17 Vector Calculus

Chapter Preview This culminating chapter of the book provides a beautiful, unifying conclusion to our study of calculus. Many ideas and themes that have appeared throughout the book come together in these final pages. First, we combine vector-valued functions (Chapter 14) and functions of several variables (Chapter 15) to form *vector fields*. Once vector fields have been introduced and illustrated through their many applications, we explore the calculus of vector fields. Concepts such as limits and continuity carry over directly. The extension of derivatives to vector fields leads to two new operations that underlie this chapter; the *curl* and the *divergence*. When integration is extended to vector fields, we discover new versions of the Fundamental Theorem of Calculus. The chapter ends with a final look at the Fundamental Theorem of Calculus and the several related forms in which it has appeared throughout the book.

17.1 Vector Fields

We live in a world filled with phenomena that can be represented by vector fields. Imagine sitting in a window seat looking out at the wing of an airliner. Although you can't see it, air is rushing over and under the wing. Focus on a point near the wing and visualize the motion of the air at that point at a single instant of time. The motion is described by a velocity vector with three components—for example, east-west, north-south, and up-down. At another point near the wing at the same time, the air is moving at a different speed and direction, and a different velocity vector is associated with that point. In general, at one instant in time, every point around the wing has a velocity vector associated with it (**Figure 17.1**). This collection of velocity vectors—a unique vector for each point in space—is a function called a *vector field*.

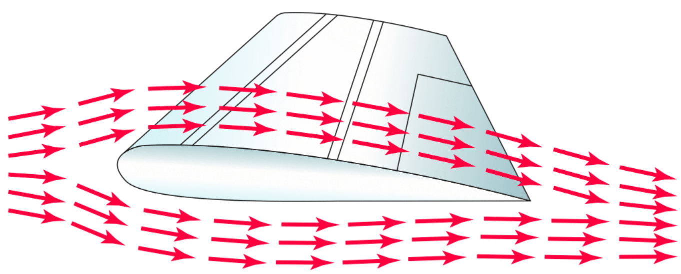
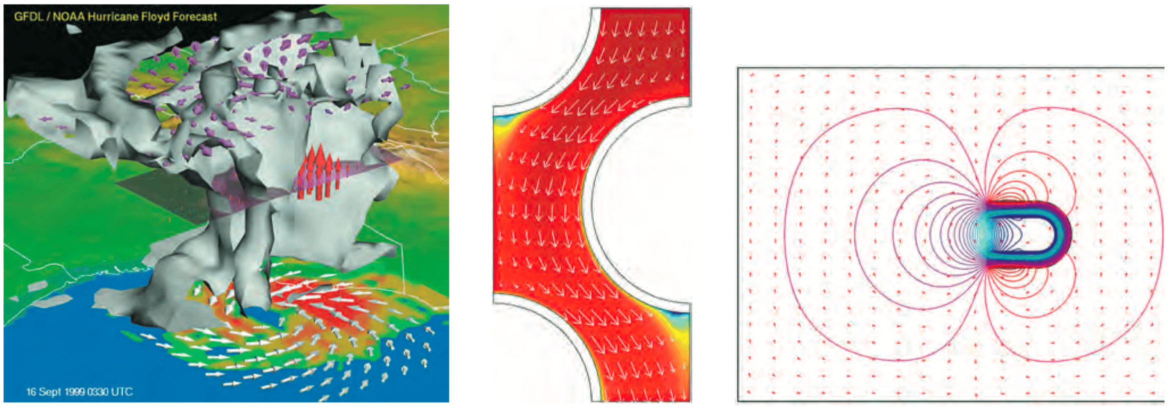


Figure 17.1

Other examples of vector fields include the wind patterns in a hurricane (**Figure 17.2a**) and the circulation of water in a heat exchanger (**Figure 17.2b**). Gravitational, magnetic, and electric force fields are represented by vector fields (**Figure 17.2c**), as are the stresses and strains in buildings and bridges. Beyond physics and engineering, the transport of a chemical pollutant in a lake or human migration patterns can be modeled by vector fields.



(a) (b) (c)

Figure 17.2

Vector Fields in Two Dimensions »

To solidify the idea of a vector field, we begin by exploring vector fields in $\mathbb{R}^2$. From there, it is a short step to vector fields in $\mathbb{R}^3$.

> **DEFINITION** **Vector Fields in Two Dimensions**
>
> Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function **F** that assigns to each point in R a vector $\langle f(x, y), g(x, y)\rangle$. The vector field is written as
>
> $$\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle \text{ or}$$
> $$\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}.$$
>
> A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Note »

A vector field cannot be represented by a single curve or surface. Instead, we plot a representative sample of vectors that illustrate the general appearance of the vector field. Consider the vector field defined by

$$\mathbf{F}(x, y) = \langle 2x, 2y\rangle = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

At selected points $P(x, y)$, we plot a vector with its tail at P equal to the value of $\mathbf{F}(x, y)$. For example, $\mathbf{F}(1, 1) = \langle 2, 2\rangle$, so we draw a vector equal to $\langle 2, 2\rangle$ with its tail at the point $(1, 1)$. Similarly, $\mathbf{F}(-2, -3) = \langle -4, -6\rangle$, so at the point $(-2, -3)$, we draw a vector equal to $\langle -4, -6\rangle$. We can make the following general observations about the vector field $\mathbf{F}(x, y) = \langle 2x, 2y\rangle$.

- For every (x, y) except $(0, 0)$, the vector $\mathbf{F}(x, y)$ points in the direction of $\langle 2x, 2y\rangle$, which is directly outward from the origin.
- The length of $\mathbf{F}(x, y)$ is $|\mathbf{F}| = |\langle 2x, 2y\rangle| = 2\sqrt{x^2 + y^2}$, which increases with distance from the origin.

The vector field $\mathbf{F} = \langle 2x, 2y\rangle$ is an example of a *radial vector field* because its vectors point radially away from the origin (**Figure 17.3**). If **F** represents the velocity of a fluid moving in two dimensions, the graph of the vector field gives a vivid image of how a small object, such as a cork, moves through the fluid. In this case, at every point of the field, a particle moves in the direction of the arrow at that point with a speed equal to the length of the arrow. For this reason, vector fields are sometimes called *flows*. When sketching vector fields, it is often useful to draw continuous curves that are aligned with the vector field. Such curves are called *flow curves* or *streamlines*; we examine their properties in greater detail later in this section.

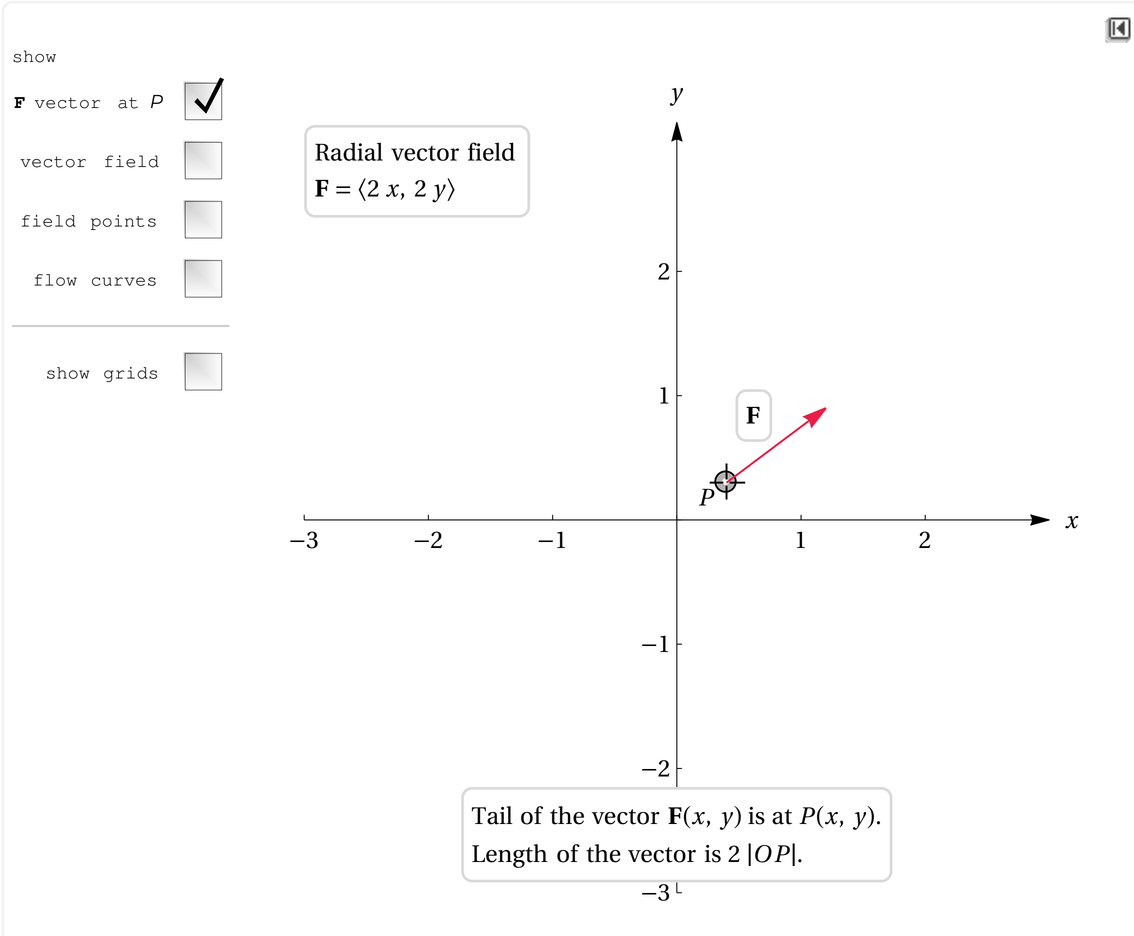


Figure 17.3

Note »

Drawing vectors with their actual length often leads to cluttered pictures of vector fields. For this reason, most of the vector fields in this chapter are illustrated with proportional scaling: All vectors are multiplied by a scalar chosen to make the vector field as understandable as possible.

EXAMPLE 1 Vector fields

Sketch representative vectors of the following vector fields.

a. $\mathbf{F}(x, y) = \langle 0, x \rangle = x\,\mathbf{j}$ (a shear field)

b. $\mathbf{F}(x, y) = \left\langle 1 - y^2, 0 \right\rangle = \left(1 - y^2\right)\mathbf{i}$, for $|y| \le 1$ (channel flow)

c. $\mathbf{F}(x, y) = \langle -y, x \rangle = -y\,\mathbf{i} + x\,\mathbf{j}$ (a rotation field)

SOLUTION »

a. This vector field is independent of y. Furthermore, because the x-component of $\mathbf{F}$ is zero, all vectors in the field (for $x \neq 0$) point in the y-direction: upward for $x > 0$ and downward for $x < 0$. The magnitudes of the vectors in the field increase with distance from the y-axis (**Figure 17.4**). The flow curves for this field are vertical lines. If **F** represents a velocity field, a particle right of the y-axis moves upward, a particle left of the y-axis moves downward, and a particle on the y-axis is stationary.

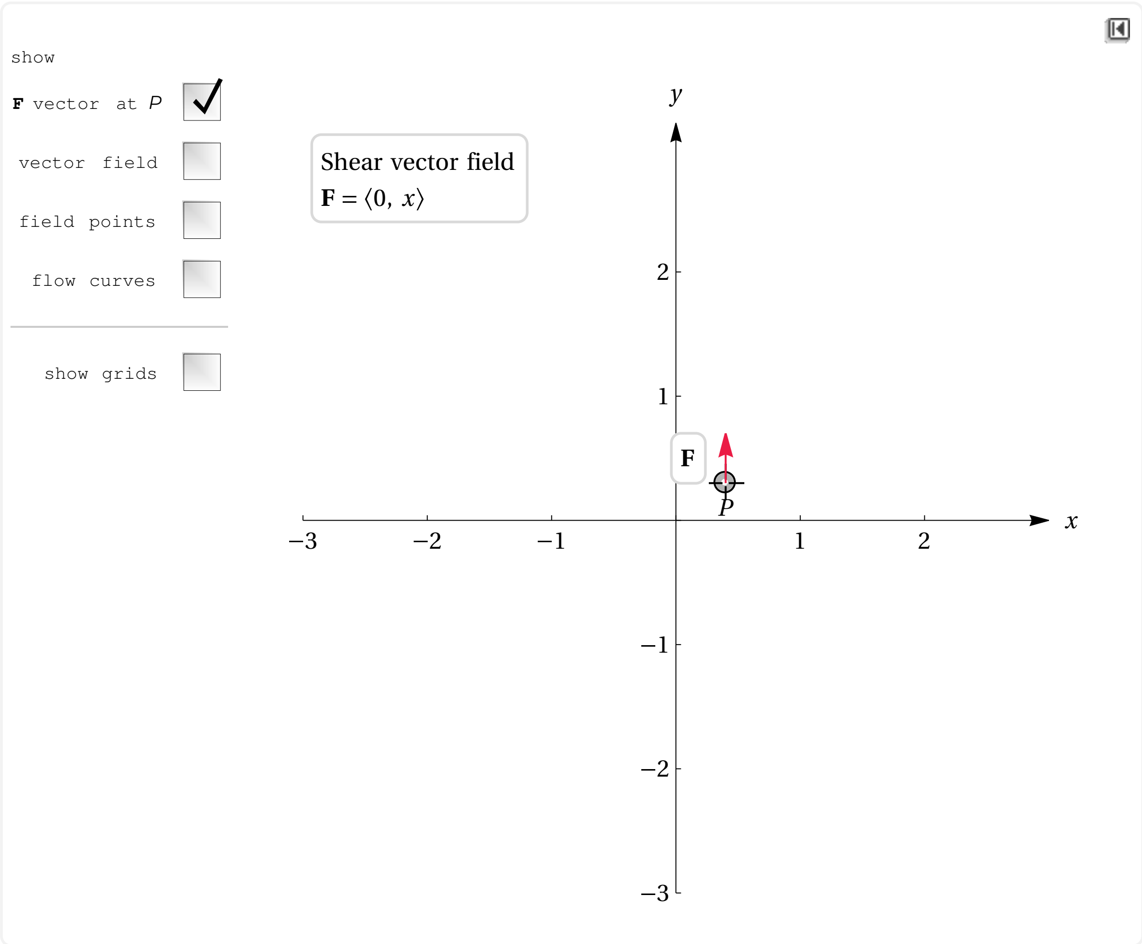


Figure 17.4

b. In this case, the vector field is independent of x and the y-component of **F** is zero. Because $1 - y^2 > 0$ for $|y| < 1$, vectors in this region point in the positive x-direction. The x-component of the vector field is zero at the boundaries $y = \pm 1$ and increases to 1 along the center of the strip, $y = 0$. The vector field might model the flow of water in a straight shallow channel (**Figure 17.5**); its flow curves are horizontal lines, indicating motion in the direction of the positive x-axis.

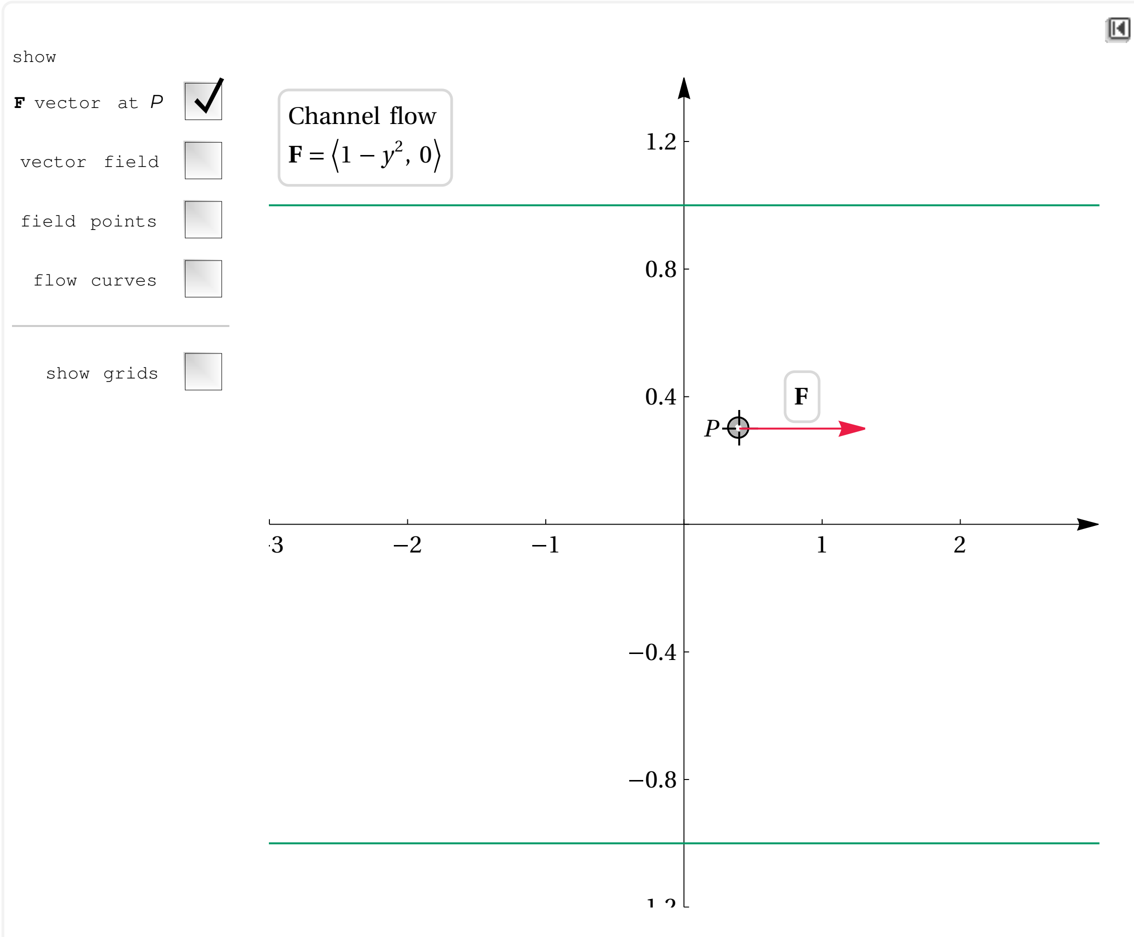


Figure 17.5

c. It often helps to determine the vector field along the coordinate axes.

- When $y = 0$ (along the x-axis), we have $\mathbf{F}(x, 0) = \langle 0, x \rangle$. With $x > 0$, this vector field consists of vectors pointing upward, increasing in length as x increases. With $x < 0$, the vectors point downward, increasing in length as $|x|$ increases.

A few more representative vectors show that the vector field has a counterclockwise rotation about the origin; the magnitudes of the vectors increase with distance from the origin (**Figure 17.6**).

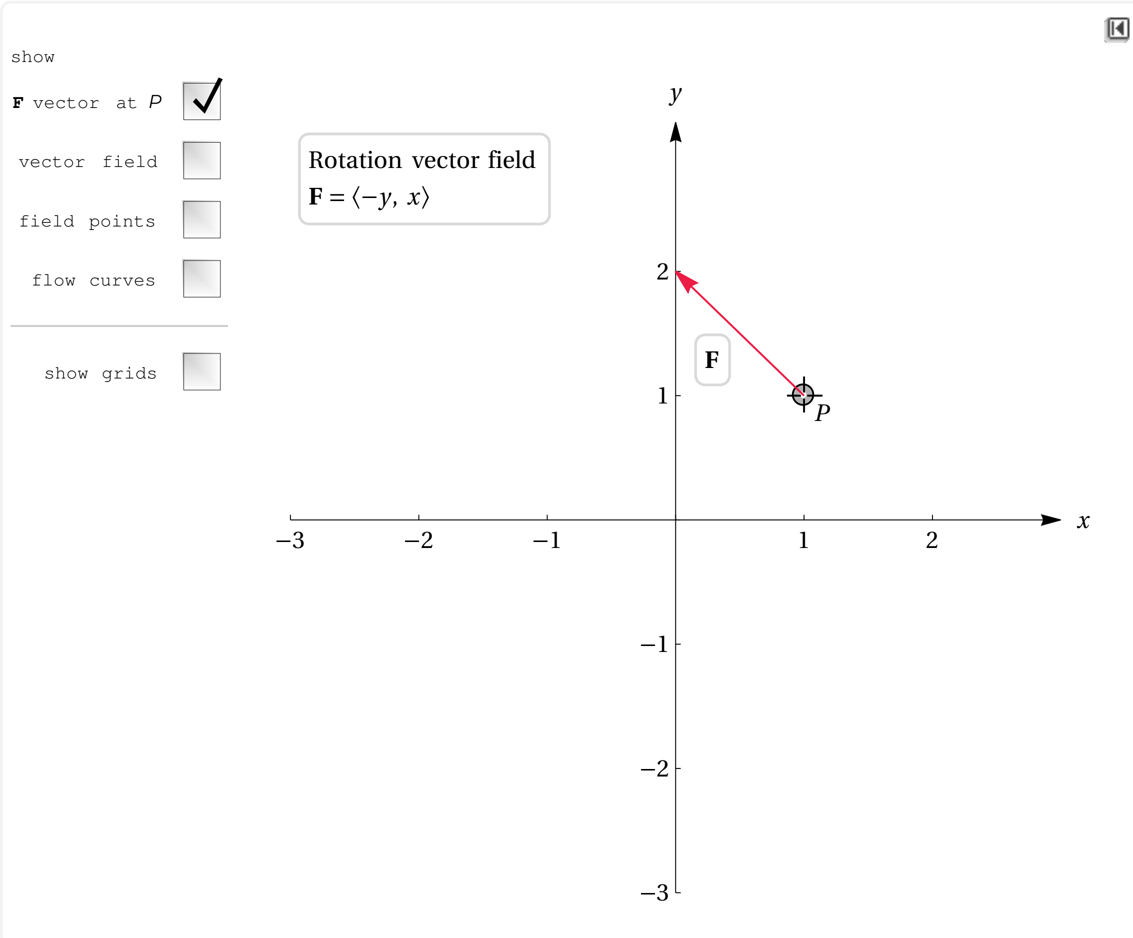


Figure 17.6

Note »

Related Exercises 10, 11, 13 ◆

Quick Check 1 If the vector field in Example 1c describes the velocity of a fluid and you place a small cork in the plane at (2, 0), what path do you think it will follow? ◆

Answer »

Radial Vector Fields in $\mathbb{R}^2$

Radial vector fields in $\mathbb{R}^2$ have the property that their vectors point directly toward or away from the origin at all points (except the origin), parallel to the position vectors $\mathbf{r} = \langle x, y \rangle$. We will work with radial vector fields of the form

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \underbrace{\frac{\mathbf{r}}{|\mathbf{r}|}}_{\text{unit vector}} \underbrace{\frac{1}{|\mathbf{r}|^{p-1}}}_{\text{magnitude}},$$

where p is a real number. **Figure 17.7** illustrates radial fields with $p = 1$ and $p = 3$. These vector fields (and their three-dimensional counterparts) play an important role in many applications. For example, central forces, such as gravitational or electrostatic forces between point masses or charges, are described by radial vector fields with $p = 3$. These forces obey an inverse square law in which the magnitude of the force is proportional to $\dfrac{1}{|\mathbf{r}|^2}$.

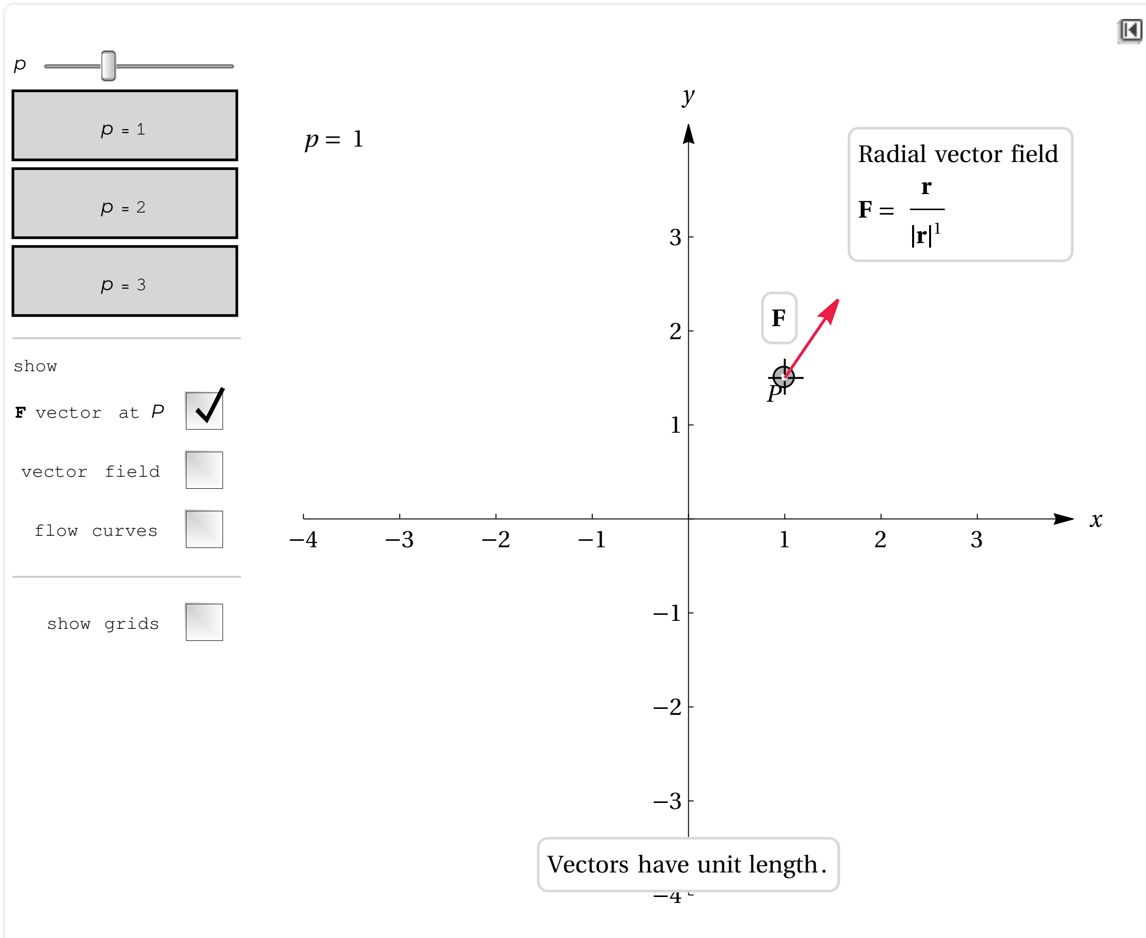


Figure 17.7

DEFINITION **Radial Vector Fields in $\mathbb{R}^2$**

Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\,\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p},$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

EXAMPLE 2 Normal and tangential vectors

Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a. Show that at each point of C, the radial vector field $\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b. Show that at each point of C, the rotation vector field $\mathbf{G}(x, y) = \frac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.

SOLUTION »

Let $g(x, y) = x^2 + y^2$. The circle C described by the equation $g(x, y) = a^2$ may be viewed as a level curve of a surface $z = x^2 + y^2$. As shown in Theorem 15.12 (Section 15.5), the gradient $\nabla g(x, y) = \langle 2x, 2y \rangle$ is orthogonal to the line tangent to C at (x, y) (**Figure 17.8**).

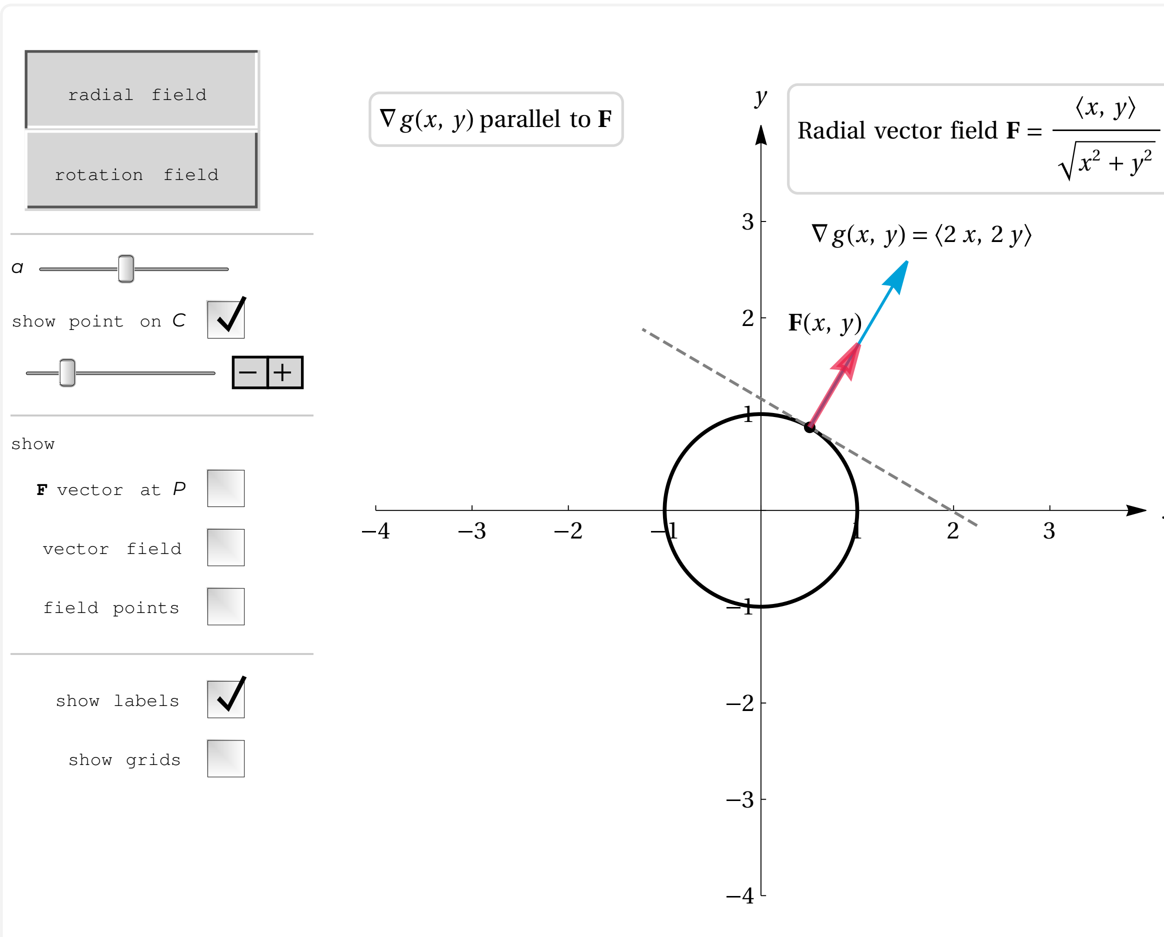


Figure 17.8

a. Notice that $\nabla g(x, y)$ is parallel to $\mathbf{F} = \frac{\langle x, y \rangle}{|\mathbf{r}|}$ at the point (x, y). It follows that $\mathbf{F}$ is also orthogonal to the line tangent to C at (x, y).

b. Notice that

$$\nabla g(x, y) \cdot \mathbf{G}(x, y) = \langle 2x, 2y \rangle \cdot \frac{\langle -y, x \rangle}{|\mathbf{r}|} = 0.$$

Therefore, $\nabla g(x, y)$ is orthogonal to the vector field **G** at (x, y), which implies that **G** is parallel to the tangent line at (x, y).

Related Exercises 27–28 ◆

Quick Check 2 In Example 2, verify that $\nabla g(x, y) \cdot \mathbf{G}(x, y) = 0$. In parts (a) and (b) of Example 2, verify that $|\mathbf{F}| = 1$ and $|\mathbf{G}| = 1$ at all points excluding the origin. ◆

Vector Fields in Three Dimensions »

Vector fields in three dimensions are conceptually the same as vector fields in two dimensions. The vector **F** now has three components, each of which depends on three variables.

DEFINITION **Vector Fields and Radial Vector Fields in $\mathbb{R}^3$**

Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function **F** that assigns to each point in D a vector $\langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$. The vector field is written as

$$\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \text{ or}$$
$$\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}.$$

A vector field $\mathbf{F} = \langle f, g, h \rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x, y, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{|\mathbf{r}|^p},$$

where p is a real number.

EXAMPLE 3 Vector fields in $\mathbb{R}^3$

Sketch and discuss the following vector fields.

a. $\mathbf{F}(x, y, z) = \langle x, y, e^{-z} \rangle$, for $z \geq 0$

b. $\mathbf{F}(x, y, z) = \left\langle 0, 0, 1 - x^2 - y^2 \right\rangle$, for $x^2 + y^2 \leq 1$

SOLUTION »

a. First consider the x- and y-components of **F** in the xy-plane $(z = 0)$, where $\mathbf{F} = \langle x, y, 1 \rangle$. This vector field looks like a radial field in the first two components, increasing in magnitude with distance from the z-axis. However, each vector also has a constant vertical component of 1. In horizontal planes $z = z_0 > 0$, the radial pattern remains the same, but the vertical component decreases as z increases. As $z \to \infty$, $e^{-z} \to 0$ and the vector field becomes a horizontal radial field (**Figure 17.9**).

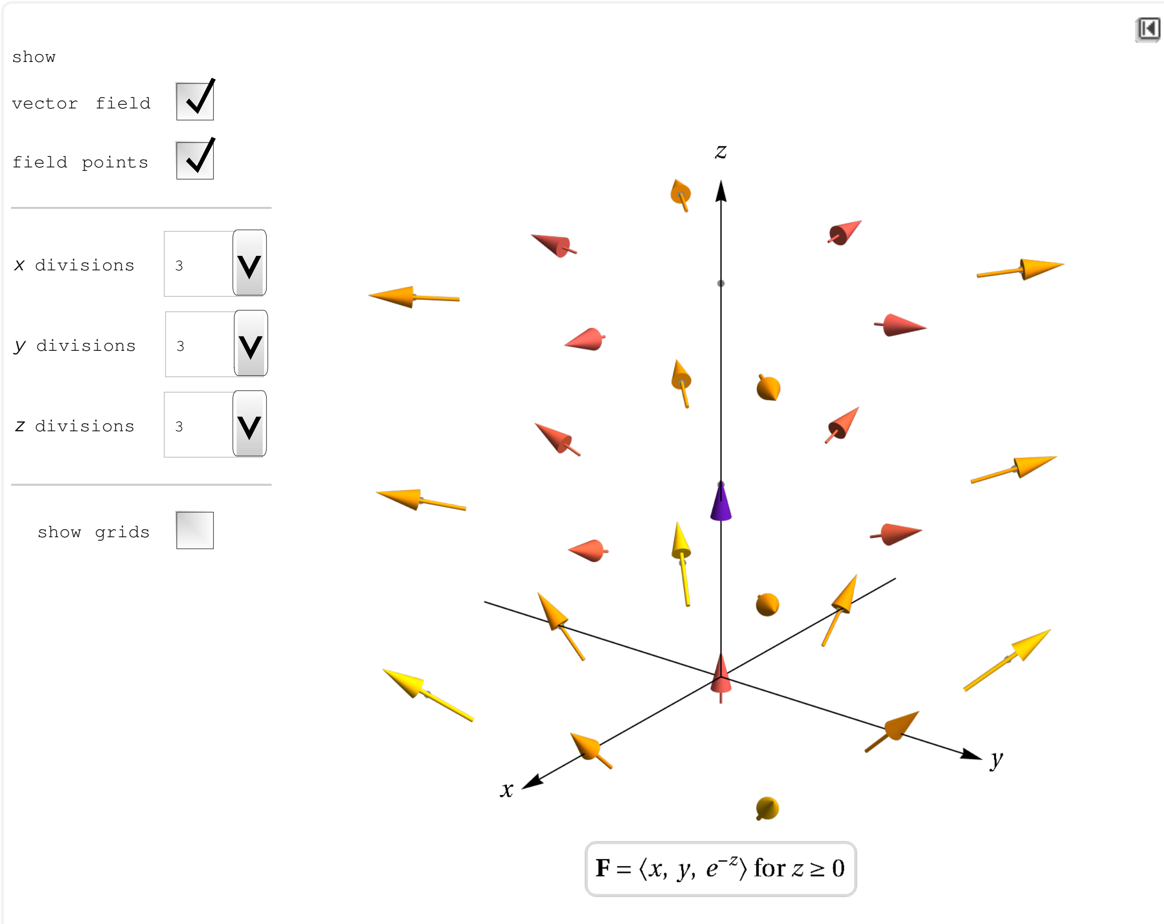


Figure 17.9

b. Regarding **F** as a velocity field, for points in and on the cylinder $x^2 + y^2 = 1$, there is no motion in the x- or y-directions. The z-component of the vector field may be written $1 - r^2$, where $r^2 = x^2 + y^2$ is the square of the distance from the z-axis. We see that the z-component increases from 0 on the boundary of the cylinder $(r = 1)$ to a maximum value of 1 along the centerline of the cylinder $(r = 0)$ (**Figure 17.10**). This vector field models the flow of a fluid inside a tube (such as a blood vessel).

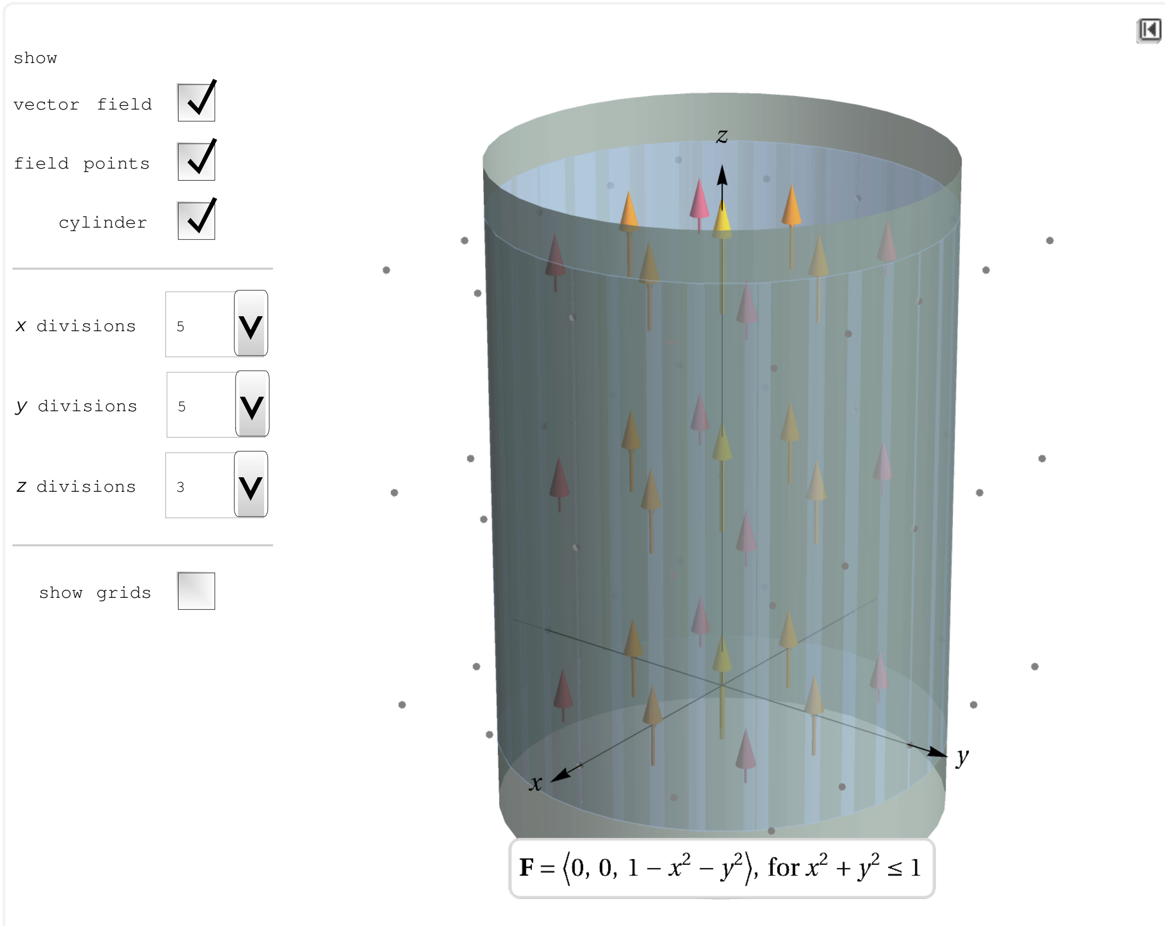


Figure 17.10

Related Exercises 20, 22 ◆

Gradient Fields and Potential Functions

One way to generate a vector field is to start with a differentiable scalar-valued function φ, take its gradient, and let $\mathbf{F} = \nabla\varphi$. A vector field defined as the gradient of a scalar-valued function φ is called a *gradient field* and the function φ is called a *potential function*.

Note »

Suppose φ is a differentiable function on a region R of $\mathbb{R}^2$ and consider the surface $z = \varphi(x, y)$. Recall from Chapter 15 that this function may also be represented by level curves in the xy-plane. At each point, (a, b) on a level curve, the gradient $\nabla\varphi(a, b) = \langle \varphi_x(a, b), \varphi_y(a, b) \rangle$ is orthogonal to the level curve at (a, b) (**Figure 17.11**). Therefore, the vectors of $\mathbf{F} = \nabla\varphi$ point in a direction orthogonal to the level curves of φ.

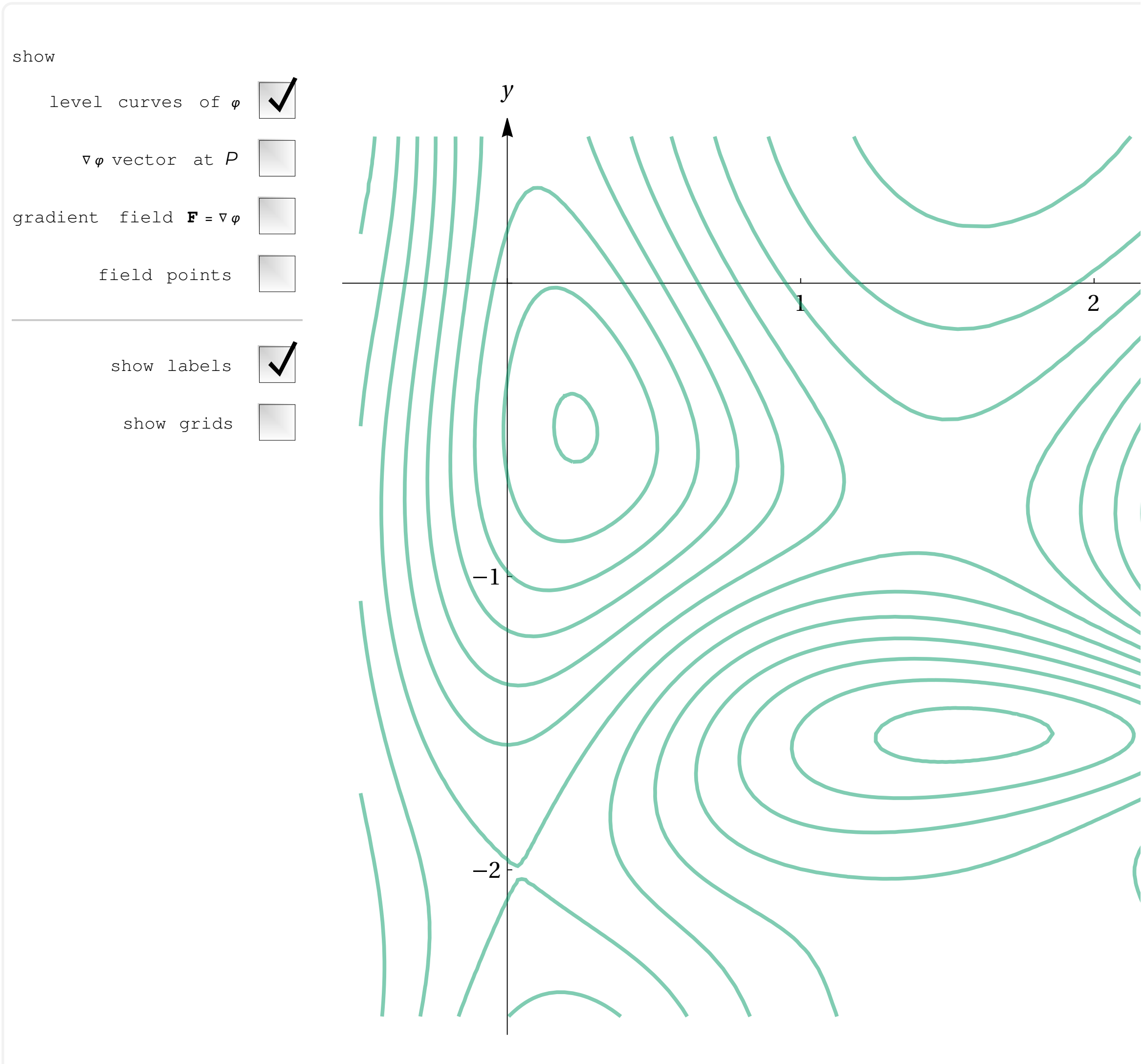


Figure 17.11

The idea extends to gradients of functions of three variables. If φ is differentiable on a region D of $\mathbb{R}^3$, then $\mathbf{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$ is a vector field that points in a direction orthogonal to the level *surfaces* of φ.

Gradient fields are useful because of the physical meaning of the gradient. For example, if φ represents the temperature in a conducting material, then the gradient field $\mathbf{F} = \nabla\varphi$ evaluated at a point indicates the direction in which the temperature increases most rapidly. According to a basic physical law, heat diffuses in the direction of the vector field $-\mathbf{F} = -\nabla\varphi$, the direction in which the temperature *decreases* most rapidly; that is, heat flows "down the gradient" from relatively hot regions to cooler regions. Similarly, water on a smooth surface tends to flow down the elevation gradient.

Quick Check 3 Find the gradient field associated with the function $\varphi(x, y, z) = x\,y\,z$. ◆

Answer »

DEFINITION **Gradient Fields and Potential Functions**

Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field**, and the function φ is a **potential function** for **F**.

Note »

EXAMPLE 4 Gradient fields

a. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the circular plate $R = \{(x, y) : x^2 + y^2 \le 25\}$.

b. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1} xy$.

SOLUTION »

a. The gradient field associated with T is

$$\mathbf{F} = \nabla T = \langle -2x, -2y \rangle = -2\langle x, y \rangle.$$

This vector field points inward toward the origin at all points of R except $(0, 0)$. The magnitudes of the vectors,

$$|\mathbf{F}| = \sqrt{(-2x)^2 + (-2y)^2} = 2\sqrt{x^2 + y^2},$$

are greatest on the edge of the disk, where $x^2 + y^2 = 25$ and $|\mathbf{F}| = 10$. The magnitudes of the vectors in the field decrease toward the center of the plate with $|\mathbf{F}(0, 0)| = 0$. **Figure 17.12** shows the level curves of the temperature function with several gradient vectors, all orthogonal to the level curves. Note that the plate is hottest at the center and coolest on the edge, so heat diffuses *outward*, in the direction opposite to that of the gradient.

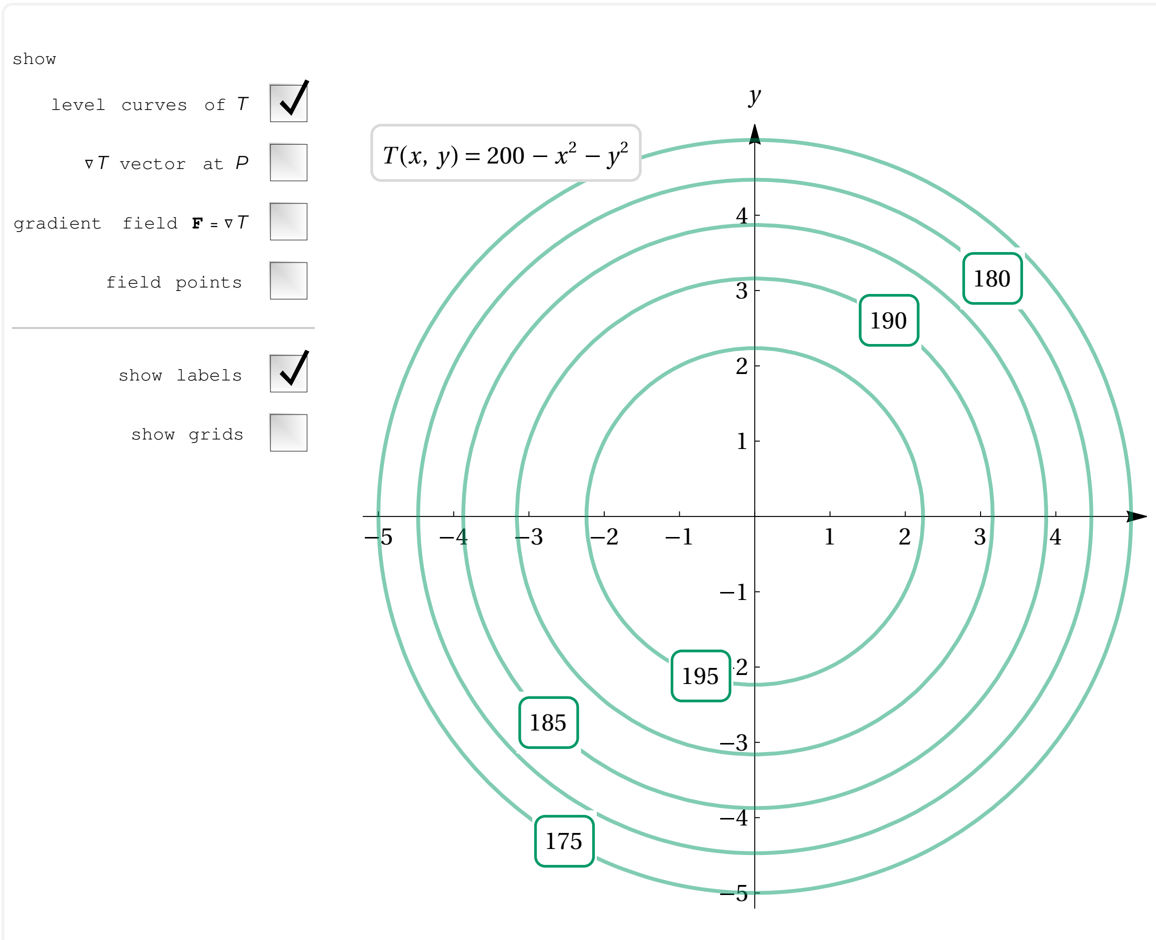


Figure 17.12

b. The gradient of a velocity potential gives the velocity components of a two-dimensional flow; that is, $\mathbf{F} = \langle u, v \rangle = \nabla\varphi$, where u and v are the velocities in the x- and y-directions, respectively. Computing the gradient, we find that

$$\mathbf{F} = \langle \varphi_x, \varphi_y \rangle = \left\langle \frac{1}{1+(xy)^2} \cdot y, \frac{1}{1+(xy)^2} \cdot x \right\rangle = \left\langle \frac{y}{1+x^2 y^2}, \frac{x}{1+x^2 y^2} \right\rangle.$$

Notice that the level curves of φ are the hyperbolas $xy = C$ or $y = C/x$. At all points, the vector field is orthogonal to the level curves (**Figure 17.13**).

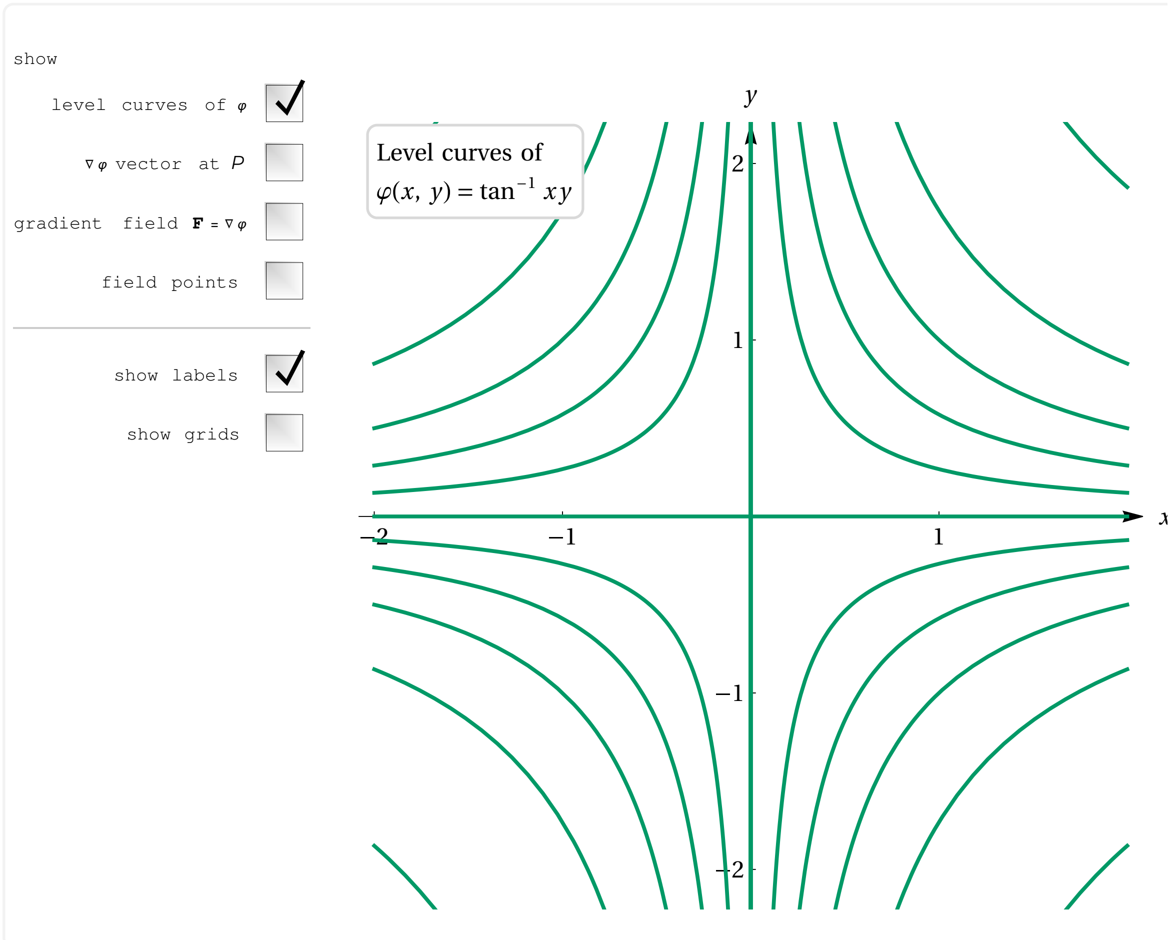


Figure 17.13

Related Exercises 38, 47 ◆

Equipotential Curves and Surfaces

The preceding example illustrates a beautiful geometric connection between a gradient field and its associated potential function. Let φ be a potential function for the vector field $\mathbf{F}$ in $\mathbb{R}^2$; that is, $\mathbf{F} = \nabla\varphi$. The level curves of a potential function are called **equipotential curves** (curves on which the potential function is constant).

Because the equipotential curves are level curves of φ, the vector field $\mathbf{F} = \nabla\varphi$ is everywhere orthogonal to the equipotential curves (**Figure 17.14**). Therefore, the vector field is visualized by drawing continuous *flow curves* or *streamlines* that are everywhere orthogonal to the equipotential curves. These ideas also apply to vector fields in $\mathbb{R}^3$ in which case the vector field is orthogonal to the **equipotential surfaces**.

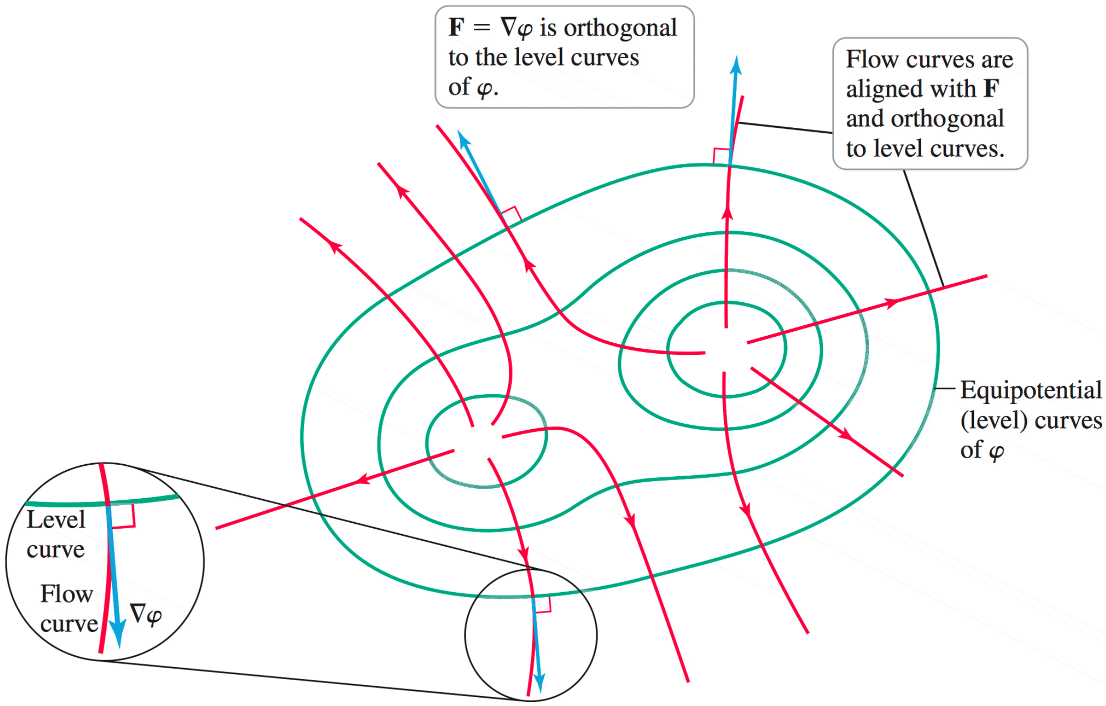


Figure 17.14

EXAMPLE 5 Equipotential curves

The equipotential curves for the potential function $\varphi(x, y) = \dfrac{x^2 - y^2}{2}$ are shown in **Figure 17.15**.

a. Find the gradient field associated with φ and verify that the gradient field is orthogonal to the equipotential curve at $(2, 1)$.

b. Verify that the vector field $\mathbf{F} = \nabla\varphi$ is orthogonal to the equipotential curves at all points (x, y).

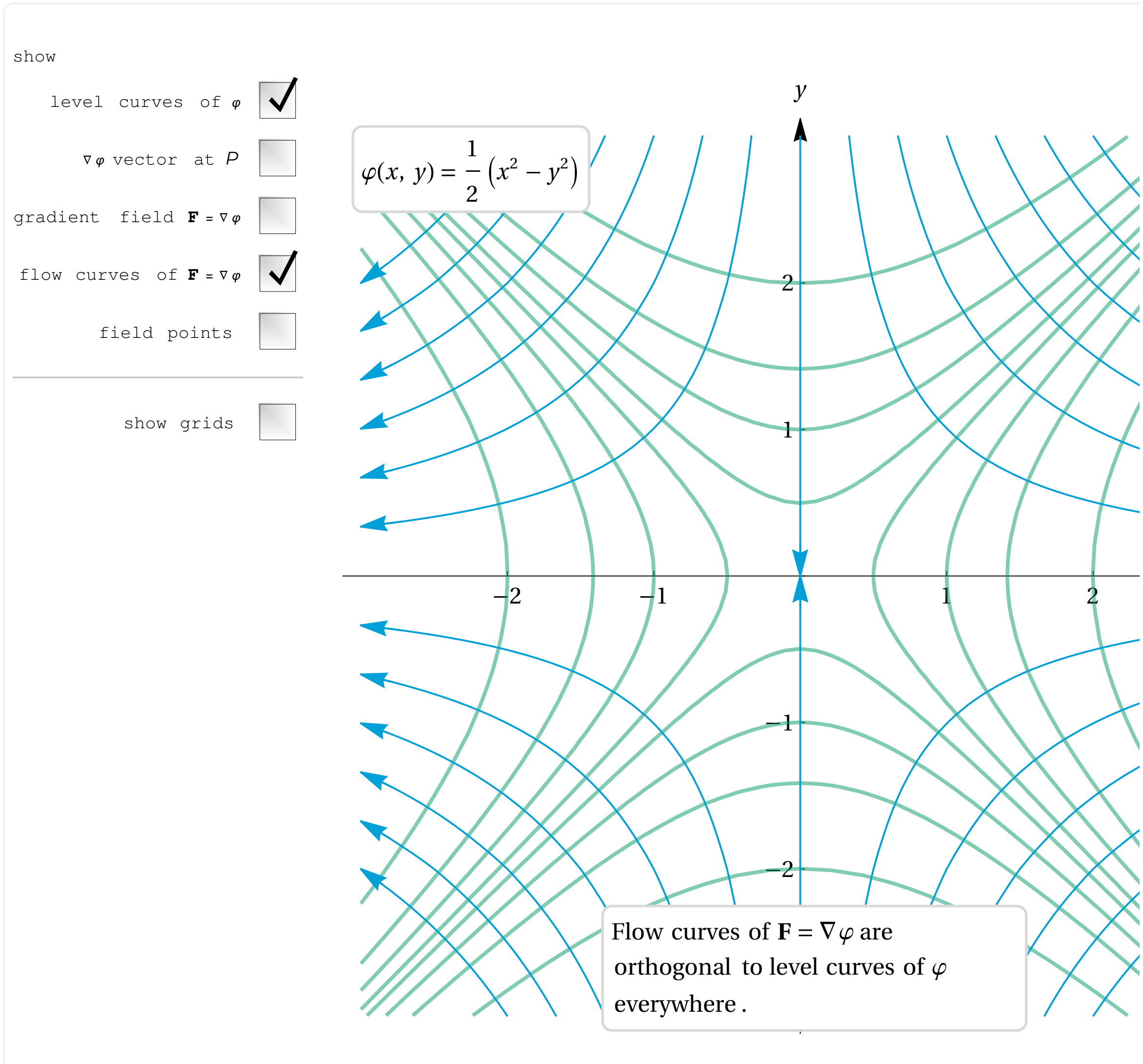


Figure 17.15

SOLUTION »

a. The level (or equipotential) curves are the hyperbolas $\frac{x^2 - y^2}{2} = C$, where C is a constant. The slope at any point on a level curve $\varphi(x, y) = C$ (Section 15.4) is

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = \frac{x}{y}.$$

At the point $(2, 1)$, the slope of the level curve is $\frac{dy}{dx} = 2$, so the vector tangent to the curve points in the direction $\langle 1, 2\rangle$. The gradient field is given by $\mathbf{F} = \nabla\varphi = \langle x, -y\rangle$, so $\mathbf{F}(2, 1) = \nabla\varphi(2, 1) = \langle 2, -1\rangle$. The dot product of the tangent vector $\langle 1, 2\rangle$ and the gradient is $\langle 1, 2\rangle \cdot \langle 2, -1\rangle = 0$; therefore, the two vectors are orthogonal.

Note »

b. In general, the line tangent to the equipotential curve at (x, y) is parallel to the vector $\langle y, x\rangle$, while the vector field at that point is $\mathbf{F} = \langle x, -y \rangle$. The vector field and the tangent vectors are orthogonal because $\langle y, x \rangle \cdot \langle x, -y \rangle = 0$.

Related Exercise 52 ◆

Exercises »

Getting Started »

Practice Exercises »

8–23. Sketching vector fields *Sketch the following vector fields.*

8. $\mathbf{F} = \langle 1, 0 \rangle$

9. $\mathbf{F} = \langle -1, 1 \rangle$

10. $\mathbf{F} = \langle 1, y \rangle$

11. $\mathbf{F} = \langle x, 0 \rangle$

12. $\mathbf{F} = \langle -x, -y \rangle$

13. $\mathbf{F} = \langle x, -y \rangle$

14. $\mathbf{F} = \langle 2\,x, 3\,y \rangle$

15. $\mathbf{F} = \langle y, -x \rangle$

16. $\mathbf{F} = \langle x + y, y \rangle$

17. $\mathbf{F} = \langle x, y - x \rangle$

18. $\mathbf{F} = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right\rangle$

19. $\mathbf{F} = \langle e^{-x}, 0 \rangle$

20. $\mathbf{F} = \langle 0, 0, 1 \rangle$

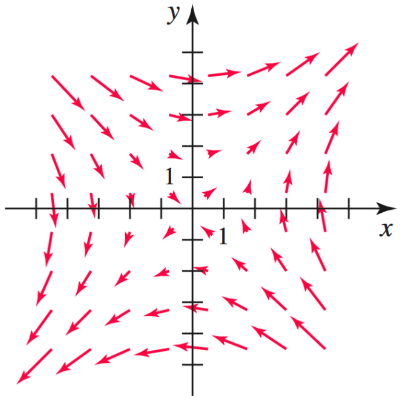
T **21.** $\mathbf{F} = \langle 1, 0, z \rangle$

T **22.** $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$

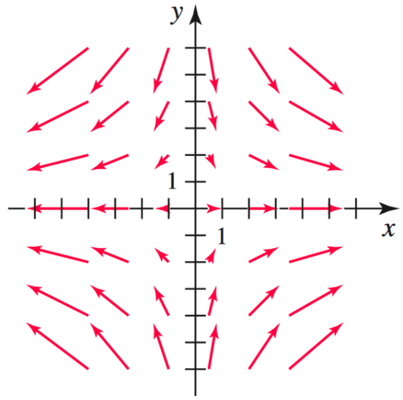
T **23.** $\mathbf{F} = \langle y, -x, 0 \rangle$

24. Matching vector fields with graphs Match vector fields a–d with graphs A–D.

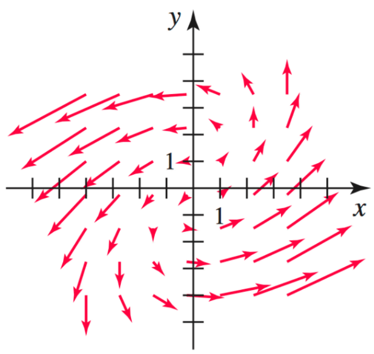
a. $\mathbf{F} = \left\langle 0, x^2 \right\rangle$
b. $\mathbf{F} = \langle x - y, x \rangle$
c. $\mathbf{F} = \langle 2\,x, -y \rangle$
d. $\mathbf{F} = \langle y, x \rangle$



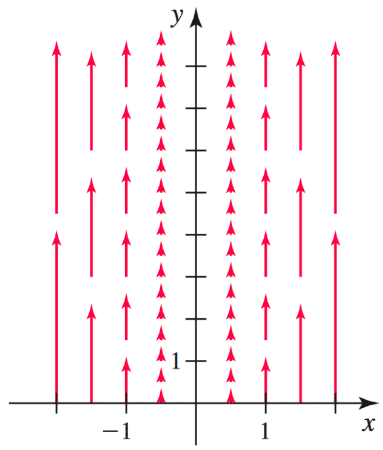
(A)



(B)



(C)



(D)

25–30. Normal and tangential components *For the vector field* **F** *and curve C, complete the following:*

- ***a.*** *Determine the points (if any) along the curve C at which the vector field* **F** *is tangent to C.*
- ***b.*** *Determine the points (if any) along the curve C at which the vector field* **F** *is normal to C.*
- ***c.*** *Sketch C and a few representative vectors of* **F** *on C.*

25. $\mathbf{F} = \left\langle \frac{1}{2}, 0 \right\rangle$; $C = \left\{(x, y) : y - x^2 = 1\right\}$

26. $\mathbf{F} = \left\langle \frac{y}{2}, -\frac{x}{2} \right\rangle$; $C = \left\{(x, y) : y - x^2 = 1\right\}$

27. $\mathbf{F} = \langle x, y \rangle$; $C = \left\{(x, y) : x^2 + y^2 = 4\right\}$

28. $\mathbf{F} = \langle y, -x \rangle$; $C = \left\{(x, y) : x^2 + y^2 = 1\right\}$

29. $\mathbf{F} = \langle x, y \rangle$; $C = \{(x, y) : x = 1\}$

30. $\mathbf{F} = \langle y, x \rangle$; $C = \left\{(x, y) : x^2 + y^2 = 1\right\}$

31–34. Design your own vector field *Specify the component functions of a vector field* **F** *in* $\mathbb{R}^2$ *with the following properties. Solutions are not unique.*

31. $\mathbf{F}$ is everywhere normal to the line $y = x$.

32. $\mathbf{F}$ is everywhere normal to the line $x = 2$.

33. At all points except $(0, 0)$, $\mathbf{F}$ has unit magnitude and points away from the origin along radial lines.

34. The flow of $\mathbf{F}$ is counterclockwise around the origin, increasing in magnitude with distance from the origin.

35–42. Gradient fields *Find the gradient field* $\mathbf{F} = \nabla\varphi$ *for the following potential functions* φ.

35. $\varphi(x, y) = x^2 y - y^2 x$

36. $\varphi(x, y) = \sqrt{x y}$

37. $\varphi(x, y) = \dfrac{x}{y}$

38. $\varphi(x, y) = \tan^{-1} \dfrac{x}{y}$

39. $\varphi(x, y, z) = \dfrac{x^2 + y^2 + z^2}{2}$

40. $\varphi(x, y, z) = \ln\left(1 + x^2 + y^2 + z^2\right)$

41. $\varphi(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$

42. $\varphi(x, y, z) = e^{-z} \sin(x + y)$

43–46. Gradient fields on curves *For the potential function* φ *and points A, B, C, and D on the level curve* $\varphi(x, y) = 0$, *complete the following steps.*

- ***a.*** *Find the gradient field* $\mathbf{F} = \nabla\varphi$.
- ***b.*** *Evaluate* $\mathbf{F}$ *at the points A, B, C, and D.*
- ***c.*** *Plot the level curve* $\varphi(x, y) = 0$ *and the vectors* $\mathbf{F}$ *at the points A, B, C, and D.*

43. $\varphi(x, y) = y - 2x$; $A(-1, -2)$, $B(0, 0)$, $C(1, 2)$, and $D(2, 4)$

44. $\varphi(x, y) = \dfrac{1}{2}x^2 - y$; $A(-2, 2)$, $B\left(-1, \dfrac{1}{2}\right)$, $C\left(1, \dfrac{1}{2}\right)$, and $D(2, 2)$

45. $\varphi(x, y) = -y + \sin x$; $A\left(\dfrac{\pi}{2}, 1\right)$, $B(\pi, 0)$, $C\left(\dfrac{3\pi}{2}, -1\right)$, and $D(2\pi, 0)$

T **46.** $\varphi(x, y) = \dfrac{32 - x^4 - y^4}{32}$; $A(2, 2)$, $B(-2, 2)$, $C(-2, -2)$, and $D(2, -2)$

T **47–48. Gradient fields** *Find the gradient field* $\mathbf{F} = \nabla\varphi$ *for the potential function* φ. *Sketch a few level curves of* φ *and a few vectors of* $\mathbf{F}$.

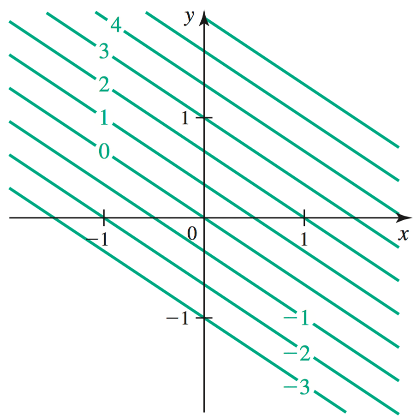
47. $\varphi(x, y) = x^2 + y^2$, for $x^2 + y^2 \le 16$

48. $\varphi(x, y) = x + y$, for $|x| \le 2$, $|y| \le 2$

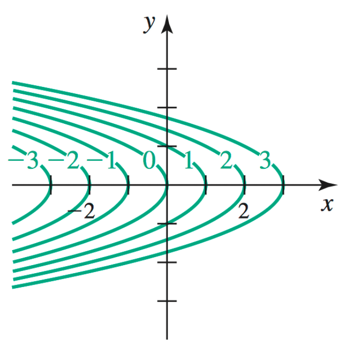
49–52. Equipotential curves *Consider the following potential functions and graphs of their equipotential curves.*

- ***a.*** *Find the associated gradient field* $\mathbf{F} = \nabla\varphi$.
- ***b.*** *Show that the vector field is orthogonal to the equipotential curve at the point* $(1, 1)$. *Illustrate this result on the figure.*
- ***c.*** *Show that the vector field is orthogonal to the equipotential curve at all points* (x, y).
- ***d.*** *Sketch two flow curves representing* $\mathbf{F}$ *that are everywhere orthogonal to the equipotential curves.*

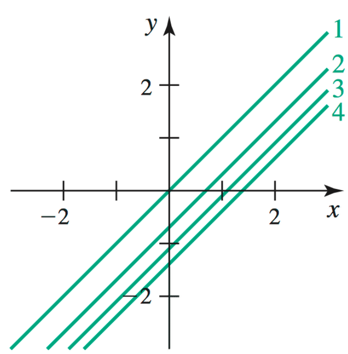
49. $\varphi(x, y) = 2x + 3y$



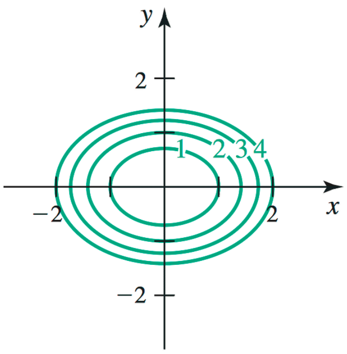
50. $\varphi(x, y) = x + y^2$



51. $\varphi(x, y) = e^{x-y}$



52. $\varphi(x, y) = x^2 + 2y^2$



53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The vector field $\mathbf{F} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\varphi_1(x, y) = x^3 + y$ and $\varphi_2(x, y) = y + x^3 + 100$.

b. The vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on the unit circle.

c. The vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

Explorations and Challenges »

54. Electric field due to a point charge The electric field in the xy-plane due to a point charge at (0, 0) is a gradient field with a potential function $V(x, y) = \dfrac{k}{\sqrt{x^2 + y^2}}$, where $k > 0$ is a physical constant.

a. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

b. Show that the vectors of the electric field point in the radial direction (outward from the origin) and the radial component of $\mathbf{E}$ can be expressed as $E_r = \dfrac{k}{r^2}$, where $r = \sqrt{x^2 + y^2}$.

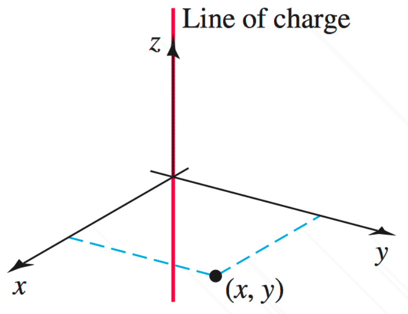
c. Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.

55. Electric field due to a line of charge The electric field in the xy-plane due to an infinite line of charge along the z-axis is a gradient field with a potential function $V(x, y) = c \ln\left(\dfrac{r_0}{\sqrt{x^2 + y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be 0 (see figure).

a. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

b. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $|\mathbf{E}| = \dfrac{c}{r}$, where $r = \sqrt{x^2 + y^2}$.

c. Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.



56. Gravitational force due to a mass The gravitational force on a point mass m due to a point mass M is a gradient field with potential $U(r) = \frac{GMm}{r}$, where G is the gravitational constant and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the masses.

a. Find the components of the gravitational force in the x-, y-, and z-directions, where $\mathbf{F}(x, y, z) = -\nabla U(x, y, z)$.

b. Show that the gravitational force points in the radial direction (outward from point mass M) and the radial component is $F(r) = \frac{GMm}{r^2}$.

c. Show that the vector field is orthogonal to the equipotential surfaces at all points in the domain of U.

57–61. Flow curves in the plane *Let* $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$ *be defined on* $\mathbb{R}^2$.

57. Explain why the flow curves or streamlines of $\mathbf{F}$ satisfy $y' = \frac{g(x, y)}{f(x, y)}$ and are everywhere tangent to the vector field.

T **58.** Find and graph the flow curves for the vector field $\mathbf{F} = \langle 1, x\rangle$.

T **59.** Find and graph the flow curves for the vector field $\mathbf{F} = \langle x, x\rangle$.

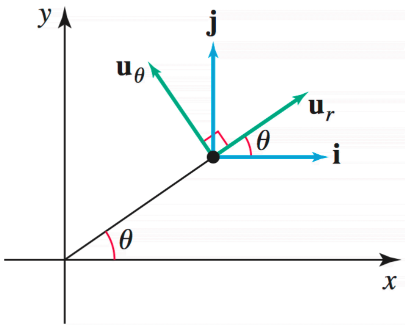
T **60.** Find and graph the flow curves for the vector field $\mathbf{F} = \langle y, x\rangle$. Note that $\frac{d}{dx}\left(y^2\right) = 2\,y\,y'(x)$.

T **61.** Find and graph the flow curves for the vector field $\mathbf{F} = \langle -y, x\rangle$.

62–63. Unit vectors in polar coordinates

62. Vectors in $\mathbb{R}^2$ may also be expressed in terms of polar coordinates. The standard coordinate unit vectors in polar coordinates are denoted $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (see figure). Unlike the coordinate unit vectors in Cartesian coordinates, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change their direction depending on the point (r, θ). Use the figure to show that for $r > 0$, the following relationships among the unit vectors in Cartesian and polar coordinates hold:

$$\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \qquad \mathbf{i} = \mathbf{u}_r\cos\theta - \mathbf{u}_\theta\sin\theta$$
$$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \qquad \mathbf{j} = \mathbf{u}_r\sin\theta + \mathbf{u}_\theta\cos\theta$$



63. Verify that the relationships in Exercise 62 are consistent when $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$.

64–66. Vector fields in polar coordinates *A vector field in polar coordinates has the form* $\mathbf{F}(r, \theta) = f(r, \theta)\,\mathbf{u}_r + g(r, \theta)\,\mathbf{u}_\theta$, *where the unit vectors are defined in Exercise 62. Sketch the following vector fields and express them in Cartesian coordinates.*

64. $\mathbf{F} = \mathbf{u}_r$

65. $\mathbf{F} = \mathbf{u}_\theta$

66. $\mathbf{F} = r\mathbf{u}_\theta$

67. **Cartesian vector field to polar vector field** Write the vector field $\mathbf{F} = \langle -y, x \rangle$ in polar coordinates and sketch the field.