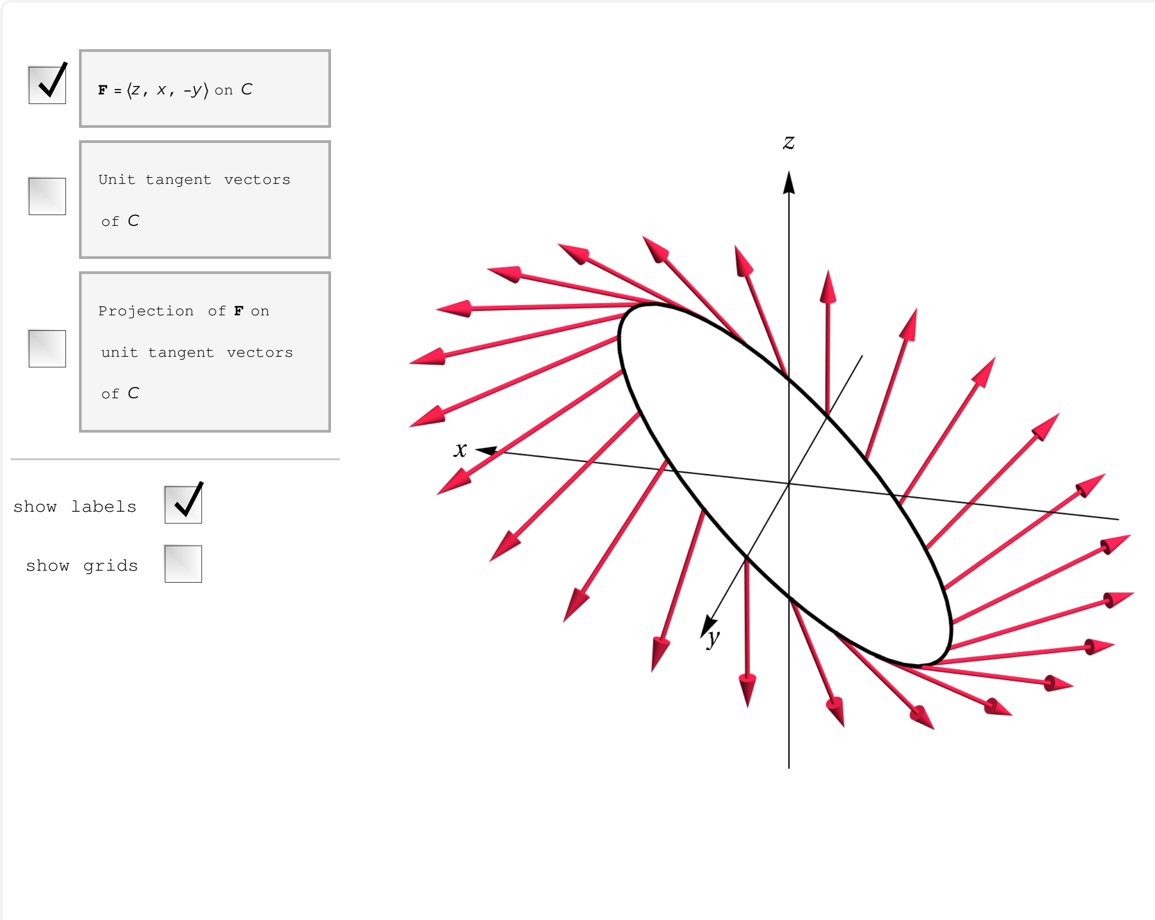


Figure 17.25

SOLUTION »

We first determine that

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle -\sin t, \cos t, -\sin t \rangle.$$

Substituting $x = \cos t$, $y = \sin t$, and $z = \cos t$ into $\mathbf{F} = \langle z, x, -y \rangle$, the circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt \qquad \text{Evaluation of a line integral}$$

$$= \int_0^{2\pi} \langle \cos t, \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle\, dt \qquad \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.$$

$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt \qquad \text{Simplify; } \sin^2 t + \cos^2 t = 1.$$

$$= 2\pi. \qquad \text{Evaluate integral.}$$

Figure 17.25 shows the projection of the vector field on the unit tangent vectors at various points on C. The circulation is the “sum” of the scalar components associated with these projections, which, in this case, is positive.

Related Exercise 53 ◆

Flux of Two-Dimensional Vector Fields

Assume $\mathbf{F} = \langle f, g \rangle$ is a continuous vector field on a region R of $\mathbb{R}^2$. We let C be a smooth oriented curve in R that does not intersect itself; C may or may not be closed. To compute the *flux* of the vector field across C, we "add up" the components of $\mathbf{F}$ *orthogonal* or *normal* to C at each point of C. Notice that every point on C has two unit vectors normal to C. Therefore, we let $\mathbf{n}$ denote the unit vector in the xy-plane normal to C in a direction to be defined momentarily. Once the direction of $\mathbf{n}$ is defined, the component of $\mathbf{F}$ normal to C is $\mathbf{F} \cdot \mathbf{n}$, and the flux is the line integral of $\mathbf{F} \cdot \mathbf{n}$ along C, which we denote $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

Note »

> In the definition of flux, the non-self-intersecting property of C means that C is a *simple* curve, as defined formally in Section 17.3.

The first step is to define the unit normal vector at a point P of C. Because C lies in the xy-plane, the unit vector $\mathbf{T}$ tangent to C at P also lies in the xy-plane. Therefore, its z-component is 0, and we let $\mathbf{T} = \langle T_x, T_y, 0 \rangle$. As always, $\mathbf{k} = \langle 0, 0, 1 \rangle$ is the unit vector in the z-direction. Because a unit vector $\mathbf{n}$ in the xy-plane normal to C is orthogonal to both $\mathbf{T}$ and $\mathbf{k}$, we determine the direction of $\mathbf{n}$ by letting $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ (**Figure 17.26**). This choice has two implications.

Note »

> Recall that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

- If C is a closed curve oriented counterclockwise (when viewed from above), the unit normal vector points *outward* along the curve (Figure 17.26). When $\mathbf{F}$ also points outward at a point on C, the angle θ between $\mathbf{F}$ and $\mathbf{n}$ satisfies $0 \le \theta < \frac{\pi}{2}$. At all such points, $\mathbf{F} \cdot \mathbf{n} > 0$ and there is a positive contribution to the flux across C. When $\mathbf{F}$ points inward at a point on C, $\frac{\pi}{2} < \theta \le \pi$ and $\mathbf{F} \cdot \mathbf{n} < 0$, which means there is a negative contribution to the flux at that point.
- If C is not a closed curve, the unit normal vector points to the right (when viewed from above) as the curve is traversed in the forward direction.

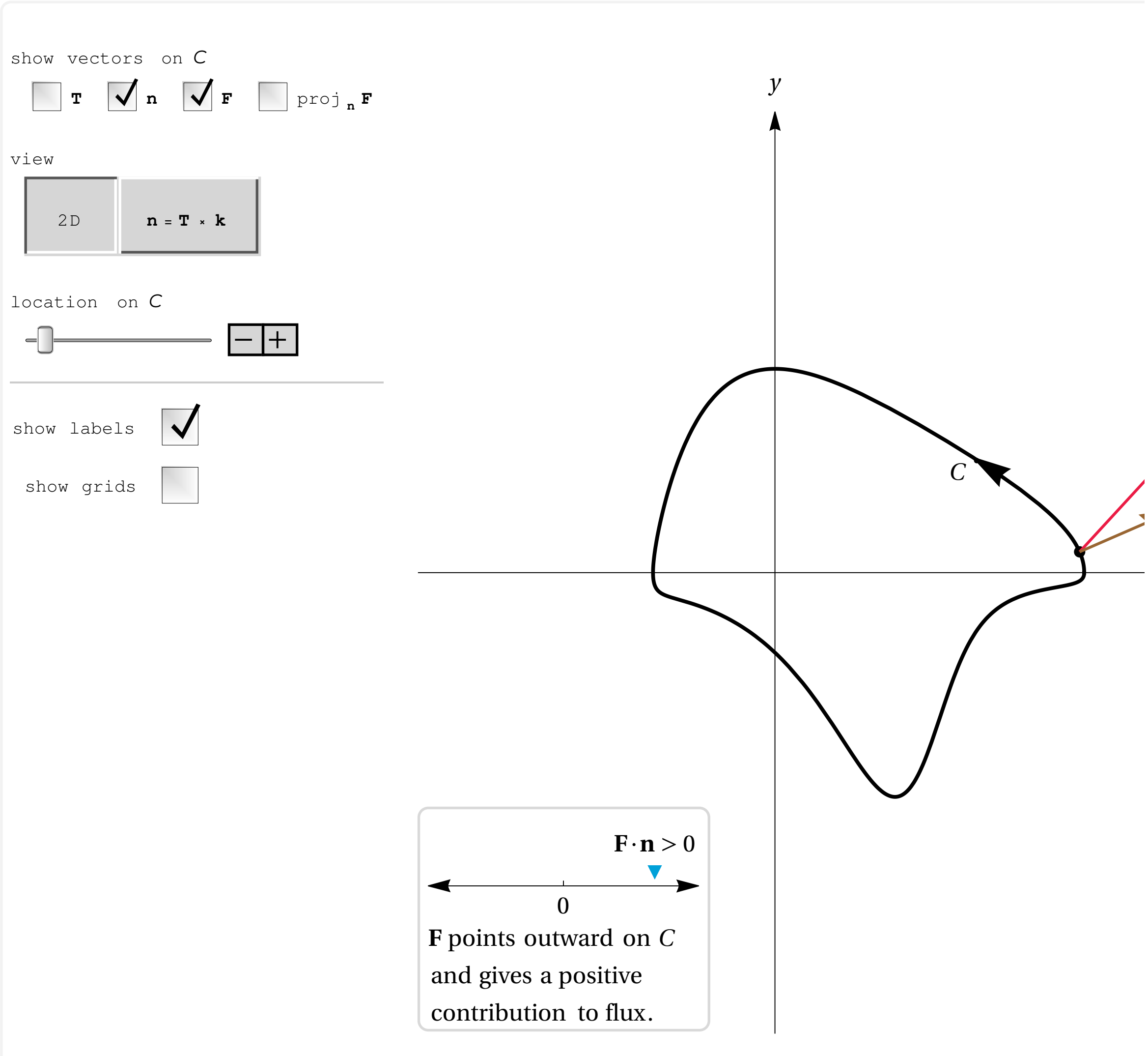


Figure 17.26

Quick Check 5 Sketch a closed curve on a sheet of paper and draw a unit tangent vector **T** on the curve pointing in the counterclockwise direction. Explain why $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is an *outward* unit normal vector. ◆

Answer »

T and **k** are unit vectors, so **n** is a unit vector. By the right-hand rule for cross-products, **n** points outward from the curve.

Calculating the cross product for the unit normal vector **n**, we find that

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\,\mathbf{i} - T_x\,\mathbf{j}.$$

Because $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, the components of **T** are

$$\mathbf{T} = \langle T_x, T_y, 0\rangle = \frac{\langle x'(t), y'(t), 0\rangle}{|\mathbf{r}'(t)|}.$$

We now have an expression for the unit normal vector:

$$\mathbf{n} = T_y\,\mathbf{i} - T_x\,\mathbf{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j} = \frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}.$$

To evaluate the flux integral $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, we make a familiar change of variables by letting $ds = |\mathbf{r}'(t)|\,dt$. The flux of $\mathbf{F} = \langle f, g\rangle$ across C is then

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \mathbf{F}\cdot\underbrace{\frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}}_{\mathbf{n}}\,\underbrace{|\mathbf{r}'(t)|\,dt}_{ds} = \int_a^b (f(t)\,y'(t) - g(t)\,x'(t))\,dt.$$

This is one useful form of the flux integral. Alternatively, we can note that $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$ and write

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx.$$

DEFINITION **Flux**

Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C: \mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (f\,y'(t) - g\,x'(t))\,dt,$$

where $\mathbf{n} = \mathbf{T}\times\mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

Note »

Like circulation integrals, flux integrals may be computed on piecewise-smooth curves by finding the flux on each piece and adding the results.

The concepts of circulation and flux can be visualized in terms of headwinds and crosswinds. Suppose the wind patterns in your neighborhood can be modeled with a vector field $\mathbf{F}$ (that doesn't change with time). Now imagine taking a walk around the block in a counterclockwise direction along a closed path. At different points along your walk, you encounter winds from various directions and with various speeds. The circulation of the wind field $\mathbf{F}$ along your path is the net amount of headwind (negative contribution) and tailwind (positive contribution) that you encounter during your walk. The flux of $\mathbf{F}$ across your path is the net amount of cross-wind (positive from your left and negative from your right) encountered on your walk.

EXAMPLE 8 **Flux of two-dimensional flows**

Find the outward flux across the unit circle with counterclockwise orientation for the following vector fields.

a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation flow field $\mathbf{F} = \langle -y, x \rangle$

SOLUTION »

a. The unit circle with counterclockwise orientation has a description $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$. Therefore, $x'(t) = -\sin t$ and $y'(t) = \cos t$. The components of **F** are $f = x(t) = \cos t$ and $g = y(t) = \sin t$. It follows that the outward flux is

$$\int_a^b (f\, y'(t) - g\, x'(t))\, dt = \int_0^{2\pi} \left(\underbrace{\cos t}_{f(t)} \underbrace{\cos t}_{y'(t)} - \underbrace{\sin t}_{g(t)} \underbrace{(-\sin t)}_{x'(t)} \right) dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi. \qquad \cos^2 t + \sin^2 t = 1$$

Because the radial vector field points outward and is aligned with the unit normal vectors on C, the outward flux is positive (**Figure 17.27**).

b. For the rotation field, $f = -y(t) = -\sin t$ and $g = x(t) = \cos t$. The outward flux is

$$\int_a^b (f\, y'(t) - g\, x'(t))\, dt = \int_0^{2\pi} \left(\underbrace{-\sin t}_{f(t)} \underbrace{\cos t}_{y'(t)} - \underbrace{\cos t}_{g(t)} \underbrace{(-\sin t)}_{x'(t)} \right) dt$$

$$= \int_0^{2\pi} 0\, dt = 0.$$

Because the rotation field is orthogonal to **n** at all points of C, the outward flux across C is zero (Figure 17.27). The results of Examples 6 and 8 are worth remembering: On a unit circle centered at the origin, the *radial* vector field $\langle x, y \rangle$ has outward flux 2π and zero circulation. The *rotation* vector field $\langle -y, x \rangle$ has zero outward flux and circulation 2π.

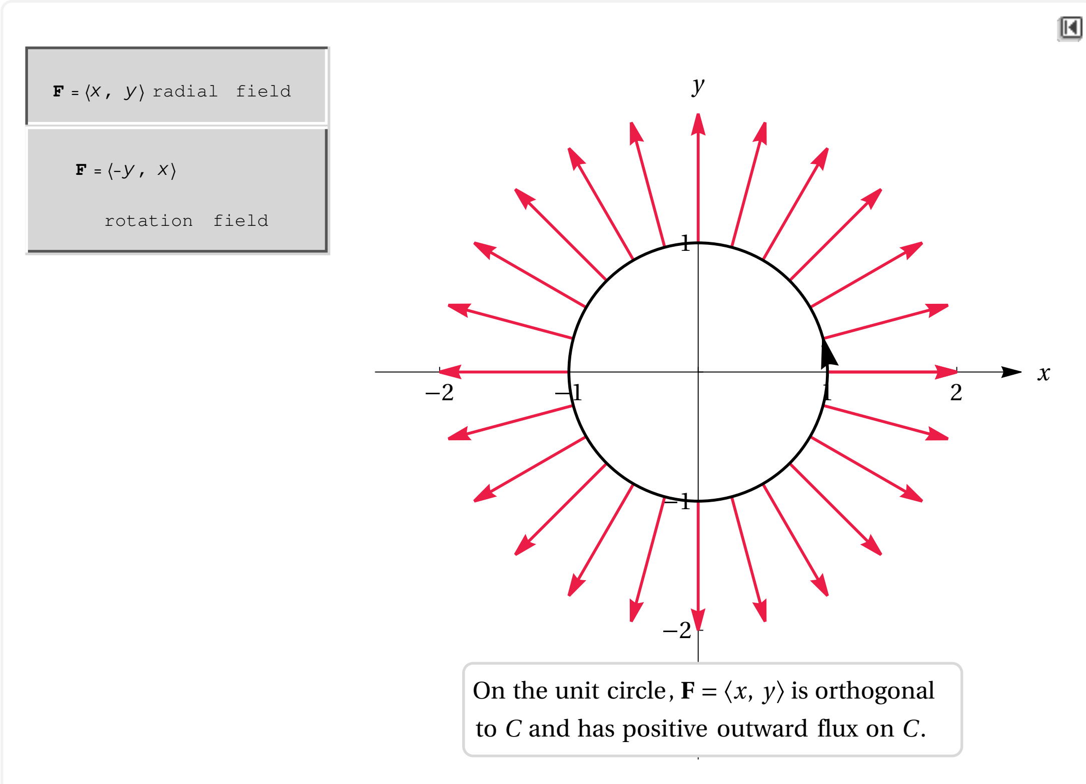


Figure 17.27

Related Exercises 59–60 ◆

Exercises »

Getting Started »

Practice Exercises »

17–34. Scalar line integrals *Evaluate the following line integrals along the curve C.*

17. $\int_C x\,y\,ds$; C is the unit circle $\mathbf{r}(t) = \langle \cos t,\ \sin t\rangle$, for $0 \le t \le 2\pi$.

18. $\int_C \left(x^2 - 2\,y^2\right) ds$; C is the line $\mathbf{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}} \right\rangle$, for $0 \le t \le 4$.

19. $\int_C (2\,x + y)\,ds$; C is the line segment $\mathbf{r}(t) = \langle 3\,t,\ 4\,t\rangle$, for $0 \le t \le 2$.

20. $\int_C x\,ds$; C is the curve $\mathbf{r}(t) = \left\langle t^3,\ 4\,t\right\rangle$, for $0 \le t \le 1$.

21. $\int_C x\,y^3\,ds$; C is the quarter-circle $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t\rangle$, for $0 \le t \le \pi/2$.

22. $\int_C 3\,x \cos y\,ds$; C is the curve $\mathbf{r}(t) = \langle \sin t,\ t\rangle$, for $0 \le t \le \pi/2$.

23. $\int_C (y - z)\, ds$; C is the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$, for $0 \le t \le 2\pi$.

24. $\int_C (x - y + 2z)\, ds$; C is the circle $\mathbf{r}(t) = \langle 1, 3\cos t, 3\sin t \rangle$, for $0 \le t \le 2\pi$.

25. $\int_C \left(x^2 + y^2\right) ds$; C is the circle of radius 4 centered at $(0, 0)$.

26. $\int_C \left(x^2 + y^2\right) ds$; C is the line segment from $(0, 0)$ to $(5, 5)$.

27. $\int_C \frac{x}{x^2 + y^2}\, ds$; C is the line segment from $(1, 1)$ to $(10, 10)$.

28. $\int_C (xy)^{1/3}\, ds$; C is the curve $y = x^2$, for $0 \le x \le 1$.

29. $\int_C xy\, ds$; C is a portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$ in the first quadrant, oriented counterclockwise.

30. $\int_C (2x - 3y)\, ds$; C is the line segment from $(-1, 0)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(1, 0)$.

31. $\int_C (x + y + z)\, ds$; C is the semicircle $\mathbf{r}(t) = \langle 2\cos t, 0, 2\sin t \rangle$, for $0 \le t \le \pi$.

32. $\int_C \frac{xy}{z}\, ds$; C is the line segment from $(1, 4, 1)$ to $(3, 6, 3)$.

33. $\int_C xz\, ds$; C is the line segment from $(0, 0, 0)$ to $(3, 2, 6)$ followed by the line segment from $(3, 2, 6)$ to $(7, 9, 10)$.

34. $\int_C x e^{yz}\, ds$; C is $\mathbf{r}(t) = \langle t, 2t, -2t \rangle$, for $0 \le t \le 2$.

35–36. Mass and density *A thin wire represented by the smooth curve C with a density ρ (units of mass per length) has a mass $M = \int_C \rho\, ds$. Find the mass of the following wires with the given density.*

T **35.** $C: \left\{(x, y) : y = 2x^2, 0 \le x \le 3\right\}$; $\rho(x, y) = 1 + xy$

36. $C: \mathbf{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$, for $0 \le \theta \le \pi$; $\rho(\theta) = \frac{2\theta}{\pi} + 1$

37–38. Average values *Find the average value of the following functions on the given curves.*

37. $f(x, y) = x + 2y$ on the line segment from $(1, 1)$ to $(2, 5)$

38. $f(x, y) = x e^y$ on the unit circle centered at the origin

39–40. Length of curves *Use a scalar line integral to find the length of the following curves.*

39. $\mathbf{r}(t) = \left\langle 20 \sin \frac{t}{4},\ 20 \cos \frac{t}{4},\ \frac{t}{2} \right\rangle$, for $0 \le t \le 2$

40. $\mathbf{r}(t) = \langle 30 \sin t,\ 40 \sin t,\ 50 \cos t \rangle$, for $0 \le t \le 2\pi$

41–46. Line integrals of vector fields in the plane *Given the following vector fields and oriented curves C, evaluate* $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

41. $\mathbf{F} = \langle x, y \rangle$ on the parabola $\mathbf{r}(t) = \langle 4t, t^2 \rangle$, for $0 \le t \le 1$

42. $\mathbf{F} = \langle -y, x \rangle$ on the semicircle $\mathbf{r}(t) = \langle 4 \cos t, 4 \sin t \rangle$, for $0 \le t \le \pi$

43. $\mathbf{F} = \langle y, x \rangle$ on the line segment from $(1, 1)$ to $(5, 10)$

44. $\mathbf{F} = \langle -y, x \rangle$ on the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

45. $\mathbf{F} = \dfrac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$ on the curve $\mathbf{r}(t) = \langle t^2, 3t^2 \rangle$, for $1 \le t \le 2$

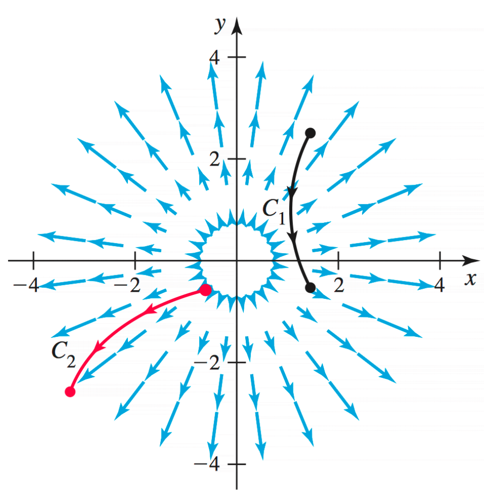
46. $\mathbf{F} = \dfrac{\langle x, y \rangle}{x^2 + y^2}$ on the line segment $\mathbf{r}(t) = \langle t, 4t \rangle$, for $1 \le t \le 10$

47–48. Line integrals from graphs *Determine whether* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *along the paths* C_1 *and* C_2 *shown in the following vector fields is positive or negative. Explain your reasoning.*

a. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$

b. $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

47.



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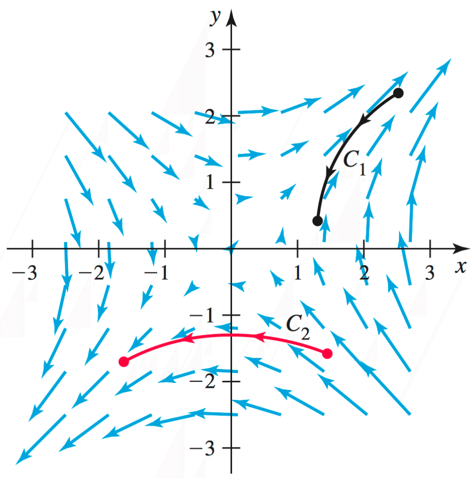


49–56. Work integrals *Given the force field* **F**, *find the work required to move an object on the given oriented curve.*

49. $\mathbf{F} = \langle y, -x \rangle$ on the line segment from (1, 2) to (0, 0) followed by the line segment from (0, 0) to (0, 4)

50. $\mathbf{F} = \langle x, y \rangle$ on the line segment from (−1, 0) to (0, 8) followed by the line segment from (0, 8) to (2, 8)

51. $\mathbf{F} = \langle y, x \rangle$ on the parabola $y = 2x^2$ from (0, 0) to (2, 8)

52. $\mathbf{F} = \langle y, -x \rangle$ on the line segment $y = 10 - 2x$ from (1, 8) to (3, 4)

53. $\mathbf{F} = \langle x, y, z \rangle$ on the tilted ellipse $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 4\cos t \rangle$, for $0 \le t \le 2\pi$

54. $\mathbf{F} = \langle -y, x, z \rangle$ on the helix $\mathbf{r}(t) = \left\langle 2\cos t, 2\sin t, \dfrac{t}{2\pi} \right\rangle$, for $0 \le t \le 2\pi$

55. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from (1, 1, 1) to (10, 10, 10)

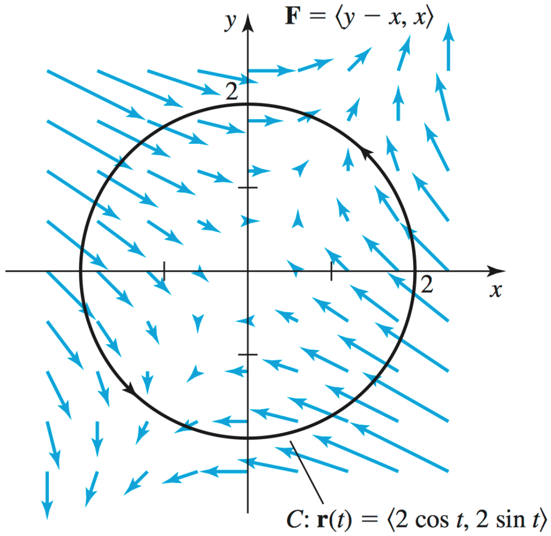
T **56.** $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$ on the line segment from (1, 1, 1) to (8, 4, 2)

57–58. Circulation *Consider the following vector fields* **F** *and closed oriented curves C in the plane (see figures).*

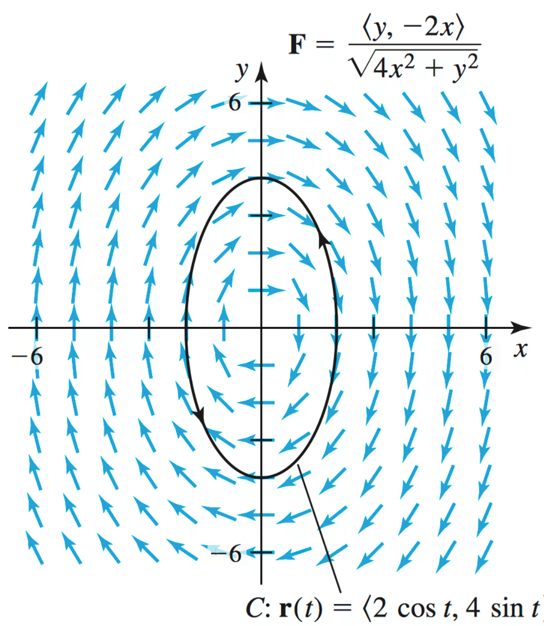
a. Based on the picture, make a conjecture about whether the circulation of **F** on C is positive, negative, or zero.

b. Compute the circulation and interpret the result.

57. $\mathbf{F} = \langle y - x, x \rangle$; $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le 2\pi$



58. $\mathbf{F} = \dfrac{\langle y, -2x \rangle}{\sqrt{4x^2 + y^2}}$; $C : \mathbf{r}(t) = \langle 2\cos t, 4\sin t \rangle$, for $0 \le t \le 2\pi$



59–60. Flux *Consider the vector fields and curves in Exercises 57–58.*

- **a.** *Based on the picture, make a conjecture about whether the outward flux of* **F** *across C is positive, negative, or zero.*
- **b.** *Compute the flux for the vector fields and curves.*

59. **F** and C given in Exercise 57

60. **F** and C given in Exercise 58

61. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If a curve has a parametric description $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, where t is the arc length, then $|\mathbf{r}'(t)| = 1$.

b. The vector field $\mathbf{F} = \langle y, x\rangle$ has both zero circulation along and zero flux across the unit circle centered at the origin.

c. If at all points of a path a force acts in a direction orthogonal to the path, then no work is done in moving an object along the path.

d. The flux of a vector field across a curve in $\mathbb{R}^2$ can be computed using a line integral.

62. Flying into a headwind An airplane flies in the xz-plane, where x increases in the eastward direction and $z \geq 0$ represents vertical distance above the ground. A wind blows horizontally out of the west, producing a force $\mathbf{F} = \langle 150, 0\rangle$. On which path between the points $(100, 50)$ and $(-100, 50)$ is the most work done overcoming the wind?

a. The straight line $\mathbf{r}(t) = \langle x(t), z(t)\rangle = \langle -t, 50\rangle$, for $-100 \leq t \leq 100$

b. The arc of a circle $\mathbf{r}(t) = \langle 100 \cos t, 50 + 100 \sin t\rangle$, for $0 \leq t \leq \pi$

63. Flying into a headwind

a. How does the result of Exercise 62 change if the force due to the wind is $\mathbf{F} = \langle 141, 50\rangle$ (approximately the same magnitude, but different direction)?

b. How does the result of Exercise 62 change if the force due to the wind is $\mathbf{F} = \langle 141, -50\rangle$ (approximately the same magnitude, but different direction)?

64. Changing orientation Let $f(x, y) = x + 2y$ and let C be the unit circle.

a. Find a parameterization of C with a counterclockwise orientation and evaluate $\int_C f\, ds$.

b. Find a parameterization of C with a clockwise orientation and evaluate $\int_C f\, ds$.

c. Compare the results of (a) and (b).

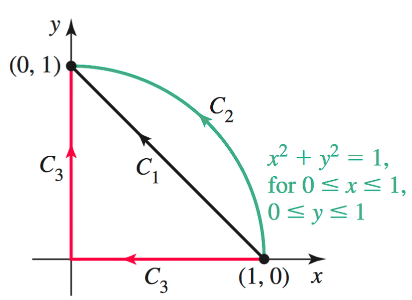
65. Changing orientation Let $f(x, y) = x$ and let C be the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

a. Find a parameterization of C in the direction from O to P. Evaluate $\int_C f\, ds$.

b. Find a parameterization of C in the direction from P to O. Evaluate $\int_C f\, ds$.

c. Compare the results of (a) and (b).

66. Work in a rotation field Consider the rotation field $\mathbf{F} = \langle -y, x\rangle$ and the three paths shown in the figure. Compute the work done on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\, ds$ is independent of the path, where C is any path from $(1, 0)$ to $(0, 1)$?



67. Work in a hyperbolic field Consider the hyperbolic force field $\mathbf{F} = \langle y, x \rangle$ (the streamlines are hyperbolas) and the three paths shown in the figure for Exercise 66. Compute the work done in the presence of **F** on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is independent of the path, where C is any path from (1, 0) to (0, 1)?

68–72. Assorted line integrals *Evaluate each line integral using the given curve C.*

68. $\int_C x^2\, dx + dy + y\, dz$; C is the curve $\mathbf{r}(t) = \left\langle t, 2t, t^2 \right\rangle$, for $0 \le t \le 3$.

T 69. $\int_C x^3 y\, dx + xz\, dy + (x+y)^2\, dz$; C is the helix $\mathbf{r}(t) = \langle 2t, \sin t, \cos t \rangle$, for $0 \le t \le 4\pi$.

T 70. $\int_C \frac{x^2}{y^4}\, ds$; C is the segment of the parabola $x = 3y^2$ from (3, 1) to (27, 3).

71. $\int_C \frac{y}{\sqrt{x^2+y^2}}\, dx - \frac{x}{\sqrt{x^2+y^2}}\, dy$; C is a quarter-circle from (0, 4) to (4, 0).

72. $\int_C (x+y)\, dx + (x-y)\, dy + x\, dz$; C is the line segment from (1, 2, 4) to (3, 8, 13).

73. Flux across curves in a vector field Consider the vector field $\mathbf{F} = \langle y, x \rangle$ shown in the figure.

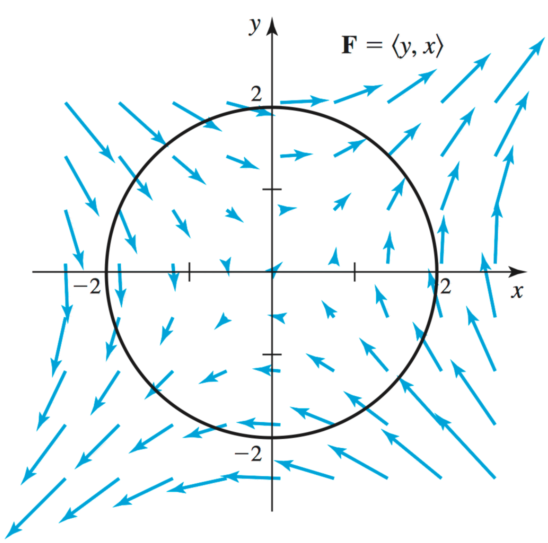
a. Compute the outward flux across the quarter circle $C: \mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le \frac{\pi}{2}$.

b. Compute the outward flux across the quarter circle $C: \mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $\frac{\pi}{2} \le t \le \pi$.

c. Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part (a).

d. Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part (b).

e. What is the outward flux across the full circle?



Explorations and Challenges »

74–75. Zero circulation fields

74. For what values of b and c does the vector field $\mathbf{F} = \langle b\,y,\ c\,x\rangle$ have zero circulation on the unit circle centered at the origin and oriented counterclockwise?

75. Consider the vector field $\mathbf{F} = \langle a\,x + b\,y,\ c\,x + d\,y\rangle$. Show that $\mathbf{F}$ has zero circulation on any oriented circle centered at the origin, for any a, b, c, and d, provided $b = c$.

76–77. Zero flux fields

76. For what values of a and d does the vector field $\mathbf{F} = \langle a\,x,\ d\,y\rangle$ have zero flux across the unit circle centered at the origin and oriented counterclockwise?

77. Consider the vector field $\mathbf{F} = \langle a\,x + b\,y,\ c\,x + d\,y\rangle$. Show that $\mathbf{F}$ has zero flux across any oriented circle centered at the origin, for any a, b, c, and d, provided $a = -d$.

78. **Heat flux in a plate** A square plate $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$ has a temperature distribution $T(x, y) = 100 - 50\,x - 25\,y$.

- **a.** Sketch two level curves of the temperature in the plate.
- **b.** Find the gradient of the temperature $\nabla T(x, y)$.
- **c.** Assume the flow of heat is given by the vector field $\mathbf{F} = -\nabla T(x, y)$. Compute $\mathbf{F}$.
- **d.** Find the outward heat flux across the boundary $\{(x, y) : x = 1,\ 0 \le y \le 1\}$.
- **e.** Find the outward heat flux across the boundary $\{(x, y) : 0 \le x \le 1,\ y = 1\}$.

79. **Inverse force fields** Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x, y, z\rangle}{|\mathbf{r}|^p}$, where $p > 1$ (the inverse square law corresponds to $p = 3$). Let C be the line from $(1, 1, 1)$ to (a, a, a), where $a > 1$, given by $\mathbf{r}(t) = \langle t, t, t\rangle$, for $1 \le t \le a$.

- **a.** Find the work done in moving an object along C with $p = 2$.
- **b.** If $a \to \infty$ in part (a), is the work finite?
- **c.** Find the work done in moving an object along C with $p = 4$.
- **d.** If $a \to \infty$ in part (c), is the work finite?
- **e.** Find the work done in moving an object along C for any $p > 1$.
- **f.** If $a \to \infty$ in part (e), for what values of p is the work finite?

80. Line integrals with respect to dx and dy Given a vector field $\mathbf{F} = \langle f, 0\rangle$ and curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$, we see that the line integral $\int_C f\,dx + g\,dy$ simplifies to $\int_C f\,dx$.

- **a.** Show that $\int_C f\,dx = \int_a^b f(t)\,x'(t)\,dt$.
- **b.** Use the vector field $\mathbf{F} = \langle 0, g\rangle$ to show that $\int_C g\,dy = \int_a^b g(t)\,y'(t)\,dt$.
- **c.** Evaluate $\int_C x\,y\,dx$, where C is the line segment from $(0, 0)$ to $(5, 12)$.

d. Evaluate $\int_C x\,y\,dy$, where C is a segment of the parabola $x = y^2$ from $(1,-1)$ to $(1,1)$.

81–82. Looking ahead: Area from line integrals *The area of a region R in the plane, whose boundary is the curve C, may be computed using line integrals with the formula*

$$\text{area of } R = \int_C x\,dy = -\int_C y\,dx.$$

81. Let R be the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), and let C be the boundary of R oriented counterclockwise. Use the formula $A = \int_C x\,dy$ to verify that the area of the rectangle is ab.

82. Let $R = \{(r, \theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ be the disk of radius a centered at the origin and let C be the boundary of R oriented counterclockwise. Use the formula $A = -\int_C y\,dx$ to verify that the area of the disk is πa^2.