

16.7 Change of Variables in Multiple Integrals

Converting double integrals from rectangular coordinates to polar coordinates (Section 16.3) and converting triple integrals from rectangular coordinates to cylindrical or spherical coordinates (Section 16.5) are examples of a general procedure known as a *change of variables*. The idea is not new: The Substitution Rule introduced in Chapter 5 with single-variable integrals is also an example of a change of variables. The aim of this section is to show how to apply a general change of variables to double and triple integrals.

Recap of Change of Variables »

Recall how a change of variables is used to simplify a single-variable integral. For example, to simplify the integral $\int_0^1 2\sqrt{2x+1} \, dx$, we choose a new variable $u = 2x + 1$, which means that $du = 2 \, dx$. Therefore,

$$\int_0^1 2\sqrt{2x+1} \, dx = \int_1^3 \sqrt{u} \, du.$$

This equality means that the area under the curve $y = 2\sqrt{2x+1}$ from $x = 0$ to $x = 1$ equals the area under the curve $y = \sqrt{u}$ from $u = 1$ to $u = 3$ (**Figure 16.76**). The relation $du = 2 \, dx$ relates the length of a small interval on the u -axis to the length of the corresponding interval on the x -axis.

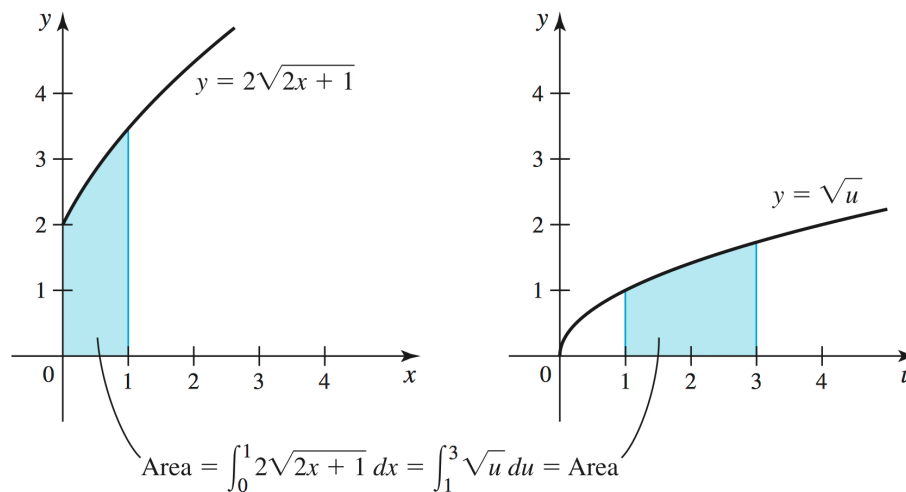


Figure 16.76

Similarly, some double and triple integrals can be simplified through a change of variables. For example, the region of integration for

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{1-x^2-y^2} \, dy \, dx$$

is the quarter disk $R = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$. Changing variables to polar coordinates with $x = r \cos \theta$, $y = r \sin \theta$, and $dy \, dx = r \, dr \, d\theta$, we have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{1-x^2-y^2} \, dy \, dx \stackrel{\substack{x=r \cos \theta \\ y=r \sin \theta}}{=} \int_0^{\pi/2} \int_0^1 e^{1-r^2} r \, dr \, d\theta.$$

In this case, the original region of integration R is transformed into a new region

$$S = \left\{ (r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2} \right\}, \text{ which is a rectangle in the } r\theta\text{-plane.}$$

Transformations in the Plane »

A change of variables in a double integral is a *transformation* that relates two sets of variables, (u, v) and (x, y) . It is written compactly as $(x, y) = T(u, v)$. Because it relates pairs of variables, T has two components,

$$T : x = g(u, v) \text{ and } y = h(u, v).$$

Geometrically, T takes a region S in the uv -plane and “maps” it point by point to a region R in the xy -plane (**Figure 16.77**). We write the outcome of this process as $R = T(S)$ and call R the **image** of S under T .

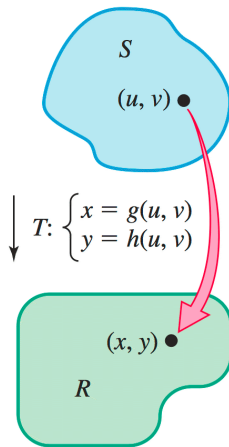


Figure 16.77

EXAMPLE 1 Image of a transformation

Consider the transformation from polar to rectangular coordinates given by

$$T : x = g(r, \theta) = r \cos \theta \text{ and } y = h(r, \theta) = r \sin \theta.$$

Find the image under this transformation of the rectangle

$$S = \left\{ (r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2} \right\}.$$

Note »

SOLUTION »

If we apply T to every point of S (**Figure 16.78**), what is the resulting set R in the xy -plane? One way to answer this question is to walk around the boundary of S , let's say counterclockwise, and determine the corresponding path in the xy -plane. In the $r\theta$ -plane, we let the horizontal axis be the r -axis and the vertical axis be the θ -axis. Starting at the origin, we denote the edges of the rectangle S as follows.

$$A = \{(r, \theta) : 0 \leq r \leq 1, \theta = 0\} \quad \text{Lower boundary}$$

$$B = \{(r, \theta) : r = 1, 0 \leq \theta \leq \frac{\pi}{2}\} \quad \text{Right boundary}$$

$$C = \{(r, \theta) : 0 \leq r \leq 1, \theta = \frac{\pi}{2}\} \quad \text{Upper boundary}$$

$$D = \{(r, \theta) : r = 0, 0 \leq \theta \leq \frac{\pi}{2}\} \quad \text{Left boundary}$$

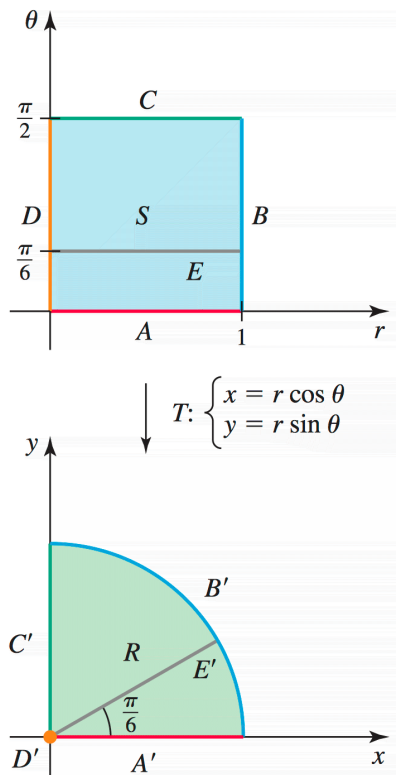


Figure 16.78

Table 16.6 shows the effect of the transformation on the four boundaries of S ; the corresponding boundaries of R in the xy -plane are denoted A' , B' , C' , and D' (Figure 16.78).

Table 16.6

Boundary of S in $r\theta$ -plane	Transformation equations	Boundary of R in xy -plane
$A: 0 \leq r \leq 1, \theta = 0$	$x = r \cos \theta = r,$ $y = r \sin \theta = 0$	$A': 0 \leq x \leq 1, y = 0$
$B: r = 1, 0 \leq \theta \leq \pi/2$	$x = r \cos \theta = \cos \theta,$ $y = r \sin \theta = \sin \theta$	$B':$ quarter unit circle
$C: 0 \leq r \leq 1, \theta = \pi/2$	$x = r \cos \theta = 0,$ $y = r \sin \theta = r$	$C': x = 0, 0 \leq y \leq 1$
$D: r = 0, 0 \leq \theta \leq \pi/2$	$x = r \cos \theta = 0,$ $y = r \sin \theta = 0$	$D':$ single point $(0, 0)$

The image of the rectangular boundary of S is the boundary of R . Furthermore, it can be shown that every point in the interior of R is the image of one point in the interior of S . (For example, the horizontal line segment E in the $r\theta$ -plane in Figure 16.78 is mapped to the line segment E' in the xy -plane.) Therefore, the image of S is the quarter disk R in the xy -plane.

Related Exercises 11–12 ♦

Quick Check 1 How would the image of S change in Example 1 if $S = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$? ♦

Answer »

The image is a semicircular disk of radius 1.

Recall that a function f is *one-to-one* on an interval I if $f(x_1) = f(x_2)$ only when $x_1 = x_2$, where x_1 and x_2 are points of I . We need an analogous property for transformations when changing variables.

DEFINITION **One-to-One Transformation**

A transformation T from a region S to a region R is *one-to-one* on S if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S .

Notice that the polar coordinate transformation in Example 1 is not one-to-one on the rectangle $S = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$ (because all points with $r = 0$ map to the point $(0, 0)$). However, this transformation is one-to-one on the interior of S .

We can now anticipate how a transformation (change of variables) is used to simplify a double integral. Suppose we have the integral $\iint_R f(x, y) \, dA$. The goal is to find a transformation to a new set of coordinates (u, v) such that the new equivalent integral $\iint_S f(x(u, v), y(u, v)) \, dA$ involves a simple region S (such as a rectangle), a simple integrand, or both. The next theorem allows us to do exactly that, but it first requires a new concept.

DEFINITION Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T : x = g(u, v)$, $y = h(u, v)$, where g and h are differentiable on a region of the uv -plane, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

Note »

The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). In some books, the Jacobian is the matrix of partial derivatives. In others, as here, the Jacobian is the determinant of the matrix of partial derivatives. Both $J(u, v)$ and $\frac{\partial(x, y)}{\partial(u, v)}$ are used to refer to the Jacobian.

The Jacobian is easiest to remember as the determinant of a 2×2 matrix of partial derivatives. With the Jacobian in hand, we can state the change-of-variables rule for double integrals.

Quick Check 2 Find $J(u, v)$ if $x = u + v$, $y = 2v$. ♦

Answer »

$$J(u, v) = 2$$

THEOREM 16.8 Change of Variables for Double Integrals

Let $T : x = g(u, v)$, $y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv -plane onto a region R in the xy -plane. Assume that T is one-to-one on the interior of S and that g and h have continuous first partial derivatives there. If f is continuous on R , then

$$\iint_R f(x, y) \, dA = \iint_S f(g(u, v), h(u, v)) |J(u, v)| \, dA.$$

Note »

The condition that g and h have continuous first partial derivatives ensures that the new integrand is integrable.

The proof of this result is technical and is found in advanced texts. The factor $|J(u, v)|$ that appears in the second integral is the absolute value of the Jacobian. Matching the area elements in the two integrals of Theorem 16.8, we see that $dx \, dy = |J(u, v)| \, du \, dv$. This expression shows that the Jacobian is a magnification (or reduction) factor: It relates the area of a small region $dx \, dy$ in the xy -plane to the area of the corresponding region $du \, dv$ in the uv -plane. If the transformation equations are linear, then this relationship is exact in the sense that $\text{area}(T(S)) = |J(u, v)| \cdot \text{area of } S$ (see Exercise 60). The way in which the Jacobian arises is explored in Exercise 61.

Note »

In the integral over R , dA corresponds to $dx dy$. In the integral over S , dA corresponds to $du dv$. The relation $dx dy = |J| du dv$ is the analog of $du = g'(x) dx$ in a change of variables with one variable.

EXAMPLE 2 Jacobian of the polar-to-rectangular transformation

Compute the Jacobian of the transformation

$$T : x = g(r, \theta) = r \cos \theta, \quad y = h(r, \theta) = r \sin \theta.$$

SOLUTION »

The necessary partial derivatives are

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r \cos \theta.$$

Therefore,

$$J(r, \theta) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r.$$

This determinant calculation confirms the change-of-variables formula for polar coordinates: $dx dy$ becomes $r dr d\theta$.

Related Exercise 20 ♦

We are now ready for a change of variables. To transform the integral $\iint_R f(x, y) dA$ into

$\iint_S f(x(u, v), y(u, v)) |J(u, v)| dA$, we must find the transformation $x = g(u, v)$ and $y = h(u, v)$, and then use it to find the new region of integration S . The next example illustrates how the region S is found, assuming the transformation is given.

EXAMPLE 3 Double integral with a change of variables given

Evaluate the integral $\iint_R \sqrt{2x(y-2x)} dA$, where R is the parallelogram in the xy -plane with vertices $(0, 0)$, $(0, 1)$, $(2, 4)$, and $(2, 5)$ (**Figure 16.79**). Use the transformation

$$T : x = 2u, \quad y = 4u + v.$$

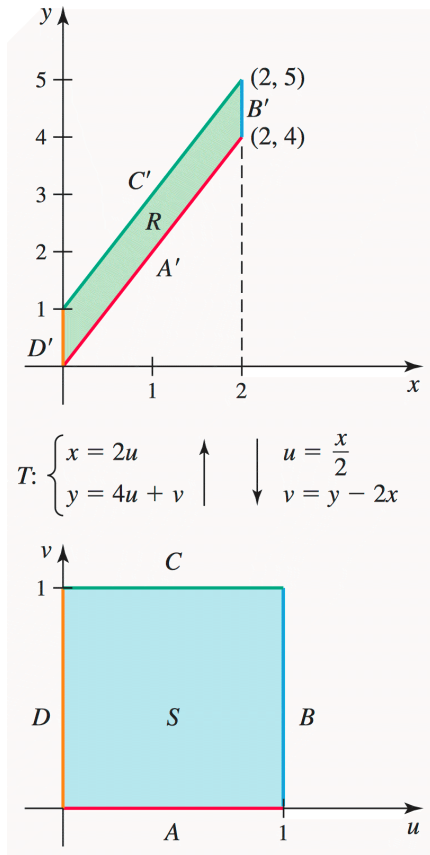


Figure 16.79

SOLUTION »

To what region S in the uv -plane is R mapped? Because T takes points in the uv -plane and assigns them to points in the xy -plane, we must reverse the process by solving $x = 2u$, $y = 4u + v$ for u and v .

$$\text{First equation : } x = 2u \Rightarrow u = \frac{x}{2}$$

$$\text{Second equation : } y = 4u + v \Rightarrow v = y - 4u = y - 2x$$

Rather than walk around the boundary of R in the xy -plane to determine the resulting region S in the uv -plane, it suffices to find the images of the vertices of R . You should confirm that the vertices map as shown in Table 16.7.

Table 16.7

(x, y)	(u, v)
(0, 0)	(0, 0)
(0, 1)	(0, 1)
(2, 5)	(1, 1)
(2, 4)	(1, 0)

Note »

The relations that “go the other direction” make up the inverse transformation, usually denoted T^{-1} .

Connecting the points in the uv -plane in order, we see that S is the unit square $\{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ (Figure 16.79). These inequalities determine the limits of integration in the uv -plane.

Note »

T is an example of a *shearing transformation*. The greater the u -coordinate of a point, the more that point is displaced in the v -direction. It also involves a uniform stretch in the u -direction.

Replacing $2x$ with $4u$ and $y - 2x$ with v , the original integrand becomes $\sqrt{2x(y - 2x)} = \sqrt{4uv}$. The Jacobian is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2.$$

The integration now follows:

$$\begin{aligned} \iint_R \sqrt{2x(y - 2x)} \, dA &= \iint_S \sqrt{4uv} \frac{|J(u, v)|}{2} \, dA && \text{Change variables.} \\ &= \int_0^1 \int_0^1 \sqrt{4uv} \, 2 \, du \, dv && \text{Convert to an iterated integral.} \\ &= 4 \int_0^1 \frac{2}{3} \sqrt{v} (u^{3/2}) \Big|_0^1 \, dv && \text{Evaluate inner integral.} \\ &= \frac{8}{3} \cdot \frac{2}{3} (v^{3/2}) \Big|_0^1 = \frac{16}{9}. && \text{Evaluate outer integral.} \end{aligned}$$

The effect of the change of variables is illustrated in **Figure 16.80**, where we see the surface $z = \sqrt{2x(y - 2x)}$ over the region R and the surface $w = 2\sqrt{4uv}$ over the region S . The volumes of the solids beneath the two surfaces are equal, but the integral over S is easier to evaluate.

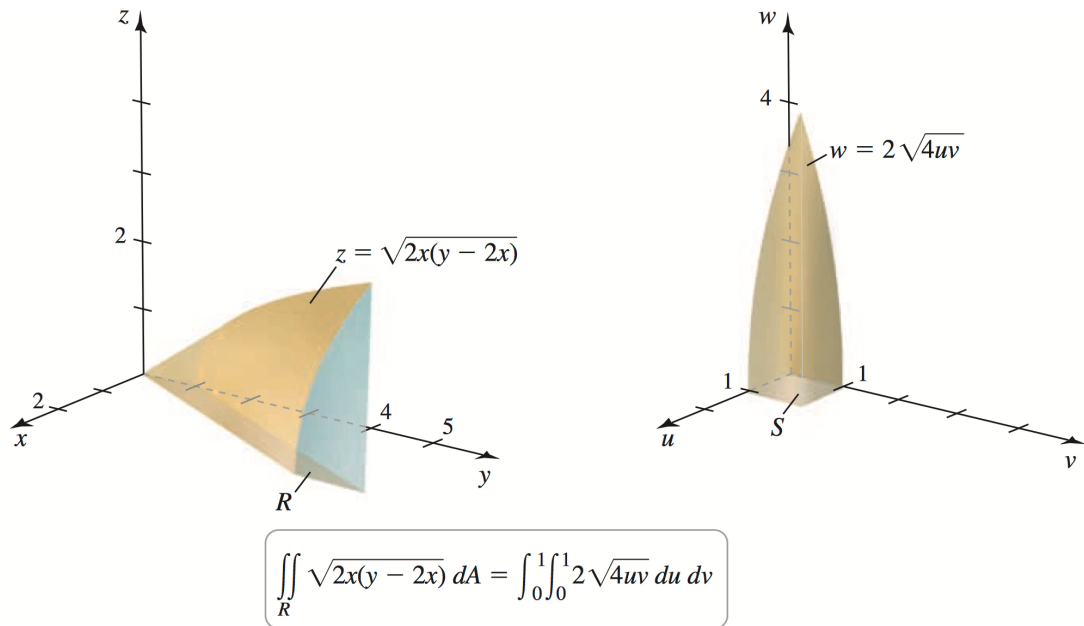


Figure 16.80

Related Exercise 29 ♦

Quick Check 3 Solve the equations $u = x + y$, $v = -x + 2y$ for x and y . ♦

Answer >

$$x = \frac{2u}{3} - \frac{v}{3}, \quad y = \frac{u}{3} + \frac{v}{3}$$

In Example 3, the required transformation was given. More practically, we must deduce an appropriate transformation from either the integrand or the region of integration.

EXAMPLE 4 Change of variables determined by the integrand

Evaluate $\iint_R \sqrt{\frac{x-y}{x+y+1}} \, dA$, where R is the square with vertices $(0, 0)$, $(1, -1)$, $(2, 0)$, and $(1, 1)$ (**Figure 16.81**).

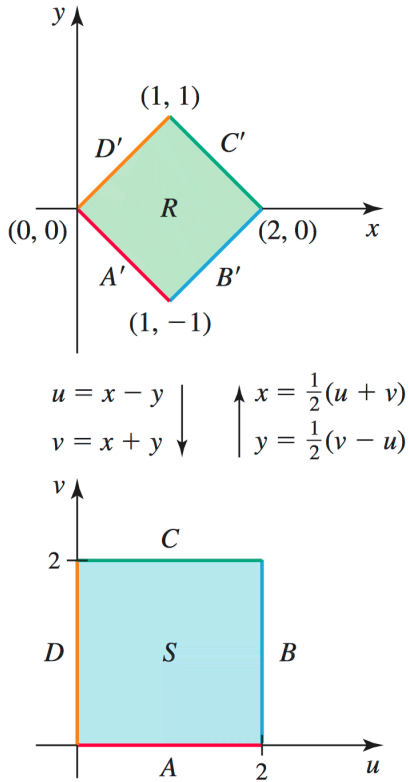


Figure 16.81

SOLUTION »

Evaluating the integral as it stands requires splitting the region R into two subregions; furthermore, the integrand presents difficulties. The terms $x + y$ and $x - y$ in the integrand suggest the new variables

$$u = x - y \quad \text{and} \quad v = x + y.$$

Note »

The transformation in Example 4 is a *rotation*. It rotates the points of R about the origin 45° in the counterclockwise direction (it also increases lengths by a factor of $\sqrt{2}$). In this example, the change of variables $u = x + y$ and $v = x - y$ would work just as well.

To determine the region S in the uv -plane that corresponds to R under this transformation, we find the images of the vertices of R in the uv -plane and connect them in order. The result is the square

$S = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 2\}$ (Figure 16.81). Before computing the Jacobian, we express x and y in terms of u

and v . Adding the two equations and solving for x , we have $x = \frac{u + v}{2}$. Subtracting the two equations and

solving for y gives $y = \frac{v - u}{2}$. The Jacobian now follows:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

With the choice of new variables, the original integrand $\sqrt{\frac{x-y}{x+y+1}}$ becomes $\sqrt{\frac{u}{v+1}}$. The integration in the uv -plane may now be done:

$$\begin{aligned} \iint_R \sqrt{\frac{x-y}{x+y+1}} dA &= \iint_S \sqrt{\frac{u}{v+1}} |J(u, v)| dA && \text{Change of variables} \\ &= \int_0^2 \int_0^2 \sqrt{\frac{u}{v+1}} \frac{1}{2} du dv && \text{Convert to an iterated integral.} \\ &= \frac{1}{2} \int_0^2 (v+1)^{-1/2} \frac{2}{3} (u^{3/2}) \Big|_0^2 dv && \text{Evaluate inner integral.} \\ &= \frac{2^{3/2}}{3} \times 2 (v+1)^{1/2} \Big|_0^2 && \text{Evaluate outer integral.} \\ &= \frac{4\sqrt{2}}{3} (\sqrt{3} - 1). && \text{Simplify.} \end{aligned}$$

Note »

An appropriate change of variables for a double integral is not always obvious. Some trial and error is often needed to come up with a transformation that simplifies the integrand and/or the region of integration. Strategies are discussed at the end of this section.

Related Exercises 32, 36 ♦

Quick Check 4 In Example 4, what is the ratio of the area of S to the area of R ? How is this ratio related to J ? ♦

Answer »

The ratio is 2, which is $\frac{1}{J(u, v)}$.

EXAMPLE 5 Change of variables determined by the region

Let R be the region in the first quadrant bounded by the parabolas $x = y^2$, $x = y^2 - 4$, $x = 9 - y^2$, and $x = 16 - y^2$

(Figure 16.82). Evaluate $\iint_R y^2 dA$.

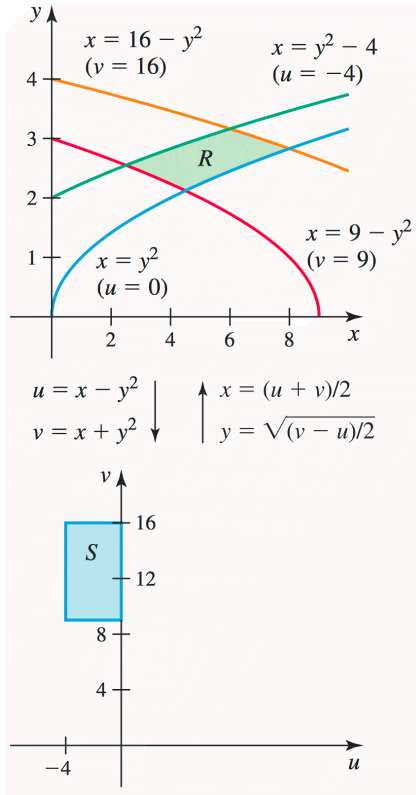


Figure 16.82

SOLUTION »

Notice that the bounding curves may be written as $x - y^2 = 0$, $x - y^2 = -4$, $x + y^2 = 9$, and $x + y^2 = 16$. The first two parabolas have the form $x - y^2 = C$, where C is a constant, which suggests the new variable $u = x - y^2$. The last two parabolas have the form $x + y^2 = C$, which suggests the new variable $v = x + y^2$. Therefore, the new variables are

$$u = x - y^2, \quad v = x + y^2.$$

The boundary curves of S are $u = -4$, $u = 0$, $v = 9$, and $v = 16$. Therefore, the new region is $S = \{(u, v) : -4 \leq u \leq 0, 9 \leq v \leq 16\}$ (Figure 16.82). To compute the Jacobian, we must find the transformation T by writing x and y in terms of u and v . Solving for x and y , and observing that $y \geq 0$ for all points in R , we find that

$$T: x = \frac{u + v}{2}, \quad y = \sqrt{\frac{v - u}{2}}.$$

The points of S satisfy $v > u$, so $\sqrt{v - u}$ is defined. Now the Jacobian may be computed:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2(v-u)}} & \frac{1}{2\sqrt{2(v-u)}} \end{vmatrix} = \frac{1}{2\sqrt{2(v-u)}}.$$

The change of variables proceeds as follows:

$$\begin{aligned}
 \iint_R y^2 \, dA &= \int_9^{16} \int_{-4}^0 \frac{v-u}{2} \frac{1}{2\sqrt{2(v-u)}} \, du \, dv && \text{Convert to an iterated integral.} \\
 &= \frac{1}{4\sqrt{2}} \int_9^{16} \int_{-4}^0 \sqrt{v-u} \, du \, dv && \text{Simplify.} \\
 &= \frac{1}{4\sqrt{2}} \frac{2}{3} \int_9^{16} \left. -(v-u)^{3/2} \right|_{-4}^0 \, dv && \text{Evaluate inner integral.} \\
 &= \frac{1}{6\sqrt{2}} \int_9^{16} \left((v+4)^{3/2} - v^{3/2} \right) \, dv && \text{Simplify.} \\
 &= \frac{1}{6\sqrt{2}} \frac{2}{5} \left. \left((v+4)^{5/2} - v^{5/2} \right) \right|_9^{16} && \text{Evaluate outer integral.} \\
 &= \frac{\sqrt{2}}{30} (32 \cdot 5^{5/2} - 13^{5/2} - 781) && \text{Simplify.} \\
 &\approx 18.79.
 \end{aligned}$$

Related Exercises 33–34 ♦

Change of Variables in Triple Integrals »

With triple integrals, we work with a transformation T of the form

$$T: x = g(u, v, w), \quad y = h(u, v, w), \quad z = p(u, v, w).$$

In this case, T maps a region S in uvw -space to a region D in xyz -space. As before, the goal is to transform the integral $\iiint_D f(x, y, z) \, dV$ into a new integral over the region S so that the integral is easier to evaluate. First, we need a Jacobian.

DEFINITION Jacobian Determinant of a Transformation of Three Variables

Given a transformation $T: x = g(u, v, w), y = h(u, v, w),$ and $z = p(u, v, w),$ where $g, h,$ and p are differentiable on a region of uvw -space, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Note »

Recall that by expanding about the first row,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

The Jacobian is evaluated as a 3×3 determinant and is a function of u , v , and w . A change of variables with respect to three variables proceeds in analogy to the two-variable case.

THEOREM 16.9 Change of Variables for Triple Integrals

Let $T: x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$ be a transformation that maps a closed bounded region S in uvw -space to a region $D = T(S)$ in xyz -space. Assume that T is one-to-one on the interior of S and that g , h , and p have continuous first partial derivatives there. If f is continuous on D , then

$$\iiint_D f(x, y, z) \, dV = \iiint_S f(g(u, v, w), h(u, v, w), p(u, v, w)) |J(u, v, w)| \, dV.$$

Note »

If we match the elements of volume in both integrals, then $dx \, dy \, dz = |J(u, v, w)| \, du \, dv \, dw$. As before, the Jacobian is a magnification (or reduction) factor, now relating the volume of a small region in xyz -space to the volume of the corresponding region in uvw -space.

To see that triple integrals in cylindrical and spherical coordinates as derived in Section 16.5 are consistent with this change of variable formulation, see Exercises 46 and 47.

EXAMPLE 6 A triple integral

Use a change of variables to evaluate $\iiint_D xz \, dV$, where D is a parallelepiped bounded by the planes

$$y = x, \quad y = x + 2, \quad z = x, \quad z = x + 3, \quad z = 0, \quad \text{and} \quad z = 4.$$

(Figure 16.83a).

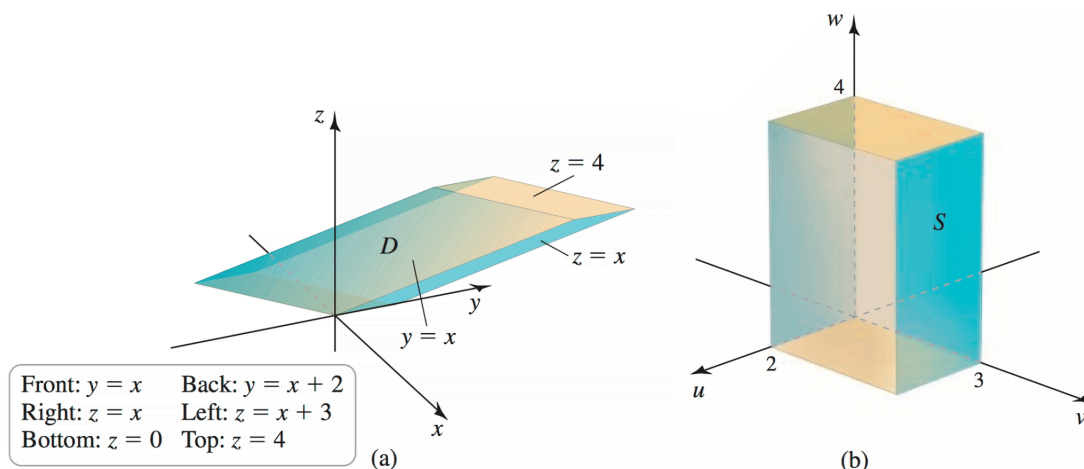


Figure 16.83

SOLUTION »

The key is to note that D is bounded by three pairs of parallel planes.

- $y - x = 0$ and $y - x = 2$

These combinations of variables suggest the new variables

$$u = y - x, \quad v = z - x, \quad \text{and} \quad w = z.$$

With this choice, the new region of integration (**Figure 16.83b**) is the rectangular box

$$S = \{(u, v, w) : 0 \leq u \leq 2, 0 \leq v \leq 3, 0 \leq w \leq 4\}.$$

To compute the Jacobian, we must express x , y , and z in terms of u , v , and w . A few steps of algebra lead to the transformation

$$T : x = w - v, \quad y = u - v + w, \quad \text{and} \quad z = w.$$

The resulting Jacobian is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Note »

It is easiest to expand the Jacobian determinant in Example 6 about the third row.

Noting that the integrand is $xz = (w - v)w = w^2 - vw$, the integral may now be evaluated:

$$\begin{aligned}
 \iiint_D xz \, dV &= \iiint_S (w^2 - vw) |J(u, v, w)| \, dV && \text{Change variables.} \\
 &= \int_0^4 \int_0^3 \int_0^2 (w^2 - vw) \frac{1}{|J(u,v,w)|} \, du \, dv \, dw && \text{Convert to an iterated integral.} \\
 &= \int_0^4 \int_0^3 2(w^2 - vw) \, dv \, dw && \text{Evaluate inner integral.} \\
 &= 2 \int_0^4 \left(vw^2 - \frac{v^2 w}{2} \right) \Big|_0^3 \, dw && \text{Evaluate middle integral.} \\
 &= 2 \int_0^4 \left(3w^2 - \frac{9w}{2} \right) \, dw && \text{Simplify.} \\
 &= 2 \left(w^3 - \frac{9w^2}{4} \right) \Big|_0^4 = 56. && \text{Evaluate outer integral.}
 \end{aligned}$$

Related Exercises 40–41 ♦

Quick Check 5 Interpret a Jacobian with a value of 1 (as in Example 6). ♦

Answer »

It means that the volume of a small region in xyz -space is unchanged when it is transformed by T to a small region in uvw -space.

Strategies for Choosing New Variables »

Sometimes a change of variables simplifies the integrand but leads to an awkward region of integration. Conversely, the new region of integration may be simplified at the expense of additional complications in the integrand. Here are a few suggestions for finding new variables of integration. The observations are made with respect to double integrals, but they also apply to triple integrals. As before, R is the original region of integration in the xy -plane and S is the new region in the uv -plane.

- 1. Aim for simple regions of integration in the uv -plane** The new region of integration in the uv -plane should be as simple as possible. Double integrals are easiest to evaluate over rectangular regions with sides parallel to the coordinate axes.
- 2. Is $(x, y) \rightarrow (u, v)$ or $(u, v) \rightarrow (x, y)$ better?** For some problems it is easiest to write (x, y) as functions of (u, v) ; in other cases the opposite is true. Depending on the problem, inverting the transformation (finding relations that go in the opposite direction) may be easy, difficult, or impossible.

Note »

- If you know (x, y) in terms of (u, v) (that is, $x = g(u, v)$ and $y = h(u, v)$), then computing the Jacobian is straightforward, as is sketching the region R given the region S . However, the transformation must be inverted to determine the shape of S .
- If you know (u, v) in terms of (x, y) (that is, $u = G(x, y)$ and $v = H(x, y)$), then sketching the region S is straightforward. However, the transformation must be inverted to compute the Jacobian.

- 3. Let the integrand suggest new variables** New variables are often chosen to simplify the integrand. For

example, the integrand $\sqrt{\frac{x-y}{x+y}}$ calls for new variables $u = x - y$ and $v = x + y$ ($u = x + y$, $v = x - y$ also

works). There is, however, no guarantee that this change of variables will simplify the region of integration. In cases in which only one combination of variables appears, let one new variable be that combination and let the other new variable be unchanged. For example, if the integrand is $(x + 4y)^{3/2}$, try letting $u = x + 4y$ and $v = y$.

- 4. Let the region suggest new variables** Example 5 illustrates an ideal situation. It occurs when the region R is bounded by two pairs of "parallel" curves in families $g(x, y) = C_1$ and $h(x, y) = C_2$ (**Figure 16.84**). In this case the new region of integration is a rectangle $S = \{(u, v) : a_1 \leq u \leq a_2, b_1 \leq v \leq b_2\}$, where $u = g(x, y)$ and $v = h(x, y)$.

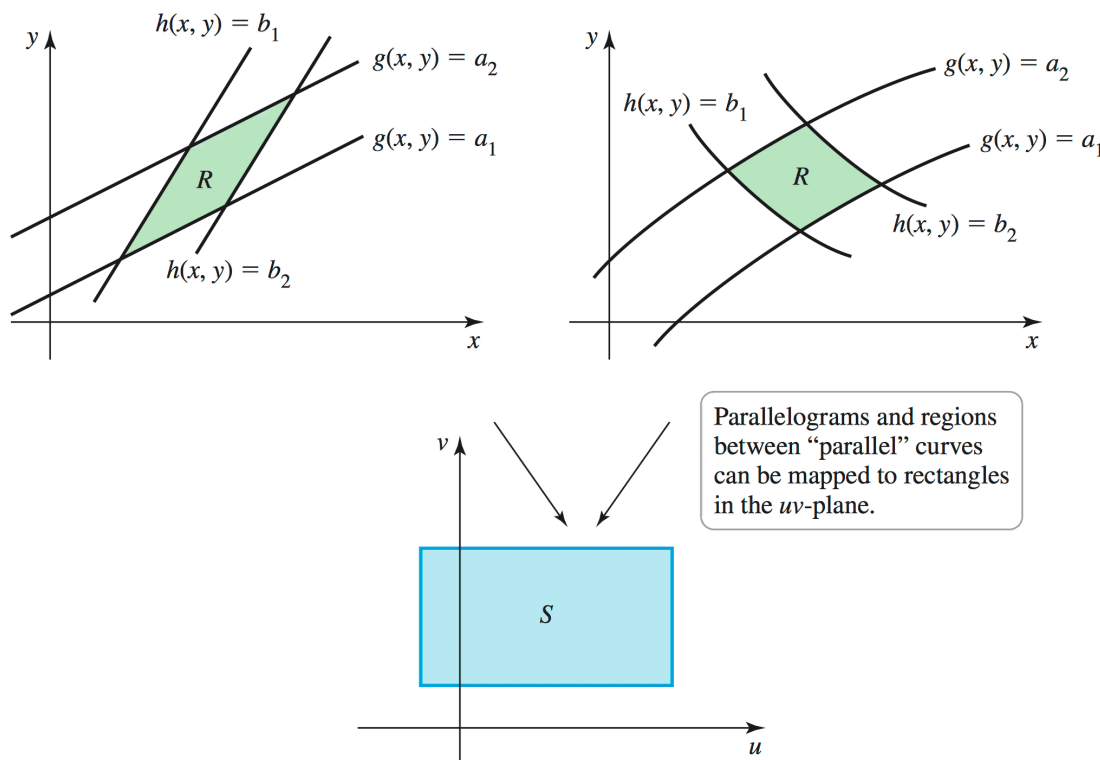


Figure 16.84

As another example, suppose the region is bounded by the lines $y = x$ (or $y/x = 1$) and $y = 2x$ (or $y/x = 2$) and by the hyperbolas $xy = 1$ and $xy = 3$. Then the new variables should be $u = xy$ and $v = y/x$ (or vice versa). The new region of integration is the rectangle $S = \{(u, v) : 1 \leq u \leq 3, 1 \leq v \leq 2\}$.

Exercises »

Getting Started »

Practice Exercises »

5–12. Transforming a square Let $S = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations.

5. $T : x = 2u, y = \frac{v}{2}$

6. $T : x = -u, y = -v$

7. $T : x = \frac{u+v}{2}, y = \frac{u-v}{2}$

8. $T : x = 2u + v, y = 2u$

9. $T : x = u^2 - v^2, y = 2uv$

10. $T : x = 2uv, y = u^2 - v^2$

11. $T : x = u \cos \pi v, y = u \sin \pi v$

12. $T : x = v \sin \pi u, y = v \cos \pi u$

13–16. Images of regions Find the image R in the xy -plane of the region S using the given transformation T . Sketch both R and S .

13. $S = \{(u, v) : v \leq 1 - u, u \geq 0, v \geq 0\}; T : x = u, y = v^2$

14. $S = \{(u, v) : u^2 + v^2 \leq 1\}; T : x = 2u, y = 4v$

15. $S = \{(u, v) : 1 \leq u \leq 3, 2 \leq v \leq 4\}; T : x = \frac{u}{v}, y = v$

16. $S = \{(u, v) : 2 \leq u \leq 3, 3 \leq v \leq 6\}; T : x = u, y = \frac{v}{u}$

17–22. Computing Jacobians Compute the Jacobian $J(u, v)$ for the following transformations.

17. $T : x = 3u, y = -3v$

18. $T : x = 4v, y = -2u$

19. $T : x = 2uv, y = u^2 - v^2$

20. $T : x = u \cos \pi v, y = u \sin \pi v$

21. $T : x = \frac{u+v}{\sqrt{2}}, y = \frac{u-v}{\sqrt{2}}$

22. $T : x = \frac{u}{v}, y = v$

23–26. Solve and compute Jacobians Solve the following relations for x and y , and compute the Jacobian $J(u, v)$.

23. $u = x + y, v = 2x - y$

24. $u = xy, v = x$

25. $u = 2x - 3y, v = y - x$

26. $u = x + 4y, v = 3x + 2y$

27–30. Double integrals—transformation given To evaluate the following integrals, carry out these steps.

- Sketch the original region of integration R in the xy -plane and the new region S in the uv -plane using the given change of variables.
- Find the limits of integration for the new integral with respect to u and v .
- Compute the Jacobian.

d. Change variables and evaluate the new integral.

27. $\iint_R xy \, dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$; use $x = u + v$,
 $y = u - v$.

28. $\iint_R x^2 y \, dA$, where $R = \{(x, y) : 0 \leq x \leq 2, x \leq y \leq x + 4\}$; use $x = 2u$, $y = 4v + 2u$.

29. $\iint_R x^2 \sqrt{x + 2y} \, dA$, where $R = \{(x, y) : 0 \leq x \leq 2, -\frac{x}{2} \leq y \leq 1 - x\}$; use $x = 2u$, $y = v - u$.

30. $\iint_R xy \, dA$, where R is bounded by the ellipse $9x^2 + 4y^2 = 36$; use $x = 2u$, $y = 3v$.

31–36. Double integrals—your choice of transformation Evaluate the following integrals using a change of variables. Sketch the original and new regions of integration, R and S .

31. $\int_0^1 \int_y^{y+2} \sqrt{x-y} \, dx \, dy$

32. $\iint_R \sqrt{y^2 - x^2} \, dA$, where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

33. $\iint_R \left(\frac{y-x}{y+2x+1} \right)^4 \, dA$, where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$,
and $y + 2x = 4$

34. $\iint_R e^{xy} \, dA$, where R is the region in the first quadrant bounded by the hyperbolas $xy = 1$ and
 $xy = 4$, and the lines $\frac{y}{x} = 1$ and $\frac{y}{x} = 3$

35. $\iint_R xy \, dA$, where R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y = 1$
and $y = 3$

36. $\iint_R (x-y) \sqrt{x-2y} \, dA$, where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and
 $x - y = 1$

37–40. Jacobians in three variables Evaluate the Jacobians $J(u, v, w)$ for the following transformations.

37. $x = v + w, y = u + w, z = u + v$

38. $x = u + v - w, y = u - v + w, z = -u + v + w$

39. $x = vw, y = uw, z = u^2 - v^2$

40. $u = x - y$, $v = x - z$, $w = y + z$ (Solve for x , y , and z first).

41–44. **Triple integrals** Use a change of variables to evaluate the following integrals.

41. $\iiint_D xy \, dV$; D is bounded by the planes $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 1$, $z = 0$, and $z = 3$.

42. $\iiint_D dV$; D is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$.

43. $\iiint_D z \, dV$; D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy -plane. Use $x = 4u \cos v$, $y = 2u \sin v$, $z = w$.

44. $\iiint_D dV$; D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy -plane. Use $x = 3u$, $y = 2v$, $z = w$.

45. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

- If the transformation $T : x = g(u, v)$, $y = h(u, v)$ is linear in u and v , then the Jacobian is a constant.
- The transformation $x = au + bv$, $y = cu + dv$ generally maps triangular regions to triangular regions.
- The transformation $x = 2v$, $y = -2u$ maps circles to circles.

46. **Cylindrical coordinates** Evaluate the Jacobian for the transformation from cylindrical coordinates (r, θ, Z) to rectangular coordinates $(x, y, z) : x = r \cos \theta$, $y = r \sin \theta$, $z = Z$. Show that $J(r, \theta, Z) = r$.

47. **Spherical coordinates** Evaluate the Jacobian for the transformation from spherical to rectangular coordinates: $x = \rho \sin \varphi \cos \theta$, $y = \rho \sin \varphi \sin \theta$, $z = \rho \cos \varphi$. Show that $J(\rho, \varphi, \theta) = \rho^2 \sin \varphi$.

48–52. **Ellipse problems** Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers. Let T be the transformation $x = au$, $y = bv$.

48. Find the area of R .

49. Evaluate $\iint_R |x|y| \, dA$.

50. Find the center of mass of the upper half of R ($y \geq 0$) assuming it has a constant density.

51. Find the average square of the distance between points of R and the origin.

52. Find the average distance between points in the upper half of R and the x -axis.

53–56. Ellipsoid problems Let D be the solid bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$, and $c > 0$ are real numbers. Let T be the transformation $x = a u$, $y = b v$, $z = c w$.

53. Find the volume of D .

54. Evaluate $\int \int \int_D |x y z| dV$.

55. Find the center of mass of the upper half of D ($z \geq 0$) assuming it has a constant density.

56. Find the average square of the distance between points of D and the origin.

Explorations and Challenges »

57. Parabolic coordinates Let T be the transformation $x = u^2 - v^2$, $y = 2 u v$.

a. Show that the lines $u = a$ in the uv -plane map to parabolas in the xy -plane that open in the negative x -direction with vertices on the positive x -axis.

b. Show that the lines $v = b$ in the uv -plane map to parabolas in the xy -plane that open in the positive x -direction with vertices on the negative x -axis.

c. Evaluate $J(u, v)$.

d. Use a change of variables to find the area of the region bounded by $x = 4 - \frac{y^2}{16}$ and $x = \frac{y^2}{4} - 1$.

e. Use a change of variables to find the area of the curved rectangle above the x -axis bounded by $x = 4 - \frac{y^2}{16}$, $x = 9 - \frac{y^2}{36}$, $x = \frac{y^2}{4} - 1$, and $x = \frac{y^2}{64} - 16$.

f. Describe the effect of the transformation $x = 2 u v$, $y = u^2 - v^2$ on horizontal and vertical lines in the uv -plane.

58. Shear transformations in \mathbb{R}^2 The transformation T in \mathbb{R}^2 given by $x = a u + b v$, $y = c v$, where a , b , and c are positive real numbers, is a *shear transformation*. Let S be the unit square $\{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$. Let $R = T(S)$ be the image of S .

a. Explain with pictures the effect of T on S .

b. Compute the Jacobian of T .

c. Find the area of R and compare it to the area of S (which is 1).

d. Assuming a constant density, find the center of mass of R (in terms of a , b , and c) and compare it to the center of mass of S (which is $(\frac{1}{2}, \frac{1}{2})$).

e. Find an analogous transformation that gives a shear in the y -direction.

59. Shear transformations in \mathbb{R}^3 The transformation T in \mathbb{R}^3 given by

$$x = a u + b v + c w, \quad y = d v + e w, \quad z = w,$$

where a , b , c , d , and e are positive real numbers, is one of many possible shear transformations in \mathbb{R}^3 . Let S be the unit cube $\{(u, v, w) : 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1\}$. Let $D = T(S)$ be the image of S .

a. Explain with pictures and words the effect of T on S .

b. Compute the Jacobian of T .

- c. Find the volume of D and compare it to the volume of S (which is 1).
- d. Assuming a constant density, find the center of mass of D and compare it to the center of mass of S (which is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$).

- 60. Linear transformations** Consider the linear transformation T in \mathbb{R}^2 given by $x = a u + b v$, $y = c u + d v$, where a, b, c , and d are real numbers, with $a d \neq b c$.
- a. Find the Jacobian of T .
 - b. Let S be the square in the uv -plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, and let $R = T(S)$. Show that $\text{area}(R) = |J(u, v)|$.
 - c. Let ℓ be the line segment joining the points P and Q in the uv -plane. Show that $T(\ell)$ (the image of ℓ under T) is the line segment joining $T(P)$ and $T(Q)$ in the xy -plane. (*Hint:* Use vectors.)
 - d. Show that if S is a parallelogram in the uv -plane and $R = T(S)$, then $\text{area}(R) = |J(u, v)| \cdot \text{area of } S$. (*Hint:* Without loss of generality, assume the vertices of S are $(0, 0)$, $(A, 0)$, (B, C) , and $(A + B, C)$, where A, B , and C are positive, and use vectors.)

- 61. Meaning of the Jacobian** The Jacobian is a magnification (or reduction) factor that relates the area of a small region near the point (u, v) to the area of the image of that region near the point (x, y) .

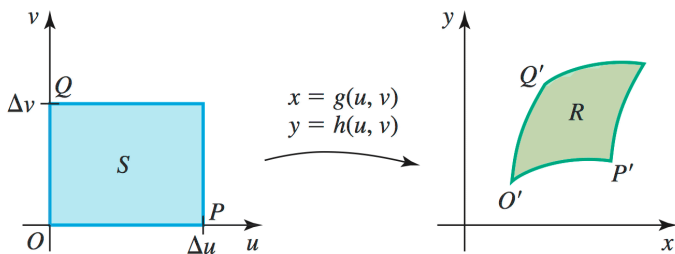
- a. Suppose S is a rectangle in the uv -plane with vertices $O(0, 0)$, $P(\Delta u, 0)$, $(\Delta u, \Delta v)$, and $Q(0, \Delta v)$ (see figure). The image of S under the transformation $x = g(u, v)$, $y = h(u, v)$ is a region R in the xy -plane. Let O' , P' and Q' be the images of O, P , and Q , respectively, in the xy -plane, where O', P' , and Q' do not all lie on the same line. Explain why the coordinates of O', P' , and Q' are $(g(0, 0), h(0, 0))$, $(g(\Delta u, 0), h(\Delta u, 0))$, and $(g(0, \Delta v), h(0, \Delta v))$, respectively.

- b. Use a Taylor series in both variables to show that

$$\begin{aligned} g(\Delta u, 0) &\approx g(0, 0) + g_u(0, 0) \Delta u, \\ g(0, \Delta v) &\approx g(0, 0) + g_v(0, 0) \Delta v, \\ h(\Delta u, 0) &\approx h(0, 0) + h_u(0, 0) \Delta u, \text{ and} \\ h(0, \Delta v) &\approx h(0, 0) + h_v(0, 0) \Delta v, \end{aligned}$$

where $g_u(0, 0)$ is $\frac{\partial x}{\partial u}$ evaluated at $(0, 0)$, with similar meanings for g_v, h_u , and h_v .

- c. Consider the parallelogram determined by the vectors $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$, two of whose sides are $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$. Use the cross product to show that the area of the parallelogram is approximately $|J(u, v)| \Delta u \Delta v$.
- d. Explain why the ratio of the area of R to the area of S is approximately $|J(u, v)|$.



- 62. Open and closed boxes** Consider the region R bounded by three pairs of parallel planes: $a x + b y = 0$, $a x + b y = 1$, $c x + d z = 0$, $c x + d z = 1$, $e y + f z = 0$, $e y + f z = 1$, where a, b, c, d, e , and f are real numbers. For the purposes of evaluating triple integrals, when do these six planes bound a finite region? Carry out the following steps.

- a. Find three vectors \mathbf{n}_1 , \mathbf{n}_2 , and \mathbf{n}_3 each of which is normal to one of the three pairs of planes.
- b. Show that the three normal vectors lie in a plane if their triple scalar product $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ is zero.
- c. Show that the three normal vectors lie in a plane if $a d e + b c f = 0$.
- d. Assuming \mathbf{n}_1 , \mathbf{n}_2 , and \mathbf{n}_3 lie in a plane P , find a vector \mathbf{N} that is normal to P . Explain why a line in the direction of \mathbf{N} does not intersect any of the six planes and therefore the six planes do not form a bounded region.
- e. Consider the change of variables $u = a x + b y$, $v = c x + d z$, $w = e y + f z$. Show that

$$J(x, y, z) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = -a d e - b c f.$$

What is the value of the Jacobian if R is unbounded?