16.3 Double Integrals in Polar Coordinates

In Chapter 12, we explored polar coordinates and saw that in certain situations they simplify problems considerably. The same is true when it comes to integration over plane regions. In this section, we learn how to formulate double integrals in polar coordinates and how to change double integrals from Cartesian coordinates to polar coordinates.

Note »

Recall the conversions from Cartesian to polar coordinates (Section 12.2):

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \text{ or}$$

$$r^2 = x^2 + y^2,\ \ \tan\theta = \frac{y}{x}.$$

Polar Rectangular Regions »

Suppose we want to find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane (**Figure 16.27**). The intersection of the paraboloid and the xy-plane ($z = 0$) is the curve $9 - x^2 - y^2 = 0$, or $x^2 + y^2 = 9$. Therefore, the region of integration R is the disk of radius 3 in the xy-plane, centered at the origin, which, when expressed in Cartesian coordinates, is $R = \left\{(x, y) : -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2},\ -3 \le x \le 3\right\}$. Using the relationship $r^2 = x^2 + y^2$ for converting Cartesian to polar coordinates, the region of integration expressed in polar coordinates is simply $R = \{(r, \theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$. Furthermore, the paraboloid expressed in polar coordinates is $z = 9 - r^2$. This problem (which is solved in Example 1) illustrates how both the integrand and the region of integration in a double integral can be simplified by working in polar coordinates.

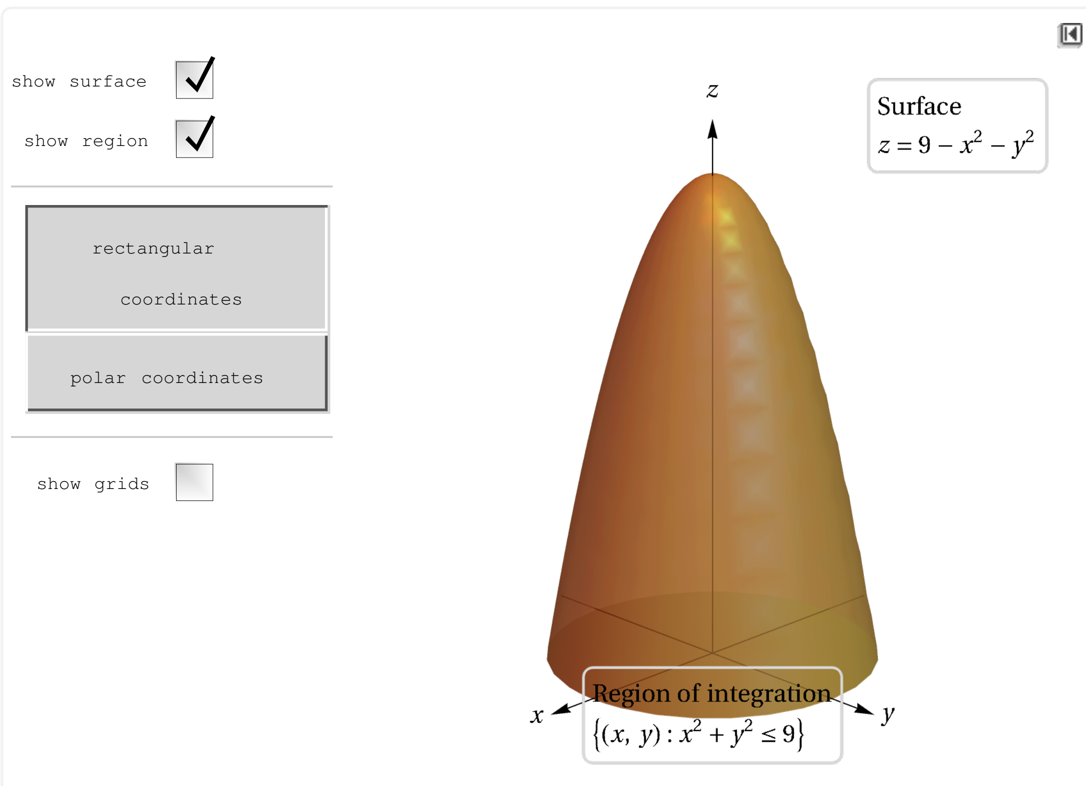


Figure 16.27

The region of integration in this problem is an example of a **polar rectangle**. In polar coordinates, it has the form $R = \{(r, \theta) : 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$ and a, b, α, and β are constants (**Figure 16.28**).

Polar rectangles are the analogs of rectangles in Cartesian coordinates. For this reason, the methods used in Section 16.1 for evaluating double integrals over rectangles can be extended to polar rectangles. The goal is to evaluate integrals of the form $\iint_R f(x, y)\, dA$, where f is a continuous function on the polar rectangle R. If f is nonnegative on R, this integral equals the volume of the solid region bounded by the surface $z = f(x, y)$ and the region R in the xy-plane.

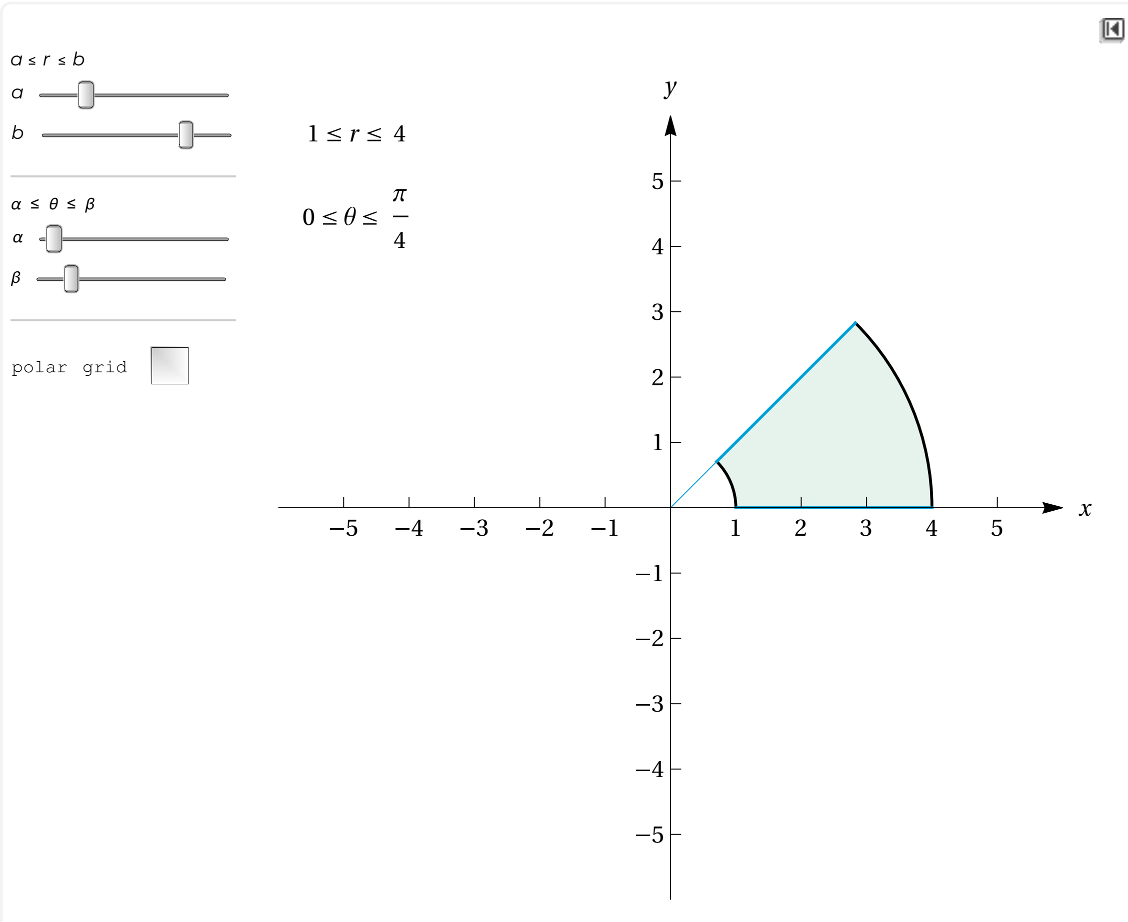


Figure 16.28

Our approach is to divide $[a, b]$ into M subintervals of equal length $\Delta r = \dfrac{b-a}{M}$. We similarly divide $[\alpha, \beta]$ into m subintervals of equal length $\Delta\theta = \dfrac{\beta - \alpha}{m}$. Now look at the arcs of the circles centered at the origin with radii

$$r = a,\ r = a + \Delta r,\ r = a + 2\,\Delta r,\ \ldots,\ r = b$$

and the rays

$$\theta = \alpha,\ \theta = \alpha + \Delta\theta,\ \theta = \alpha + 2\,\Delta\theta,\ \ldots,\ \theta = \beta$$

emanating from the origin (**Figure 16.29**). These arcs and rays divide the region R into $n = M\, m$ polar rectangles that we number in a convenient way from $k = 1$ to $k = n$. The area of the kth rectangle is denoted by ΔA_k, and we let (r_k^*, θ_k^*) be the polar coordinates of an arbitrary point in that rectangle. Note that this point also has the Cartesian coordinates $(x_k^*, y_k^*) = (r_k^* \cos \theta_k^*, r_k^* \sin \theta_k^*)$. If f is continuous on R, the volume of the solid region beneath the surface $z = f(x, y)$ and above R may be computed with Riemann sums using either ordinary rectangles (as in Sections 16.1 and 16.2) or polar rectangles. Here, we now use polar rectangles.

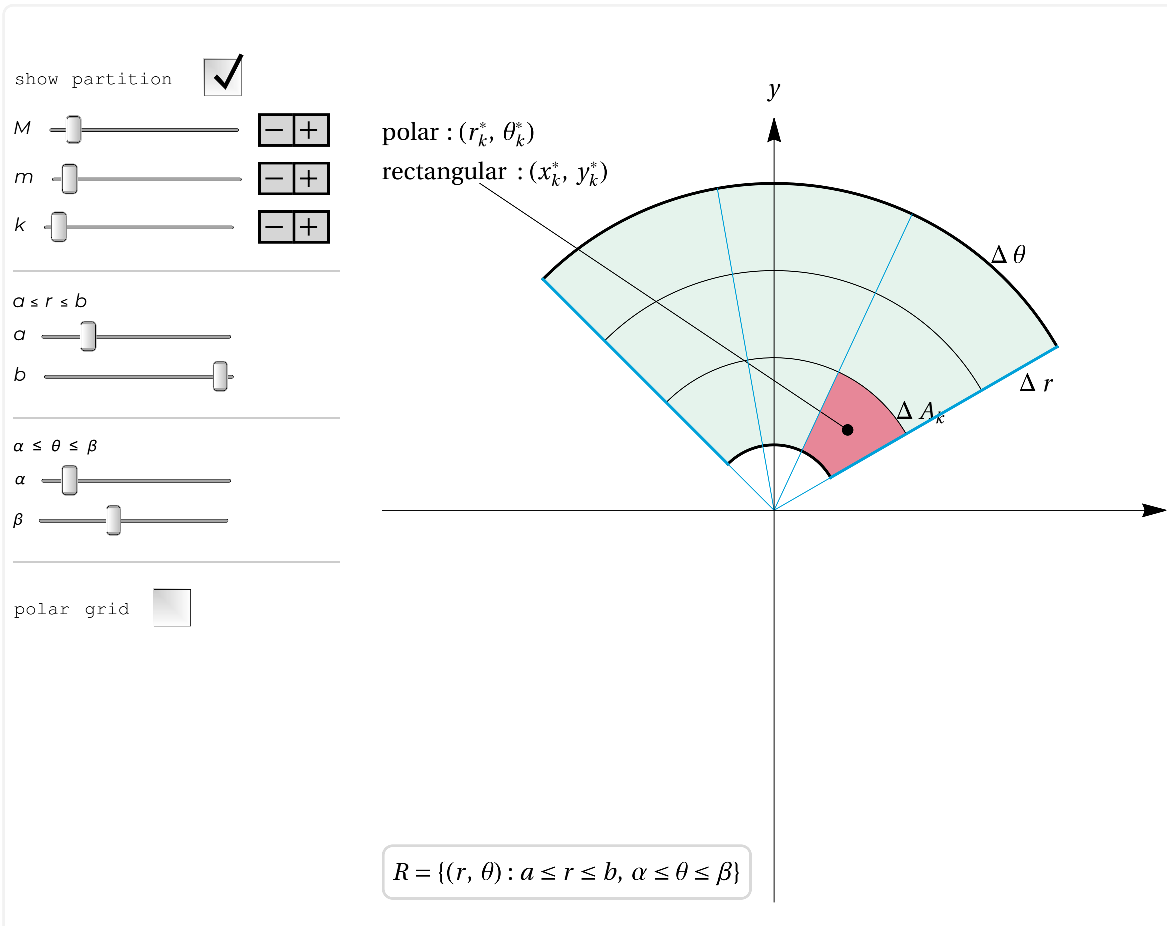


Figure 16.29

Consider the “box” whose base is the kth polar rectangle and whose height is $f(x_k^*, y_k^*)$; its volume is $f(x_k^*, y_k^*)\,\Delta A_k$, for $k = 1, \ldots, n$. Therefore, the volume of the solid region beneath the surface $z = f(x, y)$ with a base R, is approximately

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

This approximation to the volume is another Riemann sum. We let Δ be the maximum value of Δr and $\Delta\theta$. If f is continuous on R, then as $\Delta \to 0$, the sum approaches a double integral; that is,

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^* \cos \theta_k^*, r_k^* \sin \theta_k^*)\,\Delta A_k. \qquad (1)$$

The next step is to express ΔA_k in terms of Δr and $\Delta\theta$. **Figure 16.30** shows the kth polar rectangle, with an area ΔA_k. The point (r_k^*, θ_k^*) (in polar coordinates) is chosen so that the outer arc of the polar rectangle has radius $r_k^* + \dfrac{\Delta r}{2}$ and the inner arc has radius $r_k^* - \dfrac{\Delta r}{2}$. The area of the polar rectangle is

$$\begin{aligned}\Delta A_k &= (\text{area of outer sector}) - (\text{area of inner sector})\\ &= \frac{1}{2}\left(r_k^* + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k^* - \frac{\Delta r}{2}\right)^2 \Delta\theta && \text{Area of sector} = \frac{1}{2} r^2 \Delta\theta\\ &= r_k^* \Delta r \, \Delta\theta. && \text{Expand and simplify.}\end{aligned}$$

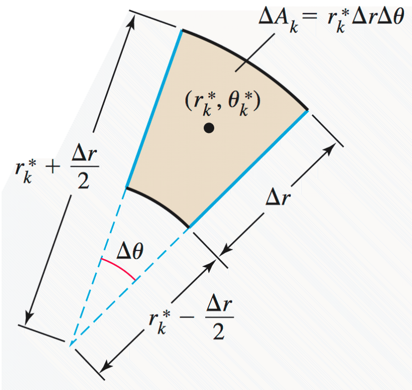
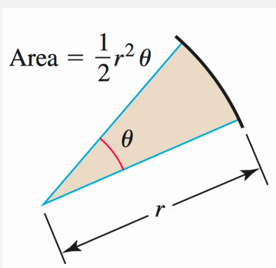


Figure 16.30

Note »

Recall that the area of a sector of a circle of radius r subtended by an angle θ is $\dfrac{1}{2} r^2 \theta$.



Substituting this expression for ΔA_k into equation (1), we have

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\, r_k^* \Delta r \, \Delta\theta.$$

This observation leads to a theorem that allows us to write a double integral in x and y as an iterated integral of $f(r\cos\theta, r\sin\theta)\, r$ in polar coordinates. It is an example of a change of variables, explained more generally in Section 16.7.

THEOREM 16.3 **Change of Variables for Double Integrals over Polar Rectangular Regions**

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as $R = \{(r, \theta): 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

Note »

The most common error in evaluating integrals in polar coordinates is to omit the factor of r that appears in the integrand. In Cartesian coordinates, the element of area is $dx\,dy$; in polar coordinates, the element of area is $r\,dr\,d\theta$, and without the factor of r, area is not measured correctly.

Quick Check 1 Describe in polar coordinates the region in the first quadrant between the circles of radius 1 and 2. ◆

Answer »

EXAMPLE 1 Volume of a paraboloid cap

Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

SOLUTION »

Using $x^2 + y^2 = r^2$, the surface is described in polar coordinates by $z = 9 - r^2$. The paraboloid intersects the xy-plane $(z = 0)$ when $z = 9 - r^2 = 0$, or $r = 3$. Therefore, the intersection curve is the circle of radius 3 centered at the origin. The resulting region of integration is the disk $R = \{(r, \theta): 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$ (**Figure 16.31**). Integrating over R in polar coordinates, the volume is

$$V = \int_0^{2\pi} \int_0^3 \underbrace{\left(9 - r^2\right)}_{z} r\, dr\, d\theta \quad \text{Iterated integral for volume}$$

$$= \int_0^{2\pi} \left(\frac{9r^2}{2} - \frac{r^4}{4} \right)\Bigg|_0^3 d\theta \quad \text{Evaluate inner integral with respect to } r.$$

$$= \int_0^{2\pi} \left(\frac{81}{4} \right) d\theta = \frac{81\pi}{2}. \quad \text{Evaluate outer integral with respect to } \theta.$$

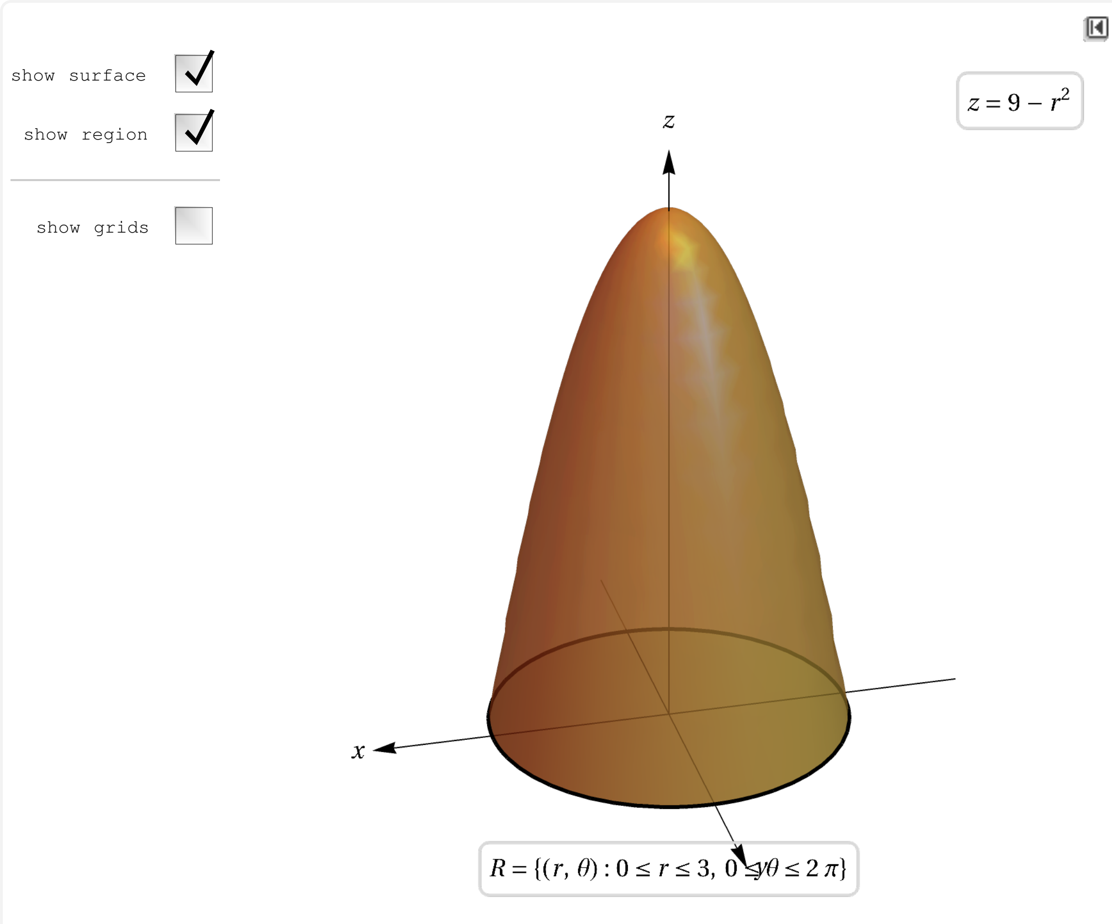


Figure 16.31

Related Exercises 12, 16 ◆

Quick Check 2 Express the functions $f(x, y) = (x^2 + y^2)^{5/2}$ and $h(x, y) = x^2 - y^2$ in polar coordinates. ◆

Answer »

r^5, $r^2(\cos^2\theta - \sin^2\theta) = r^2 \cos 2\theta$

EXAMPLE 2 Region bounded by two surfaces

Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = 2 - \sqrt{x^2 + y^2}$.

SOLUTION »

As discussed in Section 16.2, the volume of a solid bounded by two surfaces $z = f(x, y)$ and $z = g(x, y)$ over a region R in the xy-plane is given by $\iint_R (f(x, y) - g(x, y))\, dA$, where $f(x, y) \ge g(x, y)$ over R. Because the paraboloid $z = x^2 + y^2$ lies below the cone $z = 2 - \sqrt{x^2 + y^2}$ (**Figure 16.32**), the volume of the solid bounded by the surfaces is

$$V = \iint_R \left(\left(2 - \sqrt{x^2 + y^2}\right) - \left(x^2 + y^2\right)\right) dA,$$

where the boundary of R is the curve of intersection C of the surfaces projected onto the xy-plane.

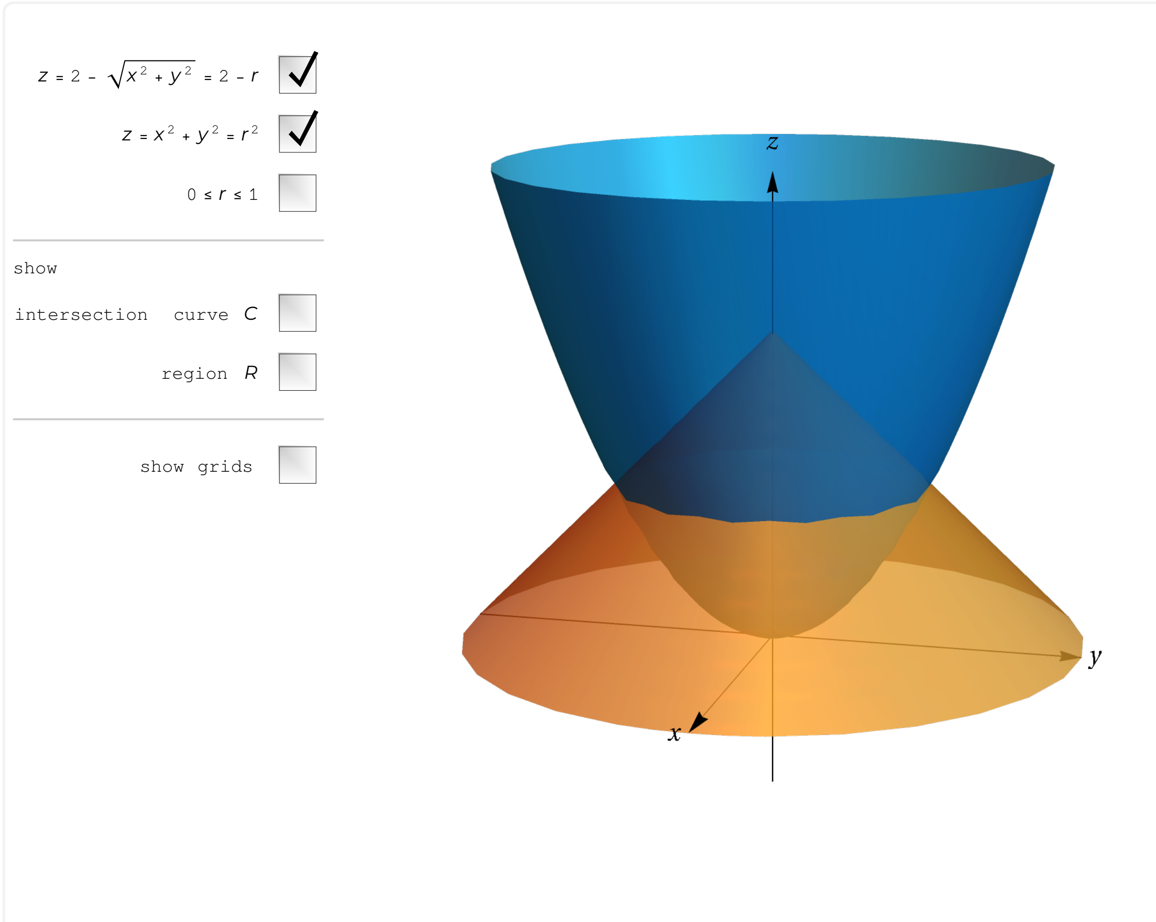


Figure 16.32

To find C, we set the equations of the surfaces equal to one another. Writing $x^2 + y^2 = 2 - \sqrt{x^2 + y^2}$ seems like a good start, but it leads to algebraic difficulties. Instead, we write the equation of the cone as $\sqrt{x^2 + y^2} = 2 - z$ and then substitute this equation into the equation for the paraboloid:

$$\begin{aligned} z &= x^2 + z^2 && \text{Parboloid} \\ z &= (2 - z)^2 && \sqrt{x^2 + y^2} = 2 - z\,(\text{cone}) \\ z^2 - 5z + 4 &= 0 && \text{Simplify.} \\ (z - 1)(z - 4) &= 0 && \text{Factor.} \\ z &= 1 \text{ or } z = 4. && \text{Solve for } z. \end{aligned}$$

The solution $z = 4$ is an extraneous root (see Quick Check 3). Setting $z = 1$ in the equation of either the paraboloid or the cone leads to $x^2 + y^2 = 1$, which is an equation of the curve C in the plane $z = 1$. Projecting C onto the xy-plane, we conclude that the region of integration (written in polar coordinates) is $R = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Converting the original volume integral into polar coordinates and evaluating it over R, we have

$$\begin{aligned} V &= \iint_R \left(\left(2 - \sqrt{x^2 + y^2}\right) - \left(x^2 + y^2\right)\right) dA && \text{Double integral for volume} \\ &= \int_0^{2\pi} \int_0^1 \left(2 - r - r^2\right) r\, dr\, d\theta && \text{Convert to polar coordinates; } x^2 + y^2 = r^2. \\ &= \int_0^{2\pi} \left(r^2 - \frac{1}{3} r^3 - \frac{1}{4} r^4\right)\Bigg|_0^1 d\theta && \text{Evaluate the inner integral.} \\ &= \int_0^{2\pi} \frac{5}{12}\, d\theta = \frac{5\pi}{6}. && \text{Evaluate the outer integral.} \end{aligned}$$

Related Exercises 33, 40 ◆

Quick Check 3 Give a geometric explanation for the extraneous root $z = 4$ found in Example 2. ◆

Answer »

$z = 2 - \sqrt{x^2 + y^2}$ is the lower half of the double-napped cone $(2 - z)^2 = x^2 + y^2$. Imagine both halves of this cone in Figure 16.32: It is apparent that the paraboloid $z = x^2 + y^2$ intersects the cone twice, once when $z = 1$ and once when $z = 4$.

EXAMPLE 3 Annular region

Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta) : 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$. (An *annulus* is the region between two concentric circles.)

SOLUTION »

The region of integration suggests working in polar coordinates (**Figure 16.33**).

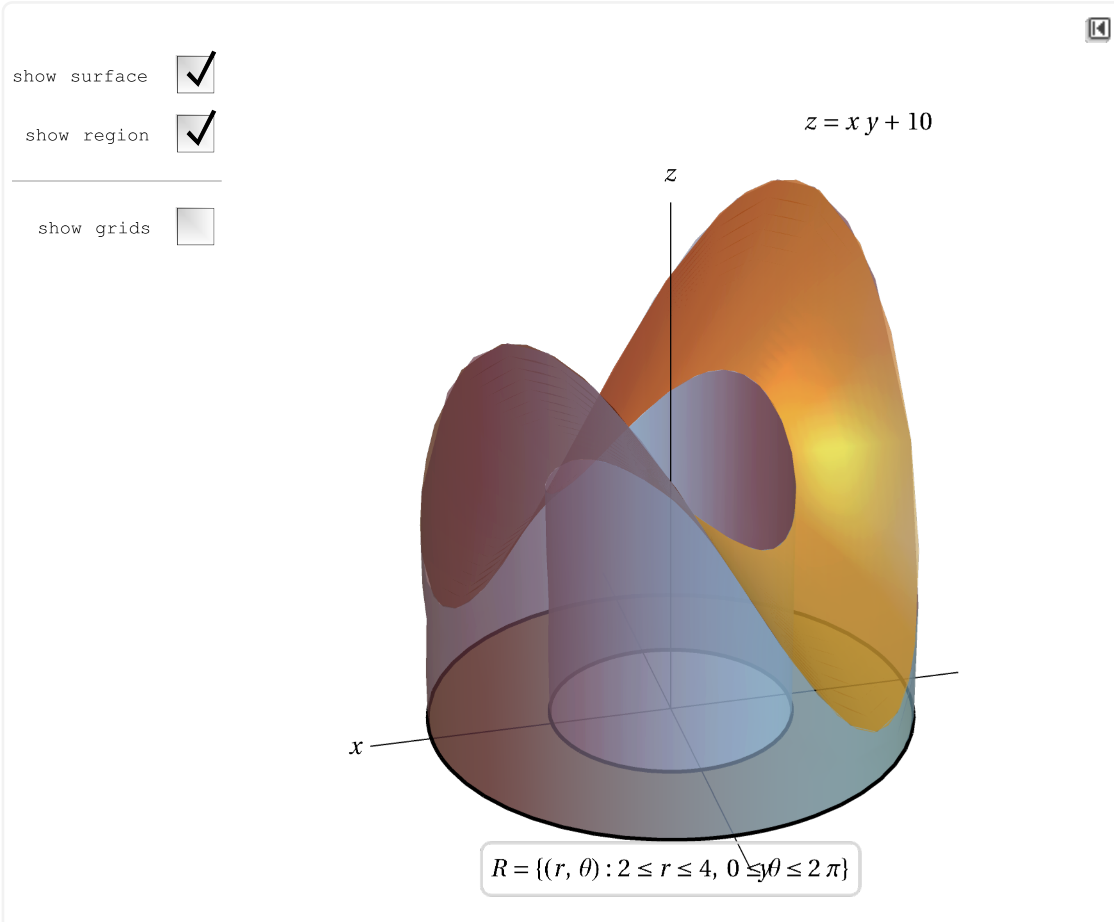


Figure 16.33

Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the integrand becomes

$$\begin{aligned} x\,y + 10 &= (r\cos\theta)(r\sin\theta) + 10 && \text{Substitute for } x \text{ and } y. \\ &= r^2 \sin\theta\cos\theta + 10 && \text{Simplify.} \\ &= \frac{1}{2} r^2 \sin 2\theta + 10. && \sin 2\theta = 2\sin\theta\cos\theta \end{aligned}$$

Substituting the integrand into the volume integral, we have

$$\begin{aligned} V &= \int_0^{2\pi}\int_2^4 \left(\frac{1}{2} r^2 \sin 2\theta + 10\right) r\,dr\,d\theta && \text{Iterated integral for volume} \\ &= \int_0^{2\pi}\int_2^4 \left(\frac{1}{2} r^3 \sin 2\theta + 10\,r\right) dr\,d\theta && \text{Simplify.} \\ &= \int_0^{2\pi} \left(\frac{r^4}{8} \sin 2\theta + 5\,r^2\right)\Bigg|_2^4 d\theta && \text{Evaluate inner integral with respect to } r. \\ &= \int_0^{2\pi} (30\sin 2\theta + 60)\,d\theta && \text{Simplify.} \\ &= (15(-\cos 2\theta) + 60\,\theta)\big|_0^{2\pi} = 120\,\pi. && \text{Evaluate outer integral with respect to } \theta. \end{aligned}$$

Related Exercises 22, 38 ◆

More General Polar Regions »

In Section 16.2 we generalized double integrals over rectangular regions to double integrals over nonrectangular regions. In an analogous way, the method for integrating over a polar rectangle may be extended to more general regions. Consider a region bounded by two rays $\theta = \alpha$ and $\theta = \beta$, where $\beta - \alpha \le 2\pi$, and two curves $r = g(\theta)$ and $r = h(\theta)$ (**Figure 16.34**):

$$R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta\}.$$

The double integral $\iint_R f(r, \theta)\, dA$ is expressed as an iterated integral in which the inner integral has limits $r = g(\theta)$ and $r = h(\theta)$, and the outer integral runs from $\theta = \alpha$ to $\theta = \beta$. If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(r, \theta)$ and R.

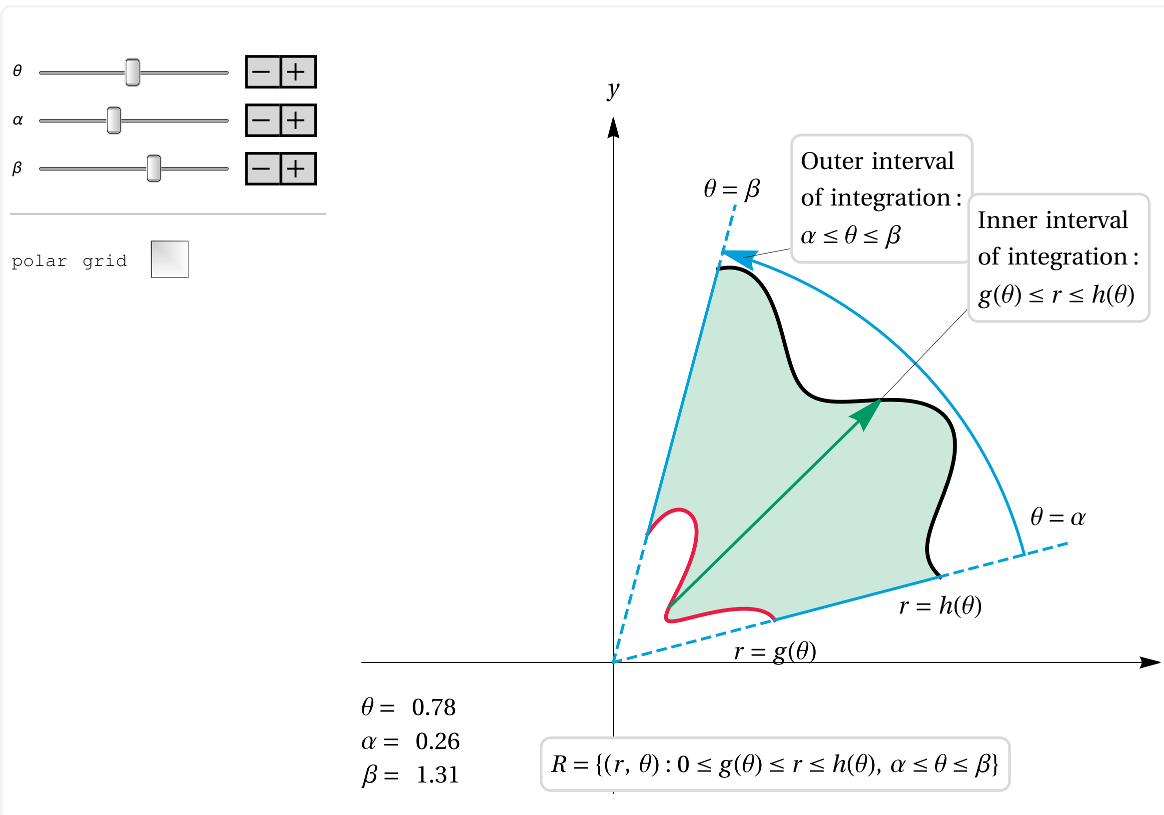


Figure 16.34

THEOREM 16.4 **Change of Variables for Double Integrals over More General Polar Regions**

Let f be continuous on the region in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta\},$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Note »

EXAMPLE 4 Specifying regions

Write an iterated integral for $\iint_R g(r, \theta)\, dA$ for the following regions R in the xy-plane.

a. The region outside the circle $r = 2$ (with radius 2 centered at $(0, 0)$) and inside the circle $r = 4\cos\theta$ (with radius 2 centered at $(2, 0)$)

b. The region inside both circles of part (a)

Note »

Recall from Section 12.2 that the polar equation $r = 2a\sin\theta$ describes a circle of radius $|a|$ with center $(0, a)$. The polar equation $r = 2a\cos\theta$ describes a circle of radius $|a|$ with center $(a, 0)$.

SOLUTION »

a. Equating the two expressions for r, we have $4\cos\theta = 2$ or $\cos\theta = \frac{1}{2}$, so the circles intersect when $\theta = \pm\frac{\pi}{3}$ (**Figure 16.35**). The inner boundary of R is the circle $r = 2$, and the outer boundary is the circle $r = 4\cos\theta$. Therefore, the region of integration is $R = \left\{(r, \theta) : 2 \le r \le 4\cos\theta,\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}\right\}$ and the iterated integral is

$$\iint_R g(r, \theta)\, dA = \int_{-\pi/3}^{\pi/3} \int_2^{4\cos\theta} g(r, \theta)\, r\, dr\, d\theta.$$

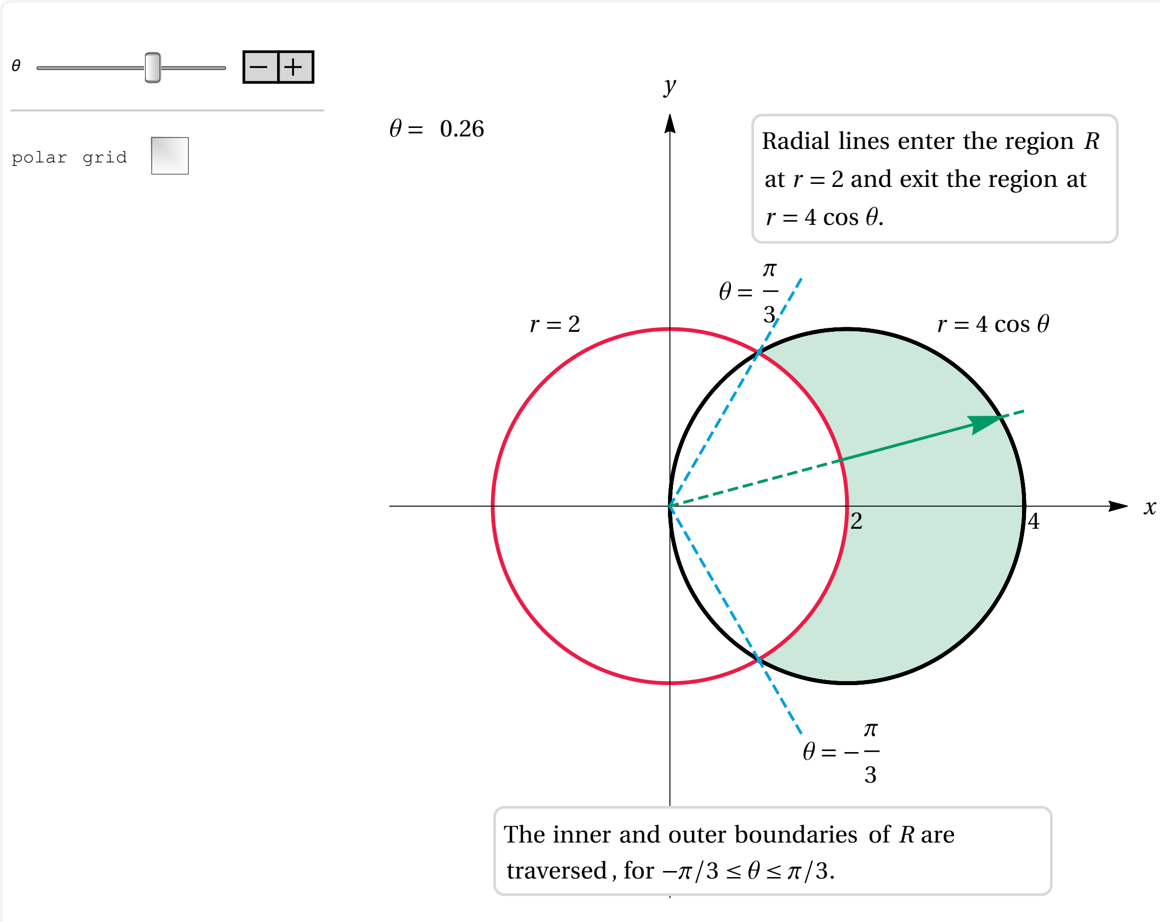


Figure 16.35

b. From part (a) we know that the circles intersect when $\theta = \pm\frac{\pi}{3}$. The region R consists of three subregions R_1, R_2, and R_3 (**Figure 16.36**).

- For $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{3}$, R_1 is bounded by $r = 0$ (inner curve) and $r = 4\cos\theta$ (outer curve).

Therefore, the double integral is expressed in three parts:

$$\iint_R g(r,\theta)\,dA = \int_{-\pi/2}^{-\pi/3}\int_0^{4\cos\theta} g(r,\theta)\,r\,dr\,d\theta + \int_{-\pi/3}^{\pi/3}\int_0^{2} g(r,\theta)\,r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{4\cos\theta} g(r,\theta)\,r\,dr\,d\theta$$

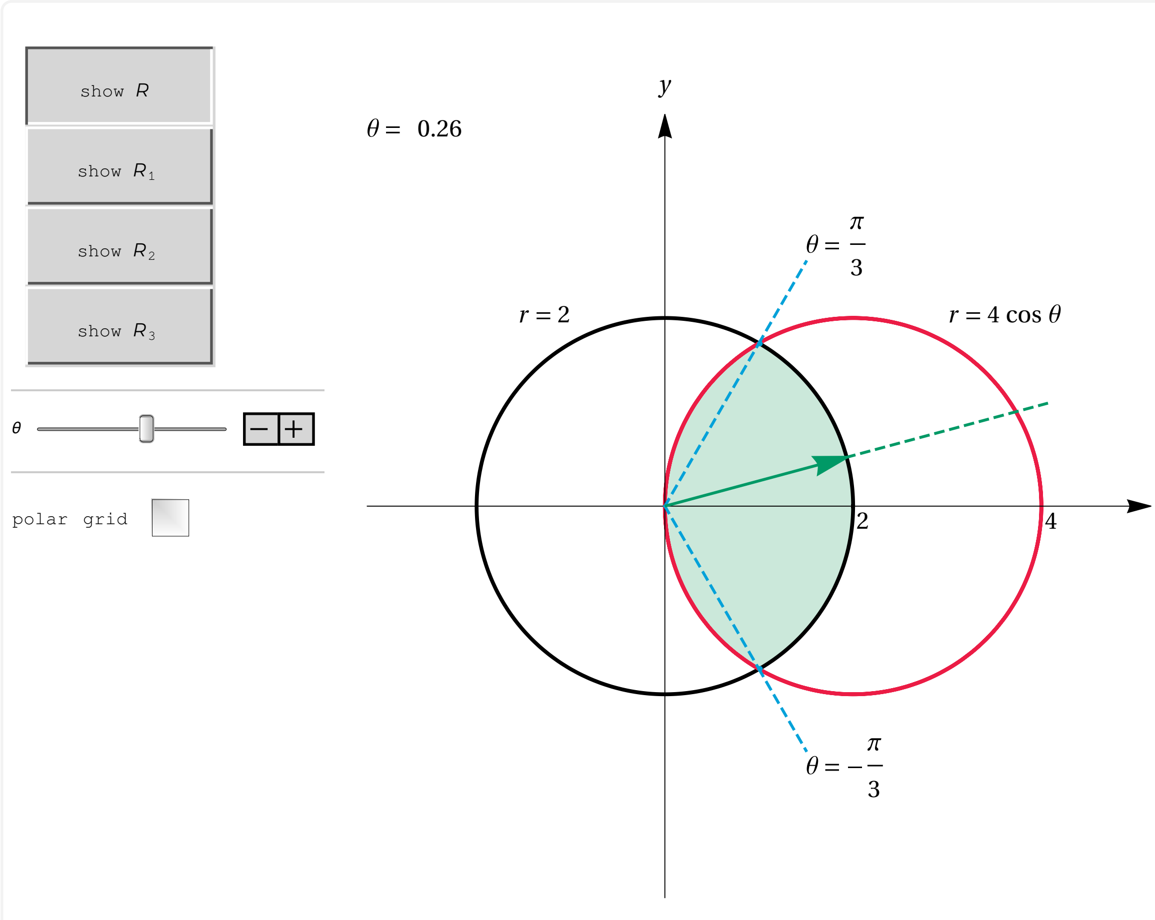


Figure 16.36

Related Exercise 44 ◆

Areas of Regions »

In Cartesian coordinates, the area of a region R in the xy-plane is computed by integrating the function $f(x, y) = 1$ over R; that is, $A = \iint_R dA$. This fact extends to polar coordinates.

Area of Polar Regions

The area of the region $R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$, is

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

Note »

Do not forget the factor of r in the area integral!

EXAMPLE 5 **Area within a lemniscate**

Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos 2\theta$ (**Figure 16.37**).

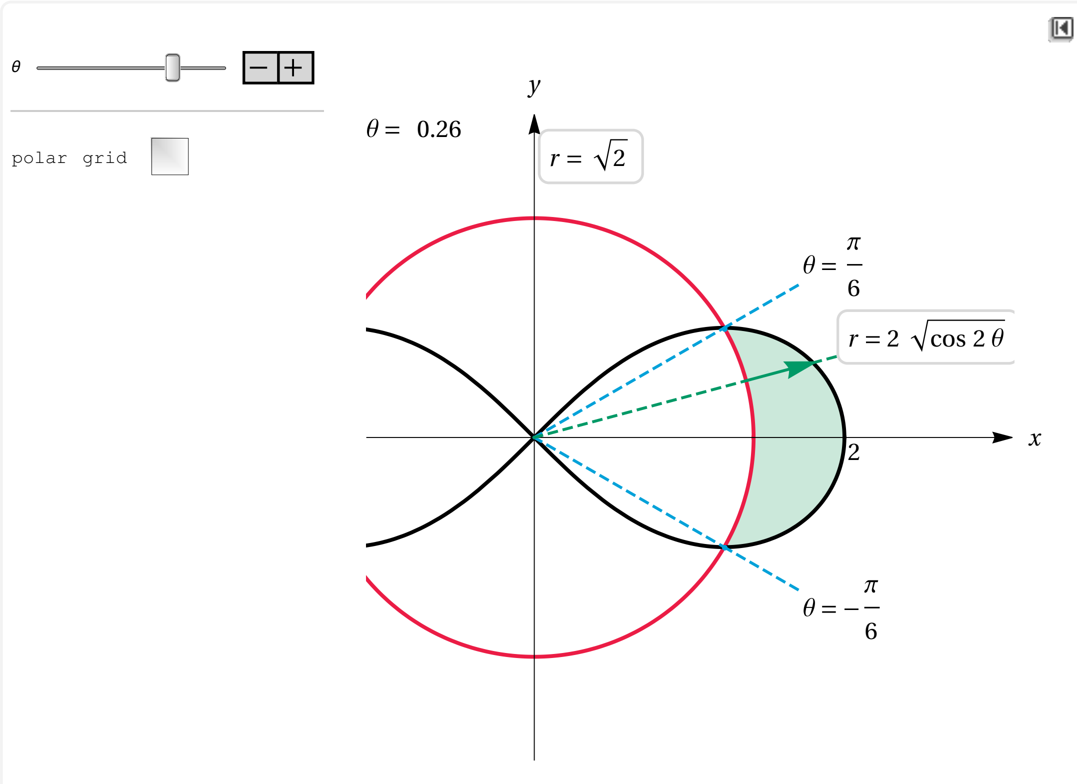


Figure 16.37

SOLUTION »

The equation of the circle can be written as $r^2 = 2$. Equating the two expressions for r^2, the circle and the lemniscate intersect when $2 = 4\cos 2\theta$, or $\cos 2\theta = \frac{1}{2}$. The angles in the first and fourth quadrants that satisfy this equation are $\theta = \pm\frac{\pi}{6}$ (Figure 16.37). The region between the two curves is bounded by the inner curve $r = g(\theta) = \sqrt{2}$ and the outer curve $r = h(\theta) = 2\sqrt{\cos 2\theta}$. Therefore, the area of the region is

$$A = \int_{-\pi/6}^{\pi/6}\int_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\left(\frac{r^2}{2}\right)\Bigg|_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} d\theta \quad \text{Evaluate inner integral with respect to } r.$$

$$= \int_{-\pi/6}^{\pi/6}(2\cos 2\theta - 1)\,d\theta \quad \text{Simplify.}$$

$$= (\sin 2\theta - \theta)\Big|_{-\pi/6}^{\pi/6} \quad \text{Evaluate outer integral with respect to } \theta.$$

$$= \sqrt{3} - \frac{\pi}{3}. \quad \text{Simplify.}$$

Related Exercises 50–51 ◆

Quick Check 4 Express the area of the disk $R = \{(r, \theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ in terms of a double integral in polar coordinates. ◆

Answer »

$$\int_0^{2\pi} \int_0^a r\, dr\, d\theta = \pi a^2$$

Average Value over a Planar Polar Region »

We have encountered the average value of a function in several different settings. To find the average value of a function over a region in polar coordinates, we again integrate the function over the region and divide by the area of the region.

EXAMPLE 6 Average y-coordinate

Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta) : 0 \le r \le a,\ 0 \le \theta \le \pi\}$.

SOLUTION »

The double integral that gives the average value we seek is $\bar{y} = \dfrac{1}{\text{area of } R} \iint_R y\, dA$. We use the fact that the area of R is $\dfrac{\pi a^2}{2}$ and the y-coordinates of points in the semicircular disk are given by $y = r \sin\theta$. Evaluating the average value integral we find that

$$\begin{aligned}
\bar{y} &= \frac{1}{\pi a^2/2} \int_0^{\pi} \int_0^a r \sin\theta\, r\, dr\, d\theta \\
&= \frac{2}{\pi a^2} \int_0^{\pi} \sin\theta \left(\frac{r^3}{3}\right)\Bigg|_0^a d\theta && \text{Evaluate inner integral with respect to } r. \\
&= \frac{2}{\pi a^2} \frac{a^3}{3} \int_0^{\pi} \sin\theta\, d\theta && \text{Simplify.} \\
&= \frac{2a}{3\pi} (-\cos\theta)\Big|_0^{\pi} && \text{Evaluate outer integral with respect to } \theta. \\
&= \frac{4a}{3\pi}. && \text{Simplify.}
\end{aligned}$$

Note that $\dfrac{4}{3\pi} \approx 0.42$, so the average value of the y-coordinates is less than half the radius of the disk.

Related Exercise 53 ◆

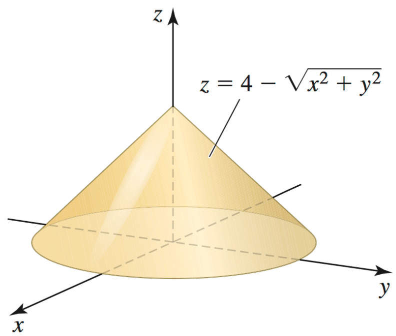
Exercises »

Getting Started »

Practice Exercises »

11–14. Volume of solids *Find the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane.*

11. $f(x, y) = 4 - \sqrt{x^2 + y^2}$



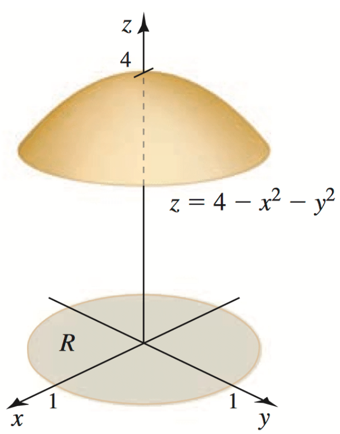
12. $f(x, y) = 16 - 4\left(x^2 + y^2\right)$

13. $f(x, y) = e^{-(x^2+y^2)/8} - e^{-2}$

14. $f(x, y) = \dfrac{20}{1 + x^2 + y^2} - 2$

15–18. Solids bounded by paraboloids *Find the volume of the solid below the paraboloid* $z = 4 - x^2 - y^2$ *and above the following polar rectangles.*

15. $R = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$



16. $R = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

17. $R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

18. $R = \left\{(r, \theta) : 1 \le r \le 2,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}\right\}$

19–20. Solids bounded by hyperboloids *Find the volume of the solid below the hyperboloid* $z = 5 - \sqrt{1 + x^2 + y^2}$ *and above the following polar rectangles.*

19. $R = \left\{(r, \theta) : \sqrt{3} \le r \le 2\sqrt{2},\ 0 \le \theta \le 2\pi\right\}$

20. $R = \left\{(r, \theta): \sqrt{3} \le r \le \sqrt{15},\ -\frac{\pi}{2} \le \theta \le \pi\right\}$

21–30. Cartesian to polar coordinates *Evaluate the the following integrals using polar coordinates. Assume* (r, θ) *are polar coordinates. A sketch is helpful.*

21. $\displaystyle\iint_R \left(x^2 + y^2\right) dA$; $R = \{(r, \theta): 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

22. $\displaystyle\iint_R 2xy\, dA$; $R = \{(r, \theta): 1 \le r \le 3,\ 0 \le \theta \le \pi/2\}$

23. $\displaystyle\iint_R 2xy\, dA$; $R = \left\{(x, y): x^2 + y^2 \le 9,\ y \ge 0\right\}$

24. $\displaystyle\iint_R \frac{dA}{1 + x^2 + y^2}$; $R = \{(r, \theta): 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

25. $\displaystyle\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}$; $R = \left\{(x, y): x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0\right\}$

26. $\displaystyle\iint_R e^{-x^2 - y^2}\, dA$; $R = \left\{(x, y): x^2 + y^2 \le 9\right\}$

27. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2 + y^2\right)^{3/2} dy\, dx$

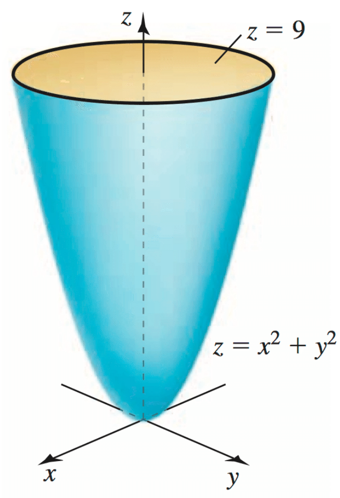
28. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\ dy\, dx$

29. $\displaystyle\iint_R \sqrt{x^2 + y^2}\ dA$; $R = \left\{(x, y): 1 \le x^2 + y^2 \le 4\right\}$

30. $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} \left(16 - x^2 - y^2\right) dx\, dy$

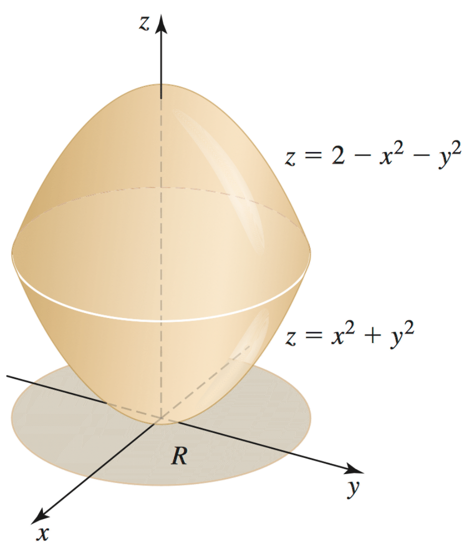
31–40. Volume between surfaces *Find the volume of the following solids.*

31. The solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

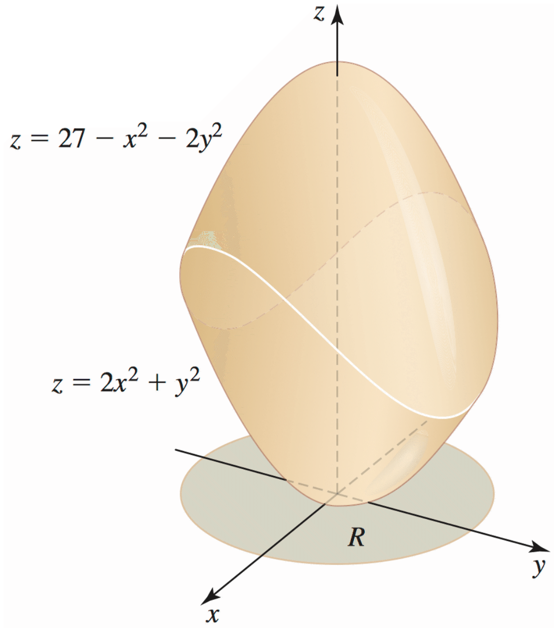


32. The solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$

33. The solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

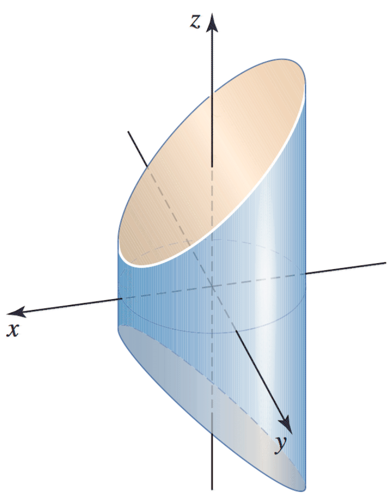


34. The solid bounded by the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$



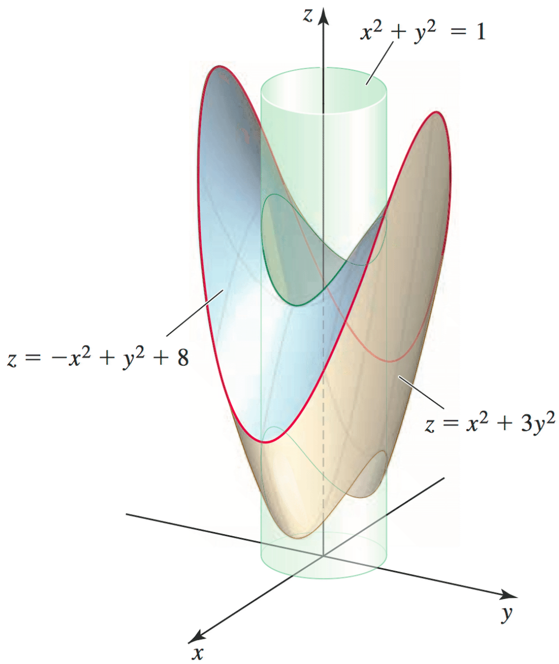
35. The solid bounded below by the paraboloid $z = x^2 + y^2 - x - y$ and above by the plane $x + y + z = 4$

36. The solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 3 - x$ and $z = x - 3$

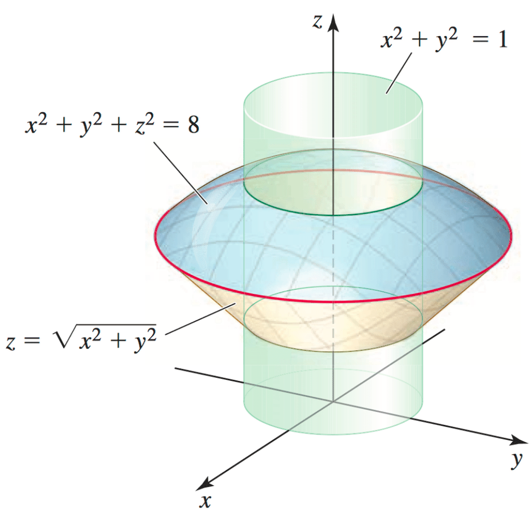


37. The solid bounded by the paraboloid $z = 18 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$

38. The solid outside the cylinder $x^2 + y^2 = 1$ that is bounded above by the hyperbolic paraboloid $z = -x^2 + y^2 + 8$ and below by the paraboloid $z = x^2 + 3y^2$



39. The solid outside the cylinder $x^2 + y^2 = 1$ that is bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the cone $z = \sqrt{x^2 + y^2}$



40. The solid bounded by the cone $z = 2 - \sqrt{x^2 + y^2}$ and the upper half of a hyperboloid of two sheets $z = \sqrt{1 + x^2 + y^2}$

41–46. Describing general regions *Sketch the following regions R. Then express* $\int\int_R g(r, \theta)\, dA$ *as an iterated integral over R in polar coordinates.*

41. The region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

42. The region inside the leaf of the rose $r = 2\sin 2\theta$ in the first quadrant

43. The region inside the lobe of the lemniscate $r^2 = 2\sin 2\theta$ in the first quadrant

44. The region outside the circle $r = 2$ and inside the circle $r = 4\sin\theta$

45. The region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in the first quadrant

46. The region outside the circle $r = \frac{1}{2}$ and inside the cardioid $r = 1 + \cos\theta$

47–52. Computing areas *Use a double integral to find the area of the following regions.*

47. The annular region $\{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

48. The region bounded by the cardioid $r = 2(1 - \sin\theta)$

49. The region bounded by all leaves of the rose $r = 2\cos 3\theta$

50. The region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

51. The region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

52. The region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis

53–54. Average values *Find the following average values.*

53. The average distance between points of the disk $\{(r, \theta) : 0 \le r \le a\}$ and the origin

54. The average value of $\frac{1}{r^2}$ over the annulus $\{(r, \theta) : 2 \le r \le 4\}$

55. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. Let R be the unit disk centered at (0, 0). Then $\iint_R (x^2 + y^2)\, dA = \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta.$

b. The average distance between the points of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the origin is 2 (calculus not required).

c. The integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ is easier to evaluate in polar coordinates than in Cartesian coordinates.

56. **Areas of circles** Use integration to show that the circles $r = 2a\cos\theta$ and $r = 2a\sin\theta$ have the same area, which is πa^2.

57. **Filling bowls with water** Which bowl holds more water if it is filled to a depth of 4 units?

- The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$
- The cone $z = \sqrt{x^2 + y^2}$, for $0 \le z \le 4$
- The hyperboloid $z = \sqrt{1 + x^2 + y^2}$, for $1 \le z \le 5$

58. **Equal volumes** To what height (above the bottom of the bowl) must the cone and paraboloid bowls of Exercise 57 be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$?

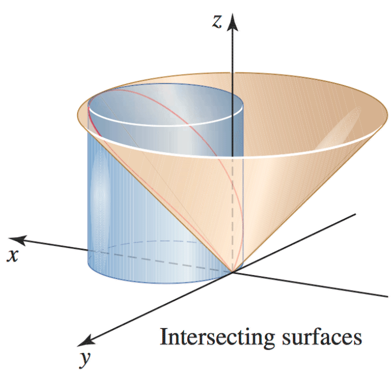
59. **Volume of a hyperbolic paraboloid** Consider the surface $z = x^2 - y^2$.

a. Find the region in the xy-plane in polar coordinates for which $z \ge 0$.

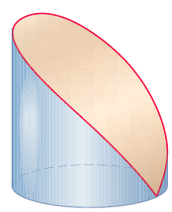
b. Let $R = \left\{(r, \theta) : 0 \le r \le a, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region R.

60. **Volume of a sphere** Use double integrals in polar coordinates to verify that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

61. **Volume** Find the volume of the solid bounded by the cylinder $(x-1)^2 + y^2 = 1$, the plane $z = 0$, and the cone $z = \sqrt{x^2 + y^2}$ (see figure). (*Hint:* Use symmetry.)



Intersecting surfaces


Corresponding solid

62. **Volume** Find the volume of the solid bounded by the paraboloid $z = 2x^2 + 2y^2$, the plane $z = 0$, and the cylinder $x^2 + (y-1)^2 = 1$. (*Hint:* Use symmetry.)

Explorations and Challenges »

63–64. Miscellaneous integrals *Evaluate the following integrals using the method of your choice. A sketch is helpful.*

63. $\displaystyle\iint_R \frac{dA}{4 + \sqrt{x^2 + y^2}}$; $R = \left\{(r, \theta) : 0 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$

64. $\displaystyle\iint_R \frac{x - y}{x^2 + y^2 + 1}\, dA$; R is the region bounded by the unit circle centered at the origin.

65–68. Improper integrals *Improper integrals arise in polar coordinates when the radial coordinate r becomes arbitrarily large. Under certain conditions, these integrals are treated in the usual way:*

$$\int_\alpha^\beta \int_a^\infty g(r, \theta)\, r\, dr\, d\theta = \lim_{b \to \infty} \int_\alpha^\beta \int_a^b g(r, \theta)\, r\, dr\, d\theta.$$

Use this technique to evaluate the following integrals.

65. $\displaystyle\int_0^{\pi/2}\int_1^{\infty}\frac{\cos\theta}{r^3}\,r\,dr\,d\theta$

66. $\displaystyle\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}};\ R=\{(r,\theta): 1\le r<\infty,\ 0\le\theta\le 2\pi\}$

67. $\displaystyle\iint_R e^{-x^2-y^2}\,dA;\ R=\left\{(r,\theta): 0\le r<\infty,\ 0\le\theta\le\frac{\pi}{2}\right\}$

68. $\displaystyle\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2}$; R is the first quadrant.

T 69. Slicing a hemispherical cake A cake is shaped like a hemisphere of radius 4 with its base on the xy-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the xy-plane and separated by an angle of φ.

a. Use a double integral to find the volume of the slice for $\varphi=\frac{\pi}{4}$. Use geometry to check your answer.

b. Now suppose the cake is sliced horizontally at $z=a>0$ and let D be the piece of cake above the plane $z=a$. For what approximate value of a is the volume of D equal to the volume in part (a)?

T 70. Mass from density data The following table gives the density (in units of g/cm^2) at selected points (in polar coordinates) of a thin semicircular plate of radius 3. Estimate the mass of the plate and explain your method.

	$\theta=0$	$\theta=\pi/4$	$\theta=\pi/2$	$\theta=3\pi/4$	$\theta=\pi$
$r=1$	2.0	2.1	2.2	2.3	2.4
$r=2$	2.5	2.7	2.9	3.1	3.3
$r=3$	3.2	3.4	3.5	3.6	3.7

71. A mass calculation Suppose the density of a thin plate represented by the polar region R is $\rho(r,\theta)$ (in units of mass per area). The mass of the plate is $\iint_R \rho(r,\theta)\,dA$. Find the mass of the thin half annulus $R=\{(r,\theta): 1\le r\le 4,\ 0\le\theta\le\pi\}$ with a density $\rho(r,\theta)=4+r\sin\theta$.

72. Area formula In Section 12.3 it was shown that the area of a region enclosed by the polar curve $r=g(\theta)$ and the rays $\theta=\alpha$ and $\theta=\beta$, where $\beta-\alpha\le 2\pi$, is $A=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$. Prove this result using the area formula with double integrals.

73. Normal distribution An important integral in statistics associated with the normal distribution is $I=\int_{-\infty}^{\infty}e^{-x^2}\,dx$. It is evaluated in the following steps.

a. In Section 8.9, it is shown that $\int_0^{\infty} e^{-x^2}\,dx$ converges (in the narrative following Example 7). Use this result to explain why $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

b. Assume

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy,$$

where we have chosen the variables of integration to be x and y and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

c. Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} x\,e^{-x^2}\,dx$, and $\int_0^{\infty} x^2\,e^{-x^2}\,dx$ (using part (a) if needed).

74. Existence of integrals For what values of p does the integral $\iint_R \frac{dA}{\left(x^2+y^2\right)^p}$ exist in the following cases? Assume (r,θ) are polar coordinates.

a. $R = \{(r,\theta) : 1 \le r < \infty,\ 0 \le \theta \le 2\pi\}$

b. $R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

75. Integrals in strips Consider the integral

$$I = \iint_R \frac{dA}{\left(1+x^2+y^2\right)^2},$$

where $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le a\}$.

a. Evaluate I for $a = 1$. (*Hint*: Use polar coordinates.)

b. Evaluate I for arbitrary $a > 0$.

c. Let $a \to \infty$ in part (b) to find I over the infinite strip $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y < \infty\}$.