16.2 Double Integrals over General Regions

Evaluating double integrals over rectangular regions is a useful place to begin our study of multiple integrals. Problems of practical interest, however, usually involve nonrectangular regions of integration. The goal of this section is to extend the methods presented in Section 16.1 so that they apply to more general regions of integration.

General Regions of Integration »

Consider a continuous function f defined over a closed bounded *nonrectangular* region R in the xy-plane. As with rectangular regions, we use a partition consisting of rectangles, but now, such a partition does not cover R exactly. In this case, only the n rectangles that lie entirely within R are considered to be in the partition (**Figure 16.9**). When f is nonnegative on R, the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane over R is approximated by the Riemann sum

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k,$$

where $\Delta A_k = \Delta x_k\,\Delta y_k$ is the area of the kth rectangle and (x_k^*, y_k^*) is any point in the kth rectangle, for $1 \le k \le n$. As before, we define Δ to be the maximum length of the diagonals of the rectangles in the partition.

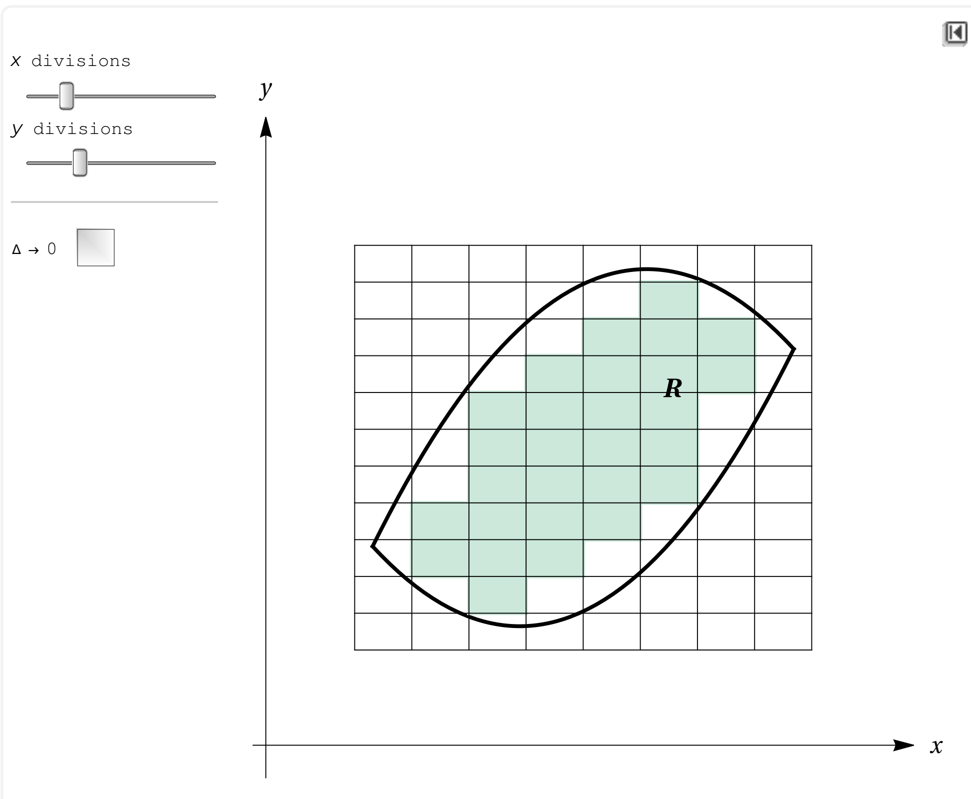


Figure 16.9

Under the assumptions that f is continuous on R and that the boundary of R consists of a finite number of smooth curves, two things occur as $\Delta \to 0$ and the number of rectangles increases $(n \to \infty)$.

- The rectangles in the partition fill R more and more completely; that is, the union of the rectangles approaches R.
- Over all partitions and all choices of (x_k^*, y_k^*) within a partition, the Riemann sums approach a (unique) limit.

The limit approached by the Riemann sums is the **double integral of *f* over *R***; that is

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k.$$

When this limit exists, f is **integrable** over R. If f is nonnegative on R, then the double integral equals the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane over R (**Figure 16.10**).

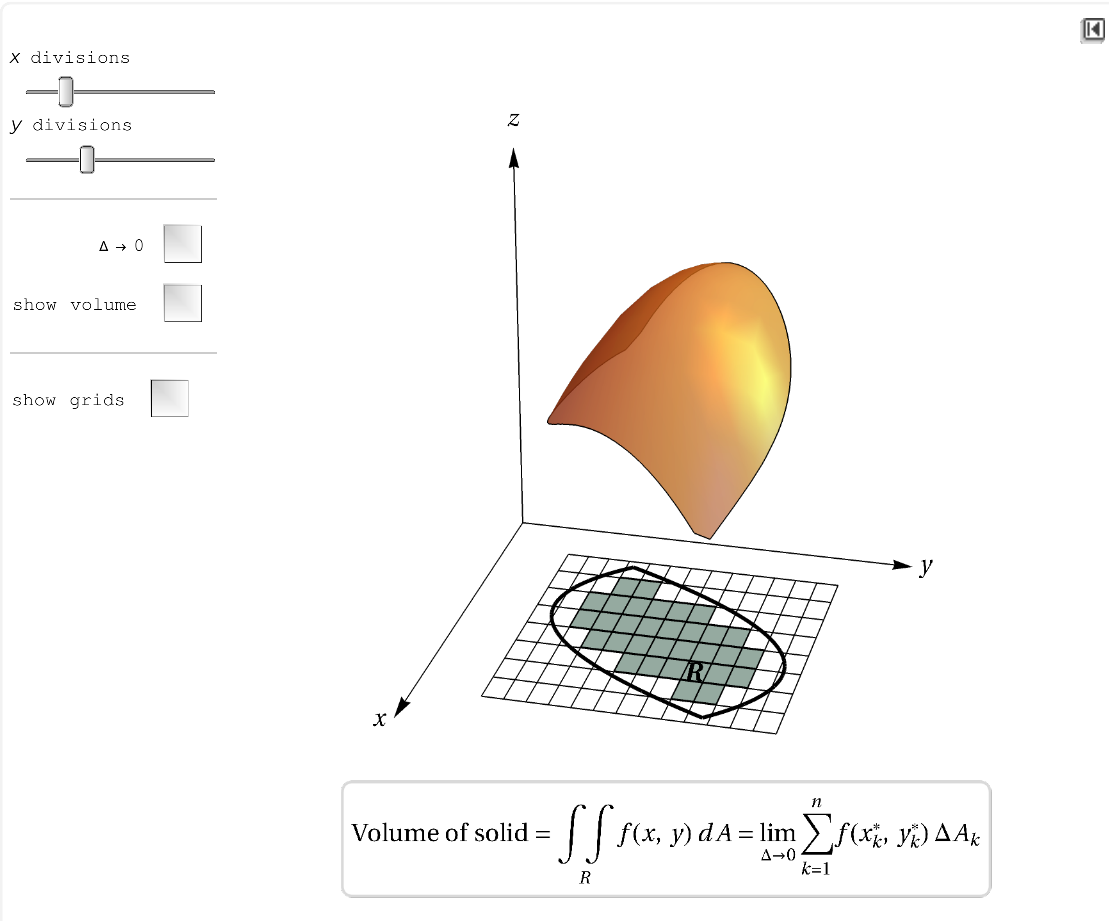


Figure 16.10

The double integral $\iint_R f(x, y)\, dA$ has another common interpretation. Suppose R represents a thin plate whose density at the point (x, y) is $f(x, y)$. The units of density are mass per unit area, so the product $f(x_k^*, y_k^*)\, \Delta A_k$ approximates the mass of the kth rectangle in R. Summing the masses of the rectangles gives an approximation to the total mass of R. In the limit as $n \to \infty$ and $\Delta \to 0$, the double integral equals the mass of the plate.

Iterated Integrals »

Double integrals over nonrectangular regions are also evaluated using iterated integrals. However, in this more general setting the order of integration is critical. Most of the double integrals we encounter fall into one of two categories determined by the shape of the region R.

The first type of region has the property that its lower and upper boundaries are the graphs of continuous functions $y = g(x)$ and $y = h(x)$, respectively, for $a \le x \le b$. Such regions have any of the forms shown in **Figure 16.11**.

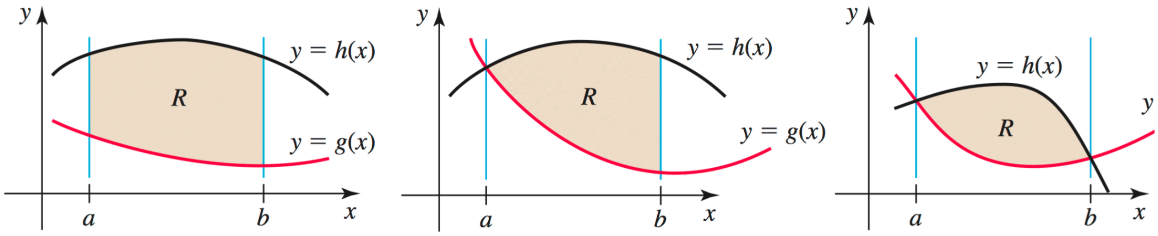


Figure 16.11

Once again, we appeal to the general slicing method. Assume for the moment that f is nonnegative on R and consider the solid bounded by the surface $z = f(x, y)$ and R (**Figure 16.12**).

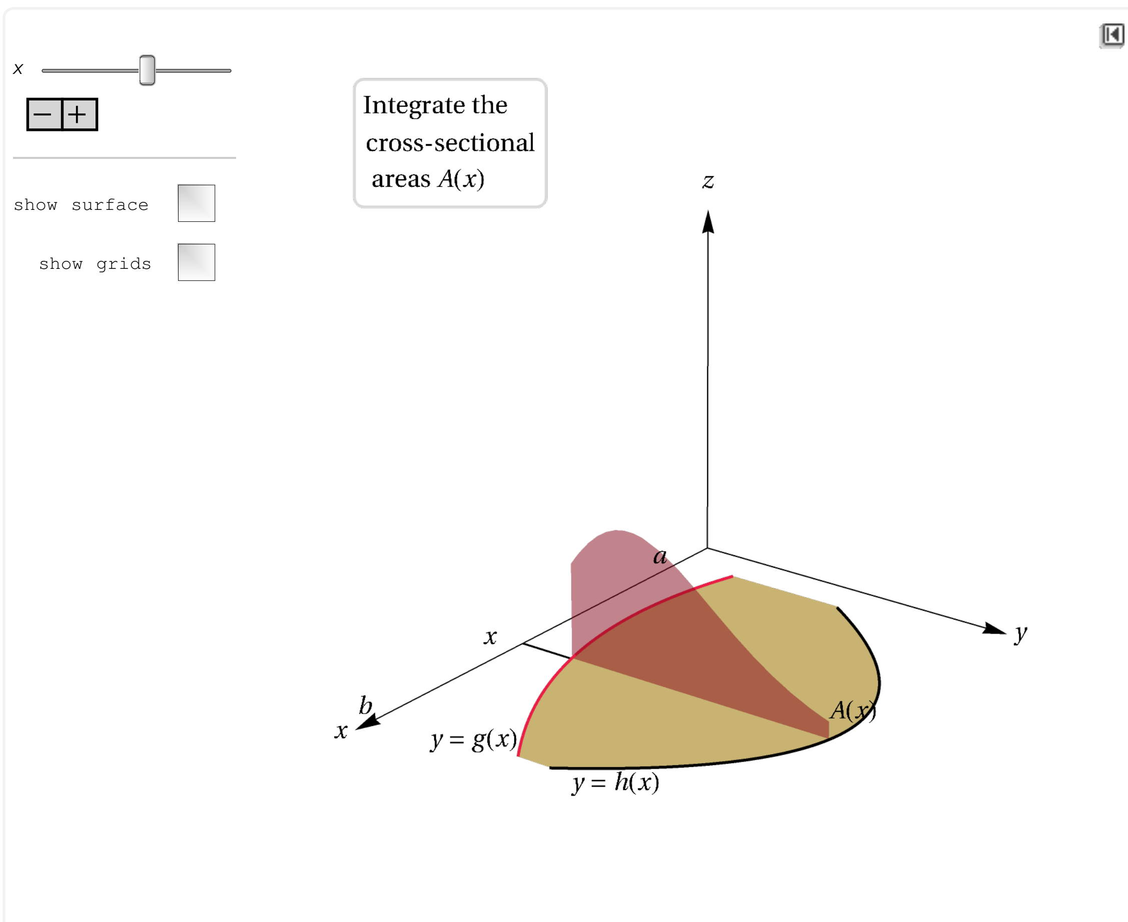


Figure 16.12

Imagine taking vertical slices through the solid parallel to the yz-plane. The cross section through the solid at a fixed value of x extends from the lower curve $y = g(x)$ to the upper curve $y = h(x)$. The area of that cross section is

$$A(x) = \int_{g(x)}^{h(x)} f(x, y)\, dy, \text{ for } a \le x \le b.$$

The volume of the region is given by a double integral; it is evaluated by integrating the cross-sectional areas $A(x)$ from $x = a$ to $x = b$:

$$\iint_R f(x, y)\, dA = \int_a^b \underbrace{\int_{g(x)}^{h(x)} f(x, y)\, dy}_{A(x)}\, dx.$$

The limits of integration in the iterated integral describe the boundaries of the region of integration R.

EXAMPLE 1 **Evaluating a double integral**

Express the integral $\iint_R 2x^2 y\, dA$ as an iterated integral, where R is the region bounded by the parabolas $y = 3x^2$ and $y = 16 - x^2$. Then evaluate the integral.

SOLUTION »

The region R is bounded below and above by the graphs of $g(x) = 3x^2$ and $h(x) = 16 - x^2$, respectively. Solving $3x^2 = 16 - x^2$, we find that these curves intersect at $x = -2$ and $x = 2$, which are the limits of integration in the x-direction (**Figure 16.13**).

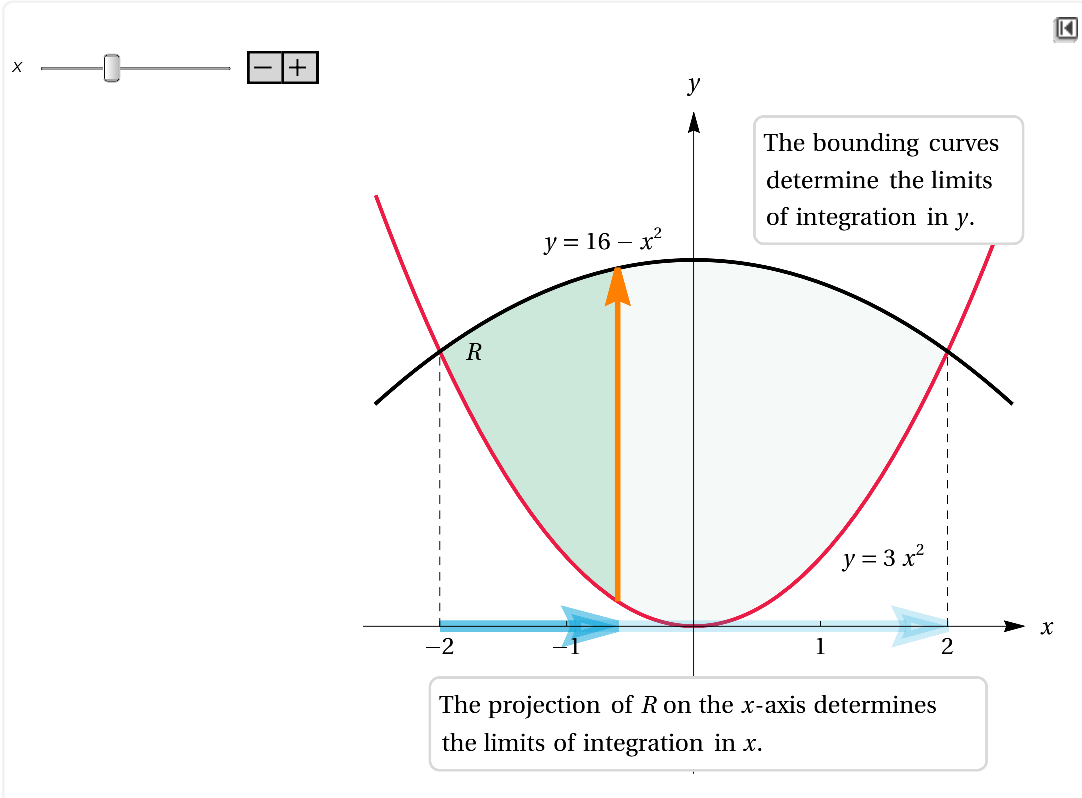


Figure 16.13

Figure 16.14 shows the solid bounded by the surface $z = 2\,x^2\,y$ and the region R. A typical vertical cross section through the solid parallel to the yz-plane at a fixed value of x has area

$$A(x) = \int_{3\,x^2}^{16-x^2} 2\,x^2\,y\,dy.$$

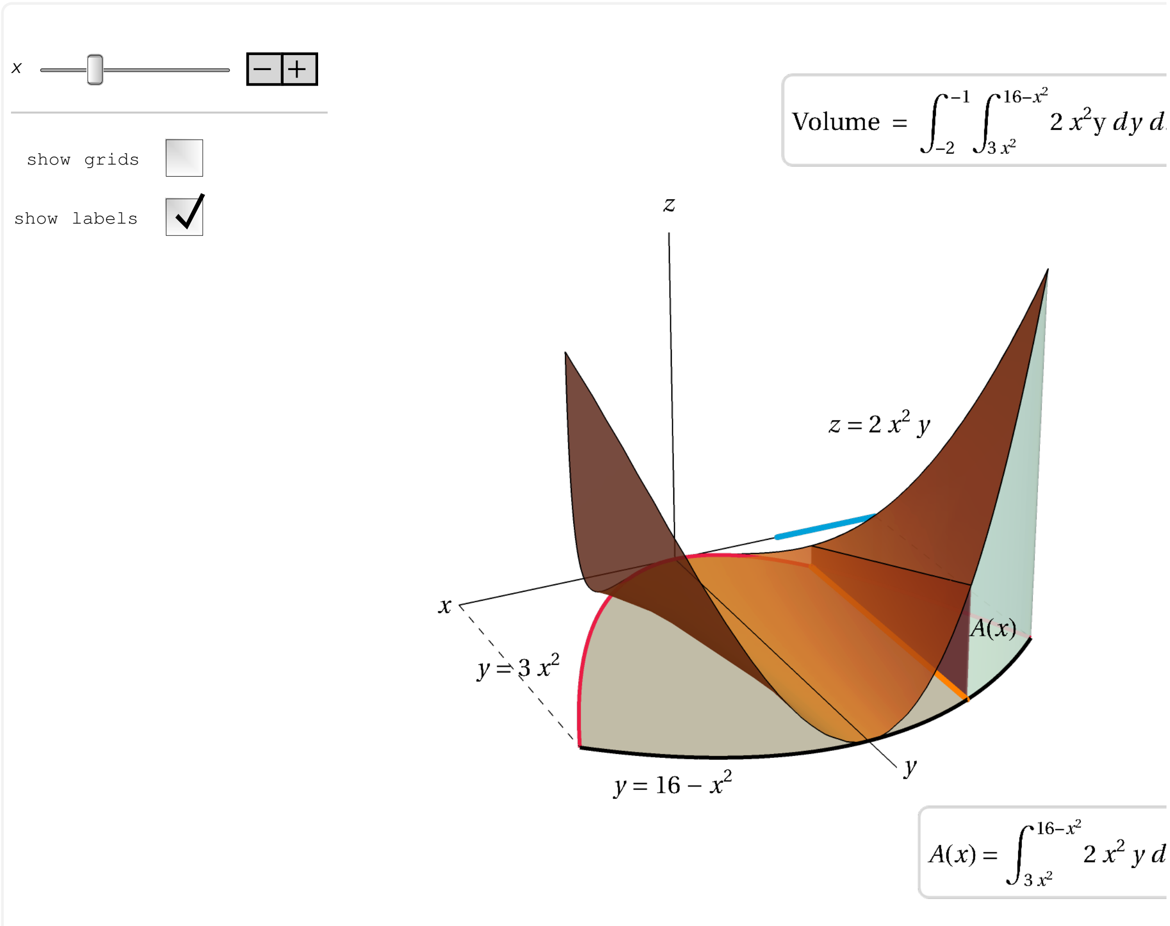


Figure 16.14

Integrating these cross-sectional areas between $x = -2$ and $x = 2$, the iterated integral becomes

$$\iint_R 2x^2 y\, dA = \int_{-2}^{2} \underbrace{\int_{3x^2}^{16-x^2} 2x^2 y\, dy}_{A(x)}\, dx \qquad \text{Convert to an iterated integral.}$$

$$= \int_{-2}^{2} x^2 y^2 \Big|_{3x^2}^{16-x^2} dx \qquad \text{Evaluate inner integral with respect to } y.$$

$$= \int_{-2}^{2} x^2\left(\left(16 - x^2\right)^2 - \left(3x^2\right)^2\right) dx \qquad \text{Simplify.}$$

$$= \int_{-2}^{2} \left(-8x^6 - 32x^4 + 256x^2\right) dx \qquad \text{Simplify.}$$

$$\approx 663.2. \qquad \text{Evaluate outer integral with respect to } x.$$

Because $z = 2x^2 y \geq 0$ on R, the value of the integral is the volume of the solid shown in Figure 16.14.

Related Exercises 12, 46 ◆

Quick Check 1 A region R is bounded by the x- and y-axes and the line $x + y = 2$. Suppose you integrate first with respect to y. Give the limits of the iterated integral over R. ◆

Answer »

Inner integral: $0 \le y \le 2 - x$. Outer integral: $0 \le x \le 2$.

Change of Perspective

Suppose that the region of integration R is bounded on the left and right by the graphs of continuous functions $x = g(y)$ and $x = h(y)$, respectively, on the interval $c \le y \le d$. Such regions may take any of the forms shown in **Figure 16.15**.

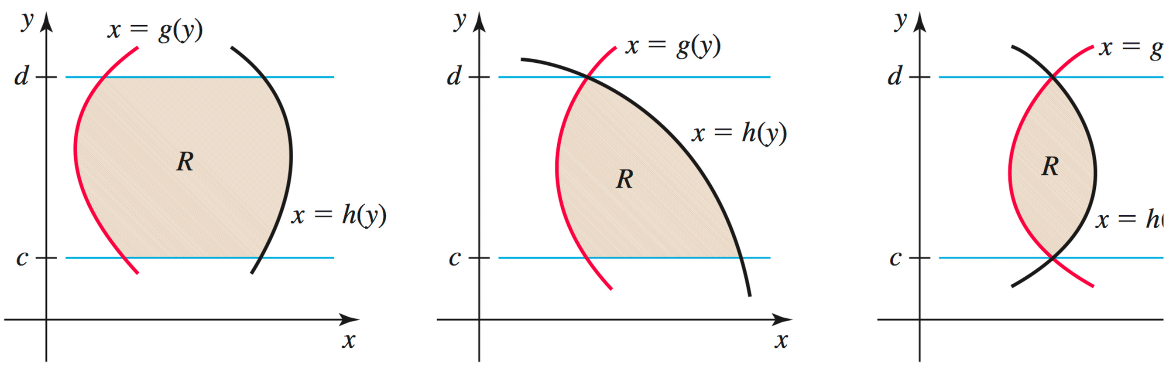


Figure 16.15

To find the volume of the solid bounded by the surface $z = f(x, y)$ and R, we now take vertical slices parallel to the xz-plane. In so doing, the double integral $\iint_R f(x, y)\, dA$ is converted to an iterated integral in which the inner integration is with respect to x over the interval $g(y) \le x \le h(y)$ and the outer integration is with respect to y over the interval $c \le y \le d$. The evaluation of double integrals in these two cases is summarized in the following theorem.

THEOREM 16.2 **Double Integrals over Nonrectangular Regions**

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$ (Figure 16.11). If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$ (Figure 16.15). If f is continuous R, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

Note »

EXAMPLE 2 **Computing a volume**

Find the volume of the solid below the surface $f(x, y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$. Notice that $f(x, y) > 0$ on R.

SOLUTION »

The region R is bounded on the left by $x = y$ and bounded on the right by $y = 8 - x$, or $x = 8 - y$ (**Figure 16.16**). These lines intersect at the point $(4, 4)$. We take vertical slices through the solid parallel to the xz-plane from $y = 1$ to $y = 4$. To visualize these slices, it helps to draw lines through R parallel to the x-axis.

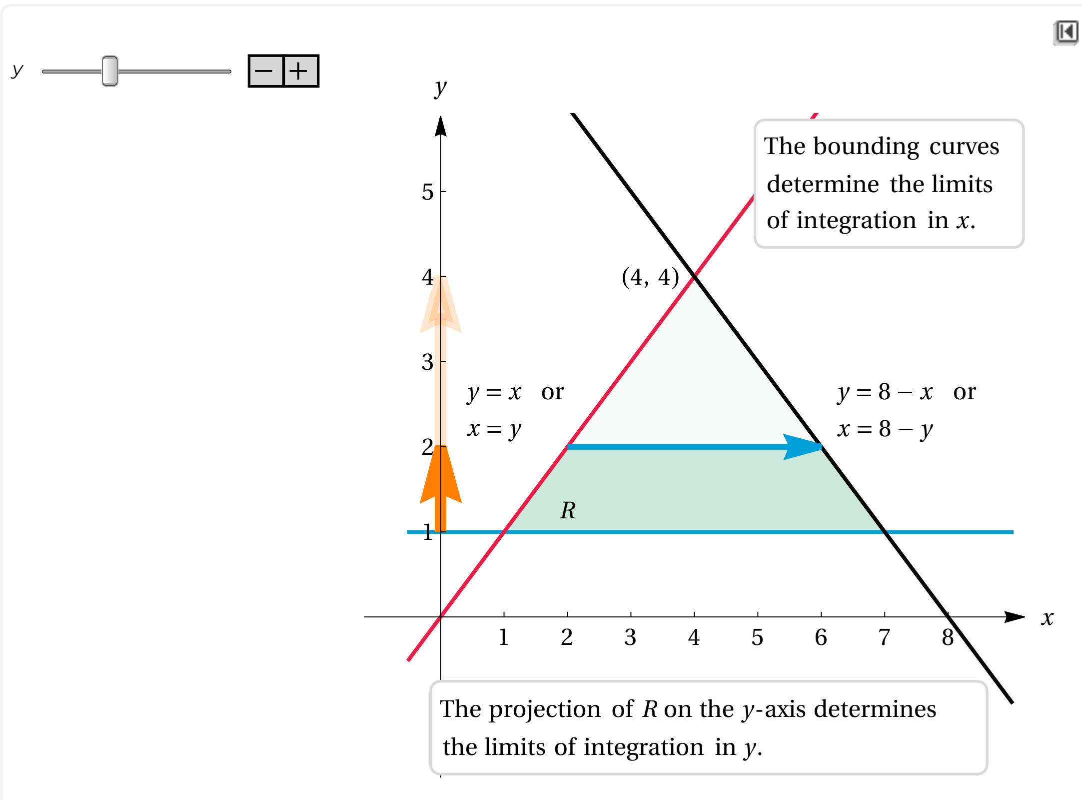


Figure 16.16

Integrating the cross-sectional areas of slices from $y = 1$ to $y = 4$, the volume of the solid beneath the graph of f and above R (**Figure 16.17**) is given by

$$\iint_R \left(2+\frac{1}{y}\right) dA = \int_1^4 \int_y^{8-y} \left(2+\frac{1}{y}\right) dx\, dy \quad \text{Convert to an iterated integral.}$$

$$= \int_1^4 \left(2+\frac{1}{y}\right) x \Big|_y^{8-y} dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \int_1^4 \left(2+\frac{1}{y}\right)(8-2y)\, dy \quad \text{Simplify.}$$

$$= \int_1^4 \left(14-4y+\frac{8}{y}\right) dy \quad \text{Simplify.}$$

$$= \left(14y-2y^2+8\ln|y|\right)\Big|_1^4 \quad \text{Evaluate outer integral with respect to } y.$$

$$= 12+8\ln 4 \approx 23.09. \quad \text{Simplify.}$$

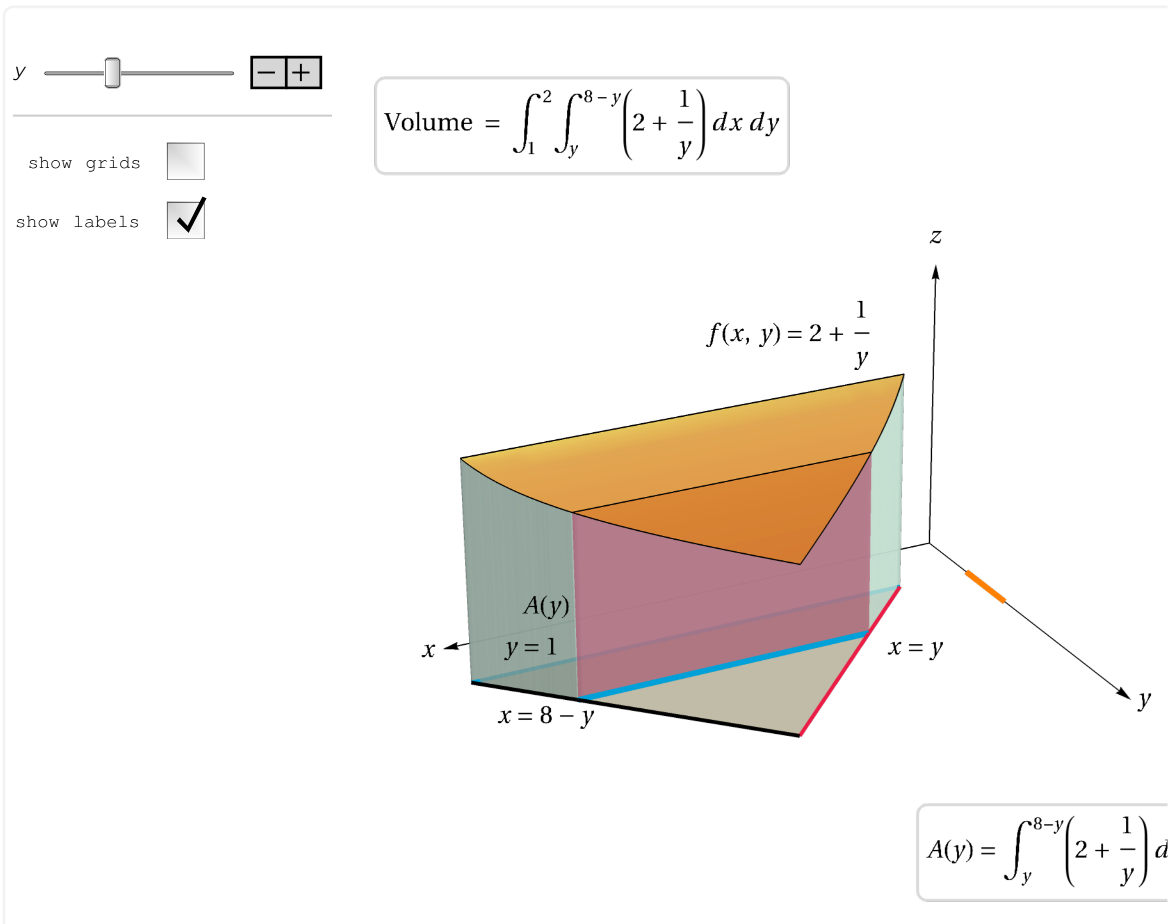


Figure 16.17

Related Exercise 74 ◆

Quick Check 2 Could the integral in Example 2 be evaluated by integrating first (inner integral) with respect to y? ◆

Answer »

Yes; however, two separate iterated integrals would be required.

Choosing and Changing the Order of Integration »

Occasionally a region of integration is bounded above and below by a pair of curves *and* the region is bounded on the right and the left by a pair of curves. For example, the region R in **Figure 16.18** is bounded above by $y = x^{1/3}$ and below by $y = x^2$, and it is bounded on the right by $x = \sqrt{y}$ and on the left by $x = y^3$. In these cases, we can choose either of two orders of integration; however, one order of integration may be preferable. The following examples illustrate the valuable techniques of choosing and changing the order of integration.

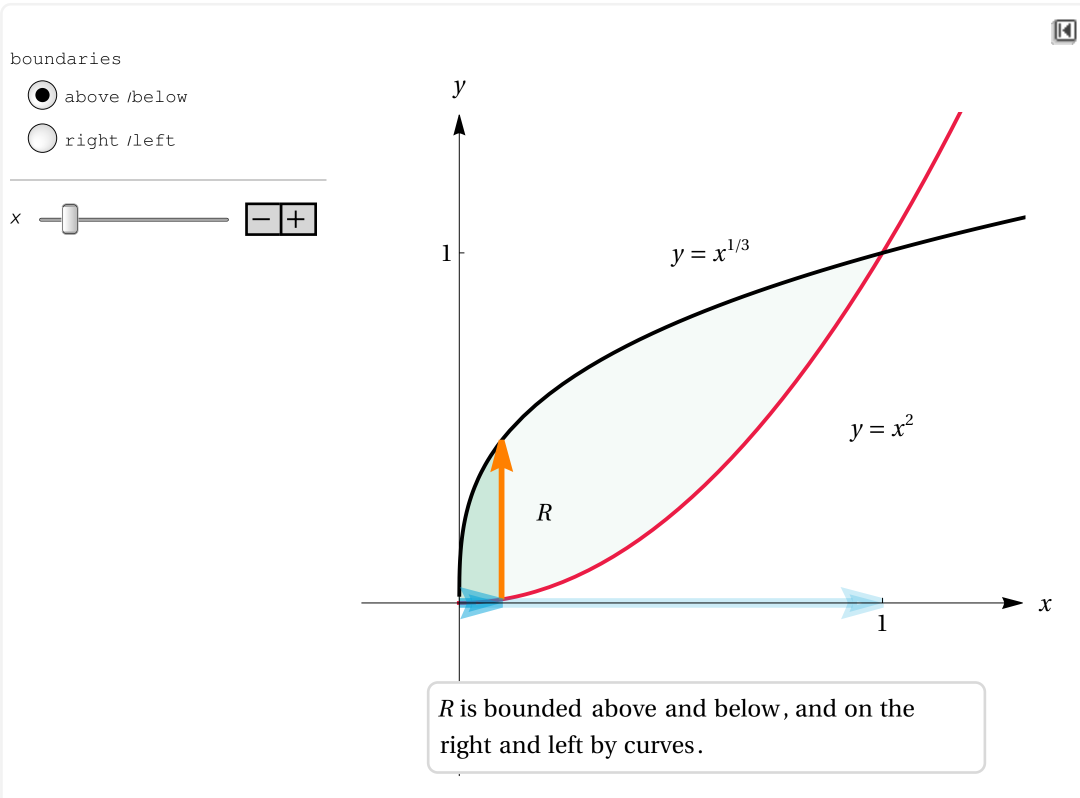


Figure 16.18

EXAMPLE 3 Volume of a tetrahedron

Find the volume of the tetrahedron (pyramid with four triangular faces) in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, $z = 0$). Assume a, b, and c are positive real numbers (**Figure 16.19**).

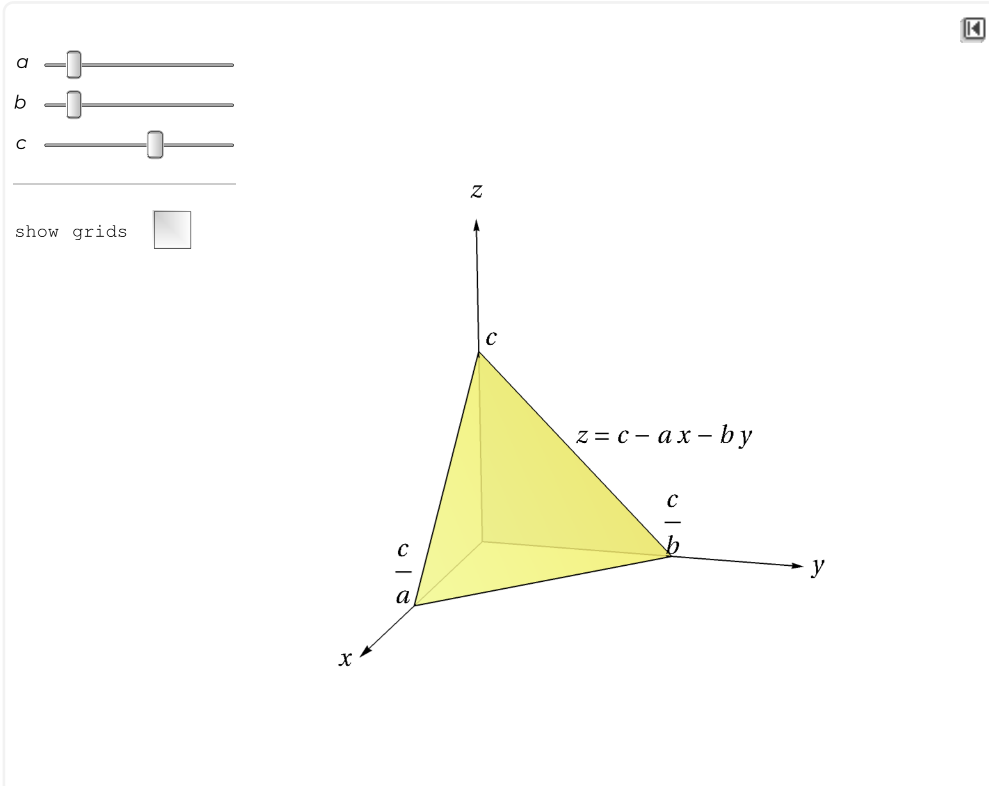


Figure 16.19

SOLUTION »

Let R be the triangular base of the tetrahedron in the xy-plane; it is bounded by the x- and y-axes and the line $a x + b y = c$ (found by setting $z = 0$ in the equation of the plane; **Figure 16.20**).

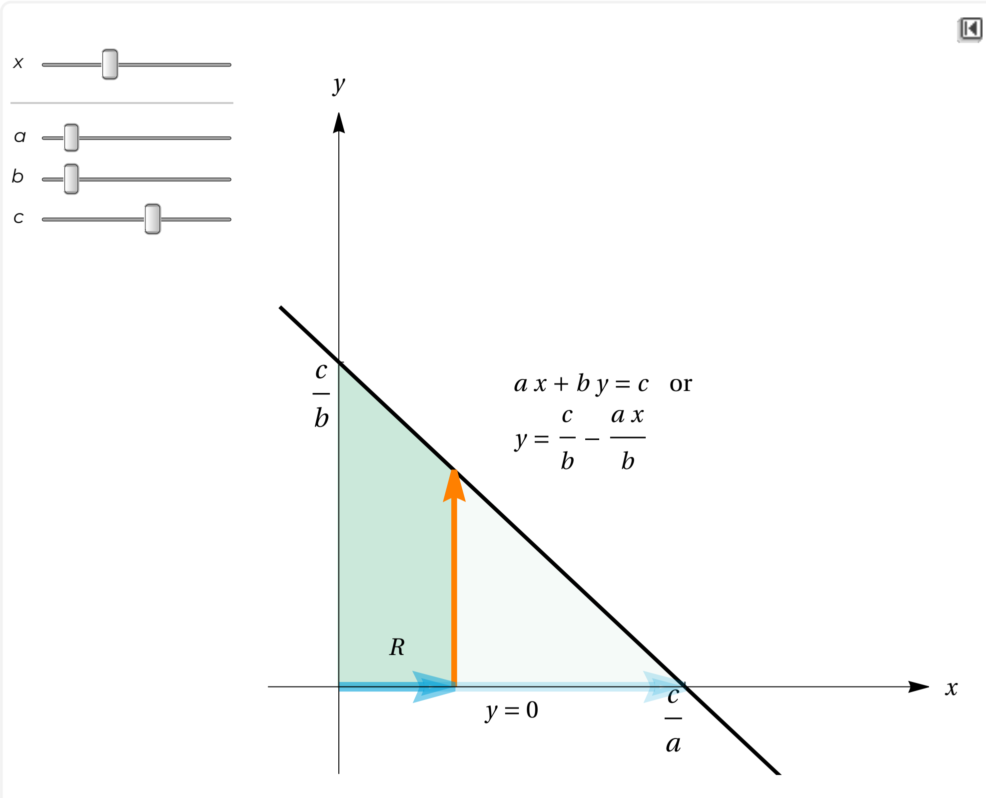


Figure 16.20

We can view R as being bounded below and above by the lines $y = 0$ and $y = \frac{c}{b} - \frac{a\,x}{b}$, respectively. The boundaries on the left and right are then $x = 0$ and $x = \frac{c}{a}$, respectively. The volume of the solid region between the plane and R is

$$\iint_R (c - a\,x - b\,y)\,dA = \int_0^{c/a} \int_0^{c/b - a\,x/b} (c - a\,x - b\,y)\,dy\,dx \quad \text{Convert to an iterated integral.}$$

$$= \int_0^{c/a} \left(c\,y - a\,x\,y - \frac{b\,y^2}{2} \right) \Bigg|_0^{c/b - a\,x/b} dx \quad \text{Evaluate inner integral with respect to } y.$$

$$= \int_0^{c/a} \frac{(a\,x - c)^2}{2\,b}\,dx \quad \text{Simplify and factor.}$$

$$= \frac{c^3}{6\,a\,b}. \quad \text{Evaluate outer integral with respect to } x.$$

Note »

In Example 3, it is just as easy to view R as being bounded on the left and the right by the lines $x = 0$ and $x = \frac{c}{a} - \frac{b\,y}{a}$, respectively, and integrating first with respect to x.

This result illustrates the volume formula for a tetrahedron. The lengths of the legs of the base are $\frac{c}{a}$ and $\frac{c}{b}$, which means the area of the base is $\frac{c^2}{2ab}$. The height of the tetrahedron is c. Therefore, the general volume formula is

$$V = \frac{c^3}{6ab} = \frac{1}{3} \underbrace{\frac{c^2}{2ab}}_{\text{area of base}} \cdot \underbrace{c}_{\text{height}} = \frac{1}{3}(\text{area of base})(\text{height}).$$

Note »

> The volume of *any* tetrahedron is $\frac{1}{3}$(area of base)(height), where any of the faces may be chosen as the base (Exercise 98).

Related Exercise 73 ◆

EXAMPLE 4 Changing the order of integration

Consider the iterated integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2 \, dx \, dy$. Sketch the region of integration determined by the limits of integration and then evaluate the iterated integral.

SOLUTION »

The region of integration is $R = \{(x, y) : y \le x \le \sqrt{\pi}, 0 \le y \le \sqrt{\pi}\}$, which is a triangle (**Figure 16.21a**). Evaluating the iterated integral as given (integrating first with respect to x) requires integrating $\sin x^2$, a function whose antiderivative is not expressible in terms of elementary functions. Therefore, this order of integration is not feasible.

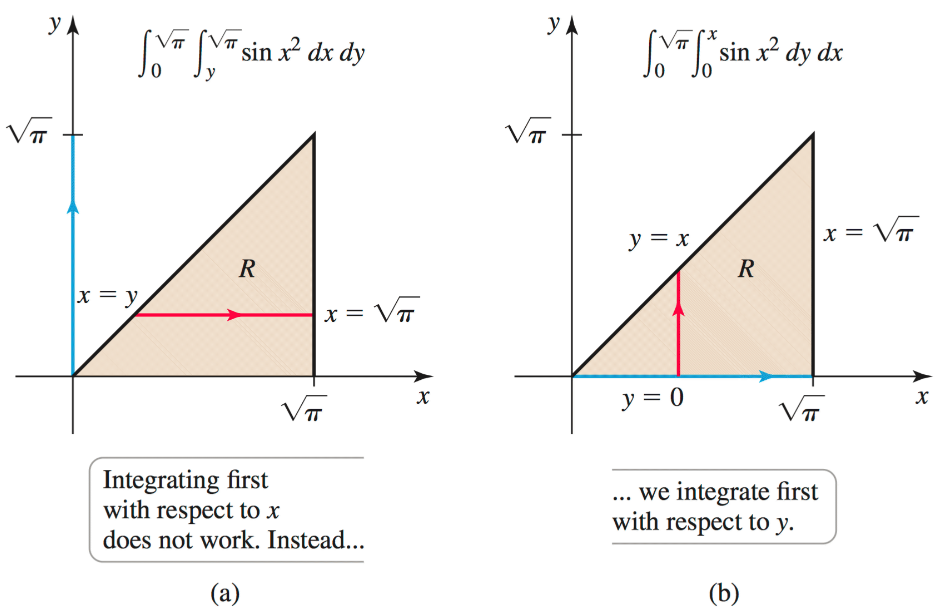


Figure 16.21

Instead, we change our perspective (**Figure 16.21b**) and integrate first with respect to y. With this order of integration, y runs from $y = 0$ to $y = x$ in the inner integral and x runs from $x = 0$ to $x = \sqrt{\pi}$ in the outer integral:

$$\iint_R \sin x^2 \, dA = \int_0^{\sqrt{\pi}} \int_0^x \sin x^2 \, dy \, dx$$

$$= \int_0^{\sqrt{\pi}} \left(y \sin x^2 \right) \Big|_0^x \, dx \qquad \text{Evaluate inner integral with respect to } y;\ \sin x^2 \text{ is constant.}$$

$$= \int_0^{\sqrt{\pi}} x \sin x^2 \, dx \qquad \text{Simplify.}$$

$$= \left(-\frac{1}{2} \cos x^2 \right) \Big|_0^{\sqrt{\pi}} \qquad \text{Evaluate outer integral with respect to } x.$$

$$= 1. \qquad \text{Simplify.}$$

This example shows that the order of integration can make a practical difference.

Related Exercises 58, 64 ◆

Quick Check 3 Change the order of integration of the integral $\int_0^1 \int_0^y f(x, y) \, dx \, dy$. ◆

Answer »

Regions Between Two Surfaces »

An extension of the preceding ideas allows us to solve more general volume problems. Let $z = f(x, y)$ and $z = g(x, y)$ be continuous functions with $f(x, y) \geq g(x, y)$ on a region R in the xy-plane. Suppose we wish to compute the volume of the solid between the two surfaces over the region R (**Figure 16.22**). Forming a Riemann sum for the volume, the height of a typical box within the solid is the vertical distance $f(x, y) - g(x, y)$ between the upper and lower surfaces. Therefore, the volume of the solid between the surfaces is

$$V = \iint_R (f(x, y) - g(x, y)) \, dA.$$

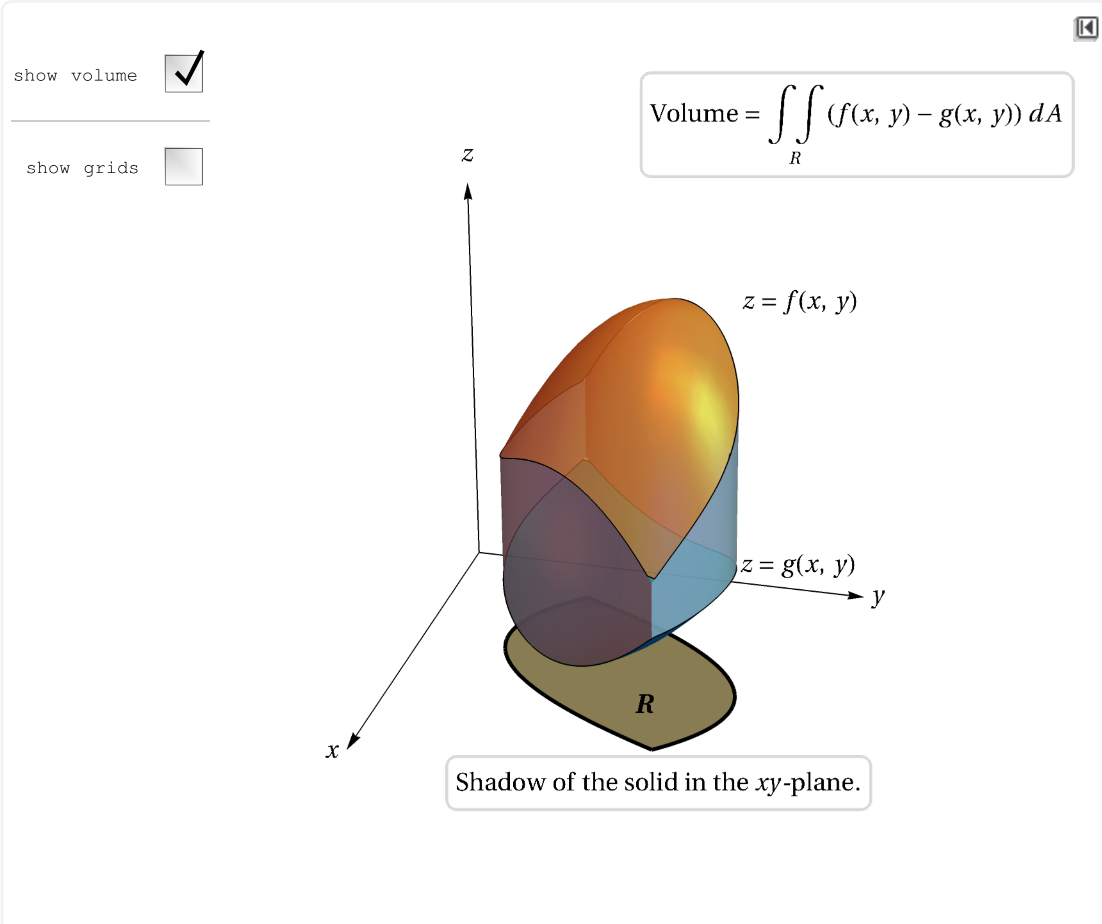


Figure 16.22

EXAMPLE 5 Region bounded by two surfaces

Find the volume of the solid region bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$ (**Figure 16.23**).

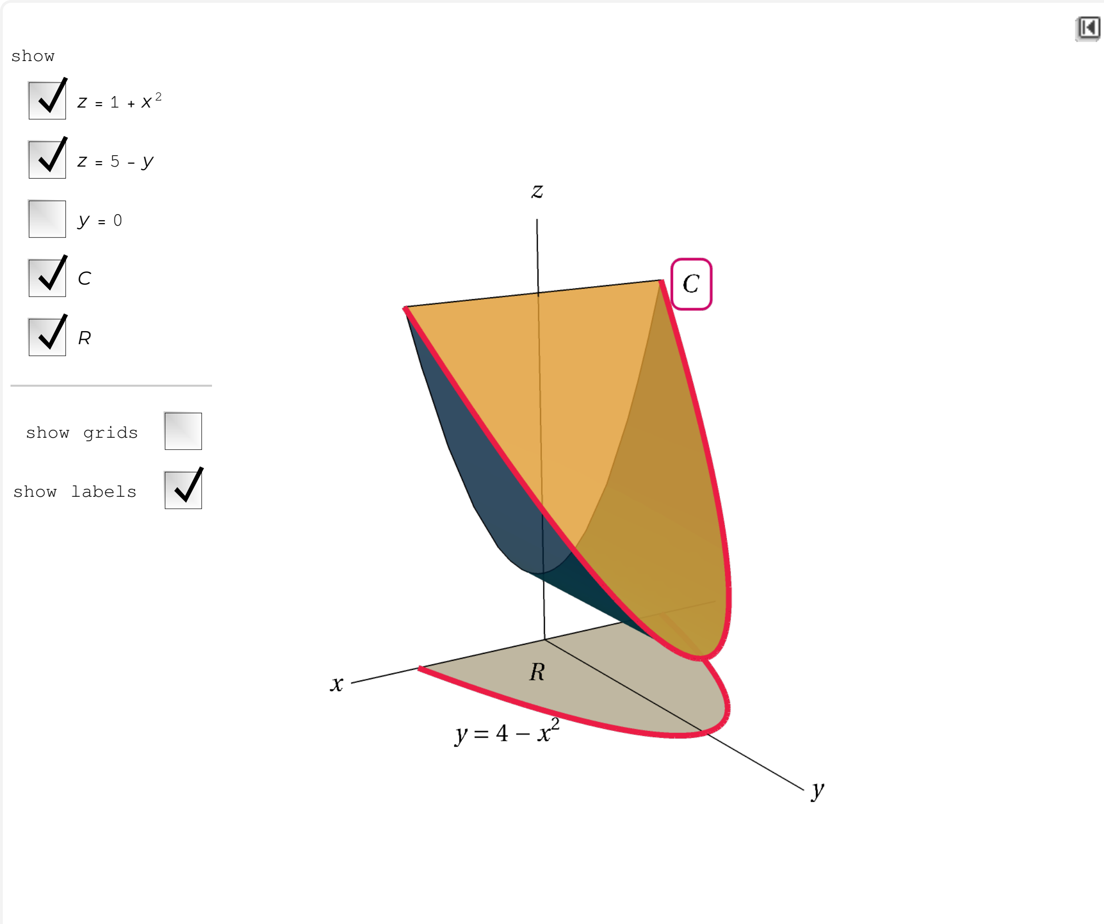


Figure 16.23

SOLUTION »

The upper surface bounding the solid is $z = 5 - y$ and the lower surface is $z = 1 + x^2$; these two surfaces intersect along a curve C. Solving $5 - y = 1 + x^2$, we find that $y = 4 - x^2$, which is the projection of C onto the xy-plane. The back wall of the solid is the plane $y = 0$, and its projection onto the xy-plane is the x-axis. This line $(y = 0)$ intersects the parabola $y = 4 - x^2$ at $x = \pm 2$. Therefore, the region of integration (Figure 16.23) is

$$R = \left\{(x, y) : 0 \le y \le 4 - x^2, \ -2 \le x \le 2\right\}.$$

Notice that R and the solid are symmetric about the yz-plane. Therefore, the volume of the entire solid is twice the volume over that part of the solid that lies the first octant. The volume of the solid is

$$2\int_0^2\int_0^{4-x^2}\left(\underbrace{(5-y)}_{f(x,y)}-\underbrace{\left(1+x^2\right)}_{g(x,y)}\right)dy\,dx = 2\int_0^2\int_0^{4-x^2}\left(4-x^2-y\right)dy\,dx \quad \text{Simplify the integrand.}$$

$$= 2\int_0^2\left(\left(4-x^2\right)y-\frac{y^2}{2}\right)\Bigg|_0^{4-x^2}dx \quad \text{Evaluate inner integral with respect to } y.$$

$$= \int_0^2\left(x^4-8x^2+16\right)dx \quad \text{Simplify.}$$

$$= \left(\frac{x^5}{5}-\frac{8x^3}{3}+16x\right)\Bigg|_0^2 \quad \text{Evaluate outer integral with respect to } x.$$

$$= \frac{256}{15}. \quad \text{Simplify.}$$

Note »

> To use symmetry to simplify a double integral, you must check that both the region of integration and the integrand have the same symmetry.

Related Exercises 78–79 ◆

Decomposition of Regions »

We occasionally encounter regions that are more complicated than those considered so far. A technique called *decomposition* allows us to subdivide a region of integration into two (or more) subregions. If the integrals over the subregions can be evaluated separately, the results are added to obtain the value of the original integral. For example, the region R in **Figure 16.24** is divided into two nonoverlapping subregions R_1 and R_2. By partitioning these regions and using Riemann sums, it can be shown that

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

This method is illustrated in Example 6. The analogue of decomposition with single variable integrals is the property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

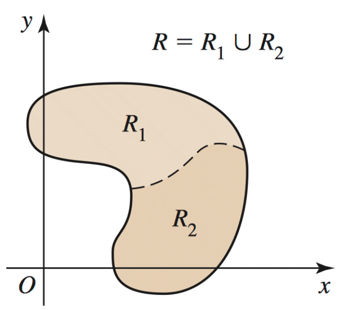


Figure 16.24

Finding Area by Double Integrals »

An interesting application of double integrals arises when the integrand is $f(x, y) = 1$. The integral $\iint_R 1\,dA$ gives the volume of the solid between the horizontal plane $z = 1$ and the region R. Because the height of this solid is 1, its volume equals (numerically) the area of R (**Figure 16.25**). Therefore, we have a way to compute areas of regions in the xy-plane using double integrals.

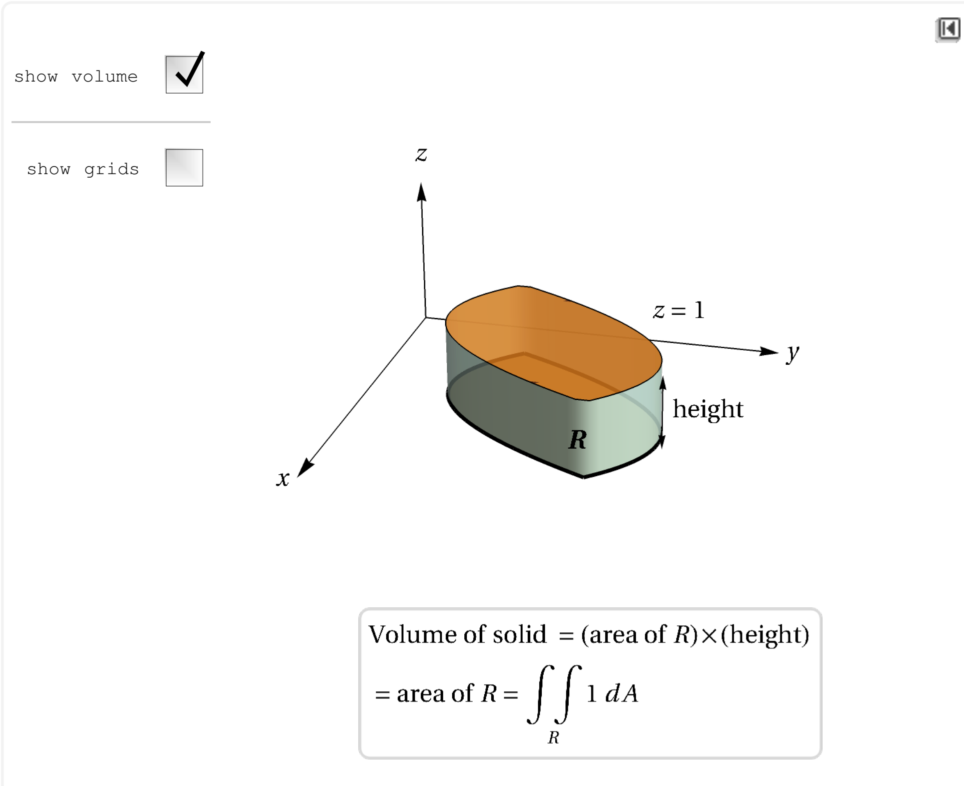


Figure 16.25

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R dA.$$

Note »

EXAMPLE 6 Area of a plane region

Find the area of the region R bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$ (**Figure 16.26**).

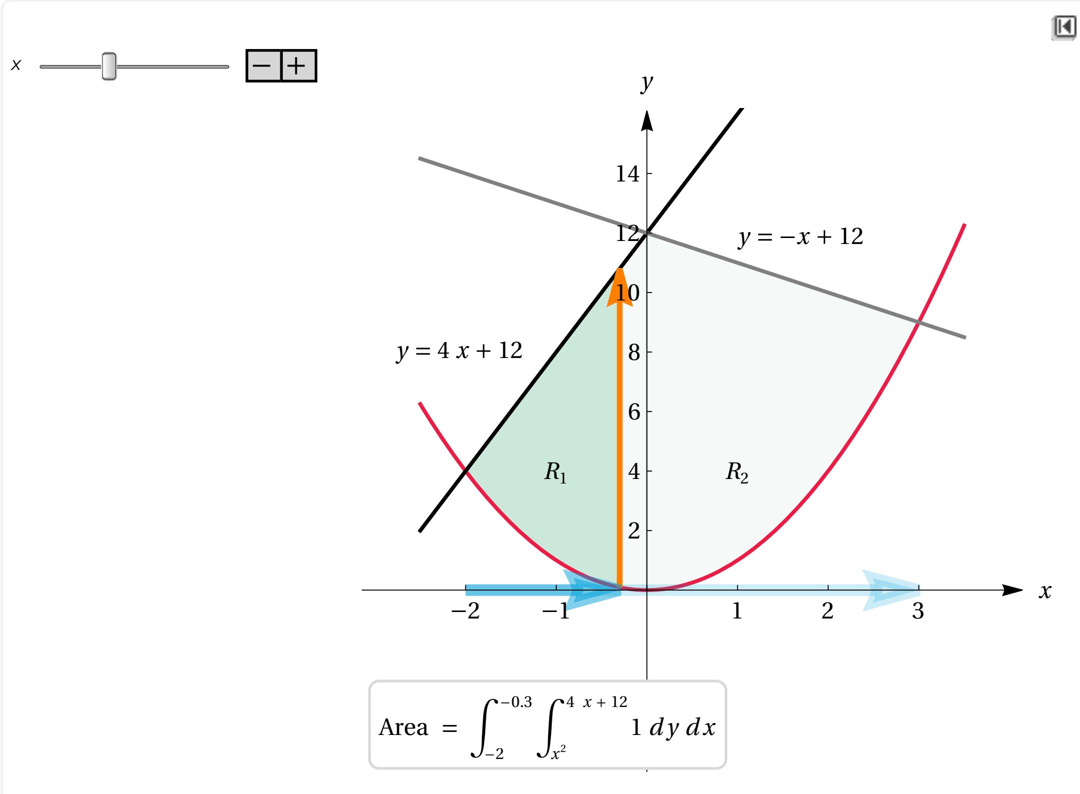


Figure 16.26

SOLUTION »

The region R in its entirety is bounded neither above and below by two curves, nor on the left and right by two curves. However, when decomposed along the y-axis, R may be viewed as two regions R_1 and R_2 that are each bounded above and below by a pair of curves. Notice that the parabola $y = x^2$ and the line $y = -x + 12$ intersect in the first quadrant at the point $(3, 9)$, while the parabola and the line $y = 4x + 12$ intersect in the second quadrant at the point $(-2, 4)$.

To find the area of R, we integrate the function $f(x, y) = 1$ over R_1 and R_2; the area is

$$\iint_{R_1} 1\,dA + \iint_{R_2} 1\,dA \qquad \text{Decompose region.}$$

$$= \int_{-2}^{0}\int_{x^2}^{4x+12} 1\,dy\,dx + \int_{0}^{3}\int_{x^2}^{-x+12} 1\,dy\,dx \qquad \text{Convert to iterated integrals.}$$

$$= \int_{-2}^{0}\left(4x + 12 - x^2\right)dx + \int_{0}^{3}\left(-x + 12 - x^2\right)dx \qquad \text{Evaluate inner integrals with respect to } y.$$

$$= \left(2x^2 + 12x - \frac{x^3}{3}\right)\Bigg|_{-2}^{0} + \left(-\frac{x^2}{2} + 12x - \frac{x^3}{3}\right)\Bigg|_{0}^{3} \qquad \text{Evaluate outer integrals with respect to } x.$$

$$= \frac{40}{3} + \frac{45}{2} = \frac{215}{6}. \qquad \text{Simplify.}$$

Related Exercise 86 ◆

Quick Check 4 Consider the triangle R with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$ as a region of integration. If we integrate first with respect to x, does R need to be decomposed? If we integrate first with respect to y, does R need to be decomposed? ◆

Answer »

No; yes

Exercises »

Getting Started »

Practice Exercises »

11–27. Evaluating integrals *Evaluate the following integrals.*

11. $\int_0^1 \int_x^1 6y\,dy\,dx$

12. $\int_0^1 \int_0^{2x} 15xy^2\,dy\,dx$

13. $\int_0^2 \int_{x^2}^{2x} xy\,dy\,dx$

14. $\int_{-\pi/4}^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx$

15. $\int_{-2}^2 \int_{x^2}^{8-x^2} x\,dy\,dx$

16. $\int_0^{\ln 2} \int_{e^x}^2 dy\,dx$

17. $\int_0^1 \int_0^x 2e^{x^2}\,dy\,dx$

18. $\int_0^{\sqrt[3]{\pi/2}} \int_0^x y\cos x^3\,dy\,dx$

19. $\int_0^{\ln 2} \int_{e^y}^2 \frac{y}{x}\,dx\,dy$

20. $\int_0^4 \int_y^{2y} xy\,dx\,dy$

21. $\int_0^{\pi/2} \int_y^{\pi/2} 6\sin(2x-3y)\,dx\,dy$

22. $\int_0^{\pi/2} \int_0^{\cos y} e^{\sin y}\,dx\,dy$

23. $\int_0^{\pi/2} \int_0^{y \cos y} dx\, dy$

24. $\int_0^1 \int_{\tan^{-1} x}^{\pi/4} 2\, x\, dy\, dx$

25. $\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2\, x\, y\, dx\, dy$

26. $\int_0^1 \int_0^x 2\, e^x\, dy\, dx$

27. $\int_{\pi/2}^{\pi} \int_0^{y^2} \cos \frac{x}{y}\, dx\, dy$

28–34. Regions of integration *Sketch each region R and write an iterated integral of a continuous function f over region R. Use the order $dy\, dx$.*

28. $R = \left\{(x,\ y) : 0 \le x \le 2,\ 3\, x^2 \le y \le -6\, x + 24\right\}$

29. $R = \{(x,\ y) : 1 \le x \le 2,\ x + 1 \le y \le 2\, x + 4\}$

30. $R = \left\{(x,\ y) : 0 \le x \le 4,\ x^2 \le y \le 8\ \sqrt{x}\right\}$

31. R is the triangular region with vertices $(0,\ 0)$, $(0,\ 2)$, and $(1,\ 0)$.

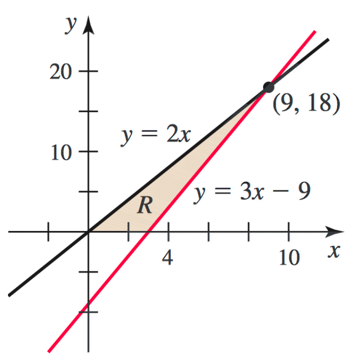
32. R is the triangular region with vertices $(0,\ 0)$, $(0,\ 2)$, and $(1,\ 1)$.

33. R is the region in the first quadrant bounded by a circle of radius 1 centered at the origin.

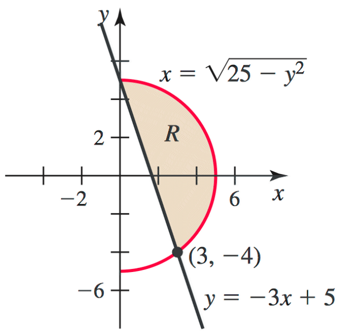
34. R is the region in the first quadrant bounded by the y-axis and the parabolas $y = x^2$ and $y = 1 - x^2$.

35–42. Regions of integration *Write an iterated integral of a continuous function f over the region R. Use the order $dy\, dx$. Start by sketching the region of integration if it is not supplied.*

35.



36.



37. R is the region bounded by $y = 4 - x$, $y = 1$, and $x = 0$

38. $R = \{(x, y) : 0 \le x \le y(1 - y)\}$

39. R is the region bounded by $y = 2x + 3$, $y = 3x - 7$, and $y = 0$

40. R is the region in quadrants 2 and 3 bounded by the semicircle with radius 3 centered at $(0, 0)$

41. R is the region bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$.

42. R is the region in the first quadrant bounded by the x-axis, the line $x = 6 - y$, and the curve $y = \sqrt{x}$

43–56. Evaluating integrals *Evaluate the following integrals. A sketch is helpful.*

43. $\iint_R x\,y\,dA$; R is bounded by $x = 0$, $y = 2x + 1$, and $y = -2x + 5$.

44. $\iint_R (x + y)\,dA$; R is the region in the first quadrant bounded by $x = 0$, $y = x^2$, and $y = 8 - x^2$.

45. $\iint_R y^2\,dA$; R is bounded by $x = 1$, $y = 2x + 2$, and $y = -x - 1$.

46. $\iint_R x^2\,y\,dA$; R is the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at $(0, 0)$.

47. $\iint_R 12\,y\,dA$; R is bounded by $y = 2 - x$, $y = \sqrt{x}$, and $y = 0$.

48. $\iint_R y^2\,dA$; R is bounded by $y = 1$, $y = 1 - x$, and $y = x - 1$.

49. $\iint_R 3\,x\,y\,dA$; R is the region in the first quadrant bounded by $y = 2 - x$, $y = 0$, and $x = 4 - y^2$.

50. $\iint_R (x + y)\,dA$; R is bounded by $y = |x|$ and $y = 4$.

51. $\iint_R 3\,x^2\,dA$; R is bounded by $y = 0$, $y = 2x + 4$, and $y = x^3$.

52. $\displaystyle\iint_R 8\,x\,y\,dA;\ R = \left\{(x,\ y) : 0 \le y \le \sec x,\ 0 \le x \le \frac{\pi}{4}\right\}$

53. $\displaystyle\iint_R (x + y)\,dA$; R is the region bounded by $y = \dfrac{1}{x}$ and $y = \dfrac{5}{2} - x$.

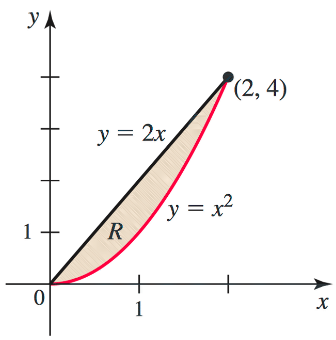
54. $\displaystyle\iint_R \frac{y}{1 + x + y^2}\,dA;\ R = \left\{(x,\ y) : 0 \le \sqrt{x} \le y,\ \ 0 \le y \le 1\right\}$

55. $\displaystyle\iint_R x\sec^2 y\,dA;\ R = \left\{(x,\ y) : 0 \le y \le x^2,\ 0 \le x \le \frac{\sqrt{\pi}}{2}\right\}$

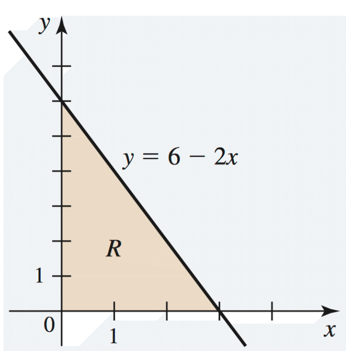
56. $\displaystyle\iint_R \frac{8\,x\,y}{1 + x^2 + y^2}\,dA;\ R = \{(x,\ y) : 0 \le y \le x,\ \ 0 \le x \le 2\}$

57–62. Changing order of integration *Reverse the order of integration in the following integrals.*

57. $\displaystyle\int_0^2 \int_{x^2}^{2\,x} f(x,\ y)\,dy\,dx$



58. $\displaystyle\int_0^3 \int_0^{6-2\,x} f(x,\ y)\,dy\,dx$



59. $\displaystyle\int_{1/2}^1 \int_0^{-\ln y} f(x,\ y)\,dx\,dy$

60. $\int_0^1 \int_1^{e^y} f(x, y)\, dx\, dy$

61. $\int_0^1 \int_0^{\cos^{-1} y} f(x, y)\, dx\, dy$

62. $\int_1^e \int_0^{\ln x} f(x, y)\, dy\, dx$

63–68. Changing order of integration *Reverse the order of integration and evaluate the integral.*

63. $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

64. $\int_0^{\pi} \int_x^{\pi} \sin y^2\, dy\, dx$

65. $\int_0^{1/2} \int_{y^2}^{1/4} y \cos\left(16\,\pi\, x^2\right) dx\, dy$

66. $\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 1}\, dy\, dx$

67. $\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos\left(x^2\, y\right) dx\, dy$

68. $\int_0^2 \int_0^{4-x^2} \frac{x\, e^{2\,y}}{4 - y}\, dy\, dx$

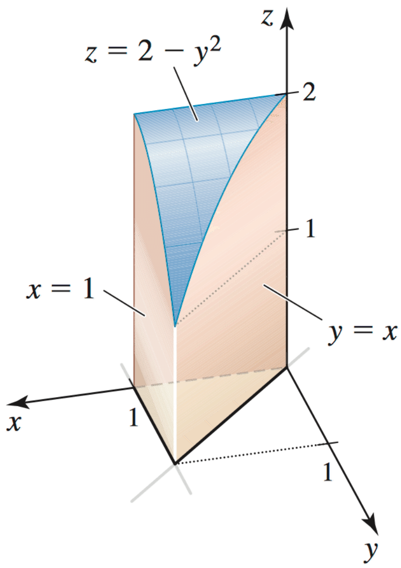
69–70. Two integrals to one *Draw the regions of integration and write the following integrals as a single iterated integral.*

69. $\int_0^1 \int_{e^y}^e f(x, y)\, dx\, dy + \int_{-1}^0 \int_{e^{-y}}^e f(x, y)\, dx\, dy$

70. $\int_{-4}^0 \int_0^{\sqrt{16-x^2}} f(x, y)\, dy\, dx + \int_0^4 \int_0^{4-x} f(x, y)\, dy\, dx$

71–80. Volumes *Find the volume of the following solids.*

71. The solid bounded by the cylinder $z = 2 - y^2$, the xy-plane, the xz-plane, and the planes $y = x$ and $x = 1$



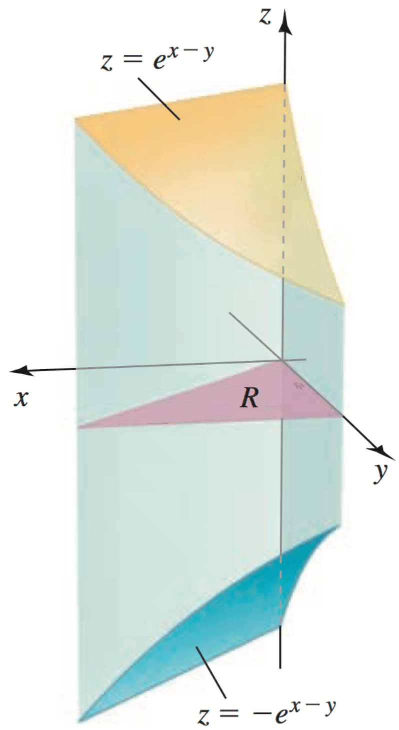
72. The solid bounded between the cylinder $z = 2\sin^2 x$ and the xy-plane over the region $R = \{(x, y) : 0 \le x \le y \le \pi\}$

73. The tetrahedron bounded by the coordinate planes ($x = 0$, $y = 0$, and $z = 0$) and the plane $z = 8 - 2x - 4y$

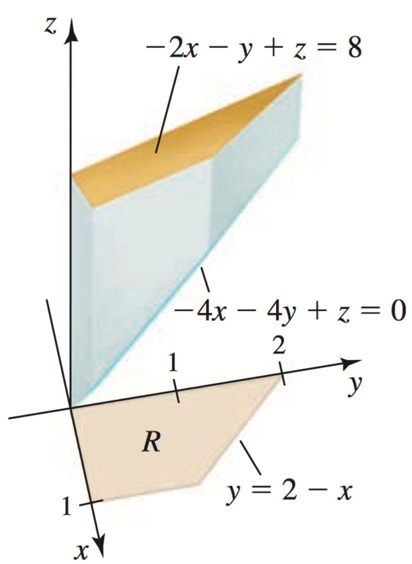
74. The solid in the first octant bounded by the coordinate planes and the surface $z = 1 - y - x^2$

75. The segment of the cylinder $x^2 + y^2 = 1$ bounded above by the plane $z = 12 + x + y$ and below by $z = 0$

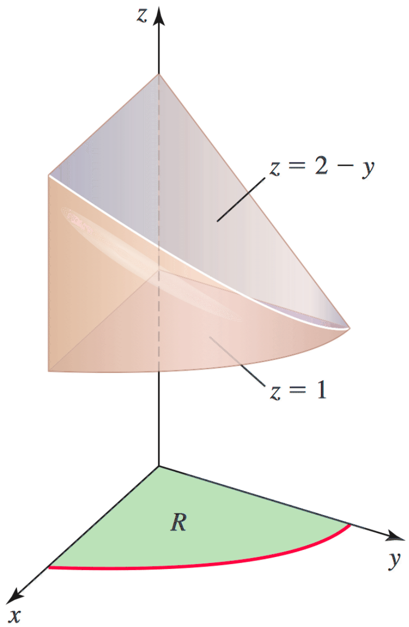
76. The solid S between the surfaces $z = e^{x-y}$ and $z = -e^{x-y}$, where S intersects the xy-plane in the region $R = \{(x, y) : 0 \le x \le y, 0 \le y \le 1\}$



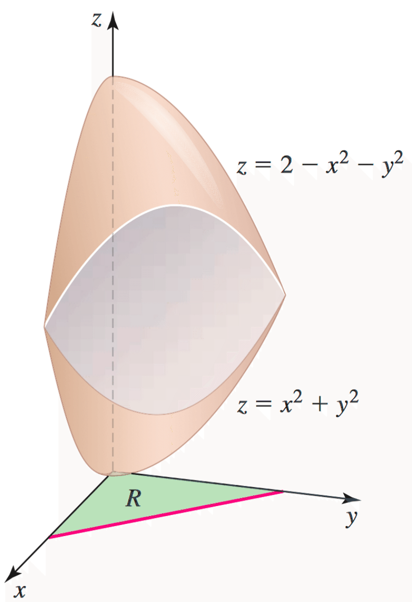
77. The solid above the region $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2 - x\}$ and between the planes $-4x - 4y + z = 0$ and $-2x - y + z = 8$



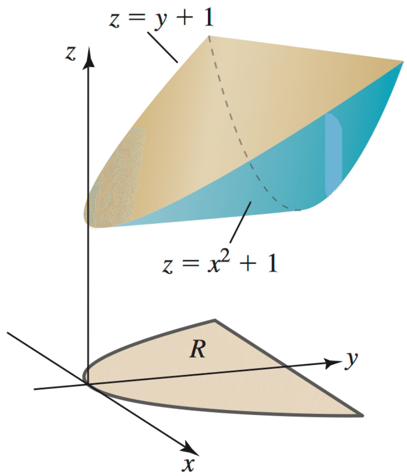
78. The solid in the first octant bounded by the planes $x = 0$, $y = 0$, $z = 1$, and $z = 2 - y$, and the cylinder $y = 1 - x^2$



79. The solid above the region $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1 - x\}$ bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and the coordinate planes in the first octant



80. The solid bounded by the parabolic cylinder $z = x^2 + 1$, and the planes $z = y + 1$ and $y = 1$



81–84. Volume using technology *Find the volume of the following solids. Use a computer algebra system to evaluate an appropriate iterated integrals.*

81. The column with a square base $R = \{(x, y) : |x| \le 1, |y| \le 1\}$ cut by the plane $z = 4 - x - y$

82. The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 1 - 2y$

83. The wedge sliced from the cylinder $x^2 + y^2 = 1$ by the planes $z = a(2 - x)$ and $z = a(x - 2)$, where $a > 0$

84. The solid bounded by the elliptical cylinder $x^2 + 3y^2 = 12$ and the planes $z = 0$, and the paraboloid $z = 3x^2 + y^2 + 1$

85–90. Area of plane regions *Use double integrals to compute the area of the following regions.*

85. The region bounded by the parabola $y = x^2$ and the line $y = 4$

86. The region bounded by the parabola $y = x^2$ and the line $y = x + 2$

87. The region in the first quadrant bounded by $y = e^x$ and $x = \ln 2$

88. The region bounded by $y = 1 + \sin x$ and $y = 1 - \sin x$ on the interval $[0, \pi]$

89. The region in the first quadrant bounded by $y = x^2$, $y = 5x + 6$, and $y = 6 - x$

90. The region bounded by the lines $x = 0$, $x = 4$, $y = x$, and $y = 2x + 1$

91. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. In the iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$, the limits a and b must be constants or functions of x.

b. In the iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$, the limits c and d must be functions of y.

c. Changing the order of integration gives $\int_0^2 \int_1^y f(x, y)\, dx\, dy = \int_1^y \int_0^2 f(x, y)\, dy\, dx$.

Explorations and Challenges »

92. Related integrals Evaluate each integral.

a. $\int_0^4 \int_0^4 (4 - x - y)\, dx\, dy$

b. $\int_0^4 \int_0^4 |4 - x - y|\, dx\, dy$

93. Sliced block Find the volume of the solid bounded by the planes $x = 0$, $x = 5$, $z = y - 1$, $z = -2y - 1$, $z = 0$, and $z = 2$.

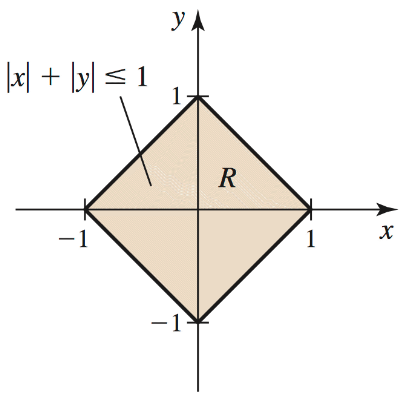
94. Square region Consider the region $R = \{(x, y) : |x| + |y| \le 1\}$ shown in the figure.

a. Use a double integral to show that the area of R is 2.

b. Find the volume of the square column whose base is R and whose upper surface is $z = 12 - 3x - 4y$.

c. Find the volume of the solid above R and beneath the cylinder $x^2 + z^2 = 1$.

d. Find the volume of the pyramid whose base is R and whose vertex is on the z-axis at $(0, 0, 6)$.



95–96. Average value *Use the definition for the average value of a function over a region R (Section 16.1),*

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

95. Find the average value of $a - x - y$ over the region $R = \{(x, y) : x + y \le a,\ x \ge 0,\ y \ge 0\}$, where $a > 0$.

96. Find the average value of $z = a^2 - x^2 - y^2$ over the region $R = \{(x, y) : x^2 + y^2 \le a^2\}$, where $a > 0$.

T **97–98. Area integrals** *Consider the following regions R. Use a computer algebra system to evaluate the integrals.*

a. *Sketch the region R.*

b. *Evaluate* $\iint_R dA$ *to determine the area of the region.*

c. *Evaluate* $\iint_R xy\, dA$.

97. R is the region between both branches of $y = \frac{1}{x}$ and the lines $y = x + \frac{3}{2}$ and $y = x - \frac{3}{2}$.

98. R is the region bounded by the ellipse $\frac{x^2}{18} + \frac{y^2}{36} = 1$ with $y \le \frac{4x}{3}$.

99–102. Improper integrals *Many improper double integrals may be handled using the techniques for improper integrals in one variable (Section 8.9). For example, under suitable conditions on f,*

$$\int_a^\infty \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx = \lim_{b \to \infty} \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Use or extend the one-variable methods for improper integrals to evaluate the following integrals.

99. $\int_1^\infty \int_0^{e^{-x}} x\, y\, dy\, dx$

100. $\int_1^\infty \int_0^{1/x^2} \frac{2y}{x}\, dy\, dx$

101. $\int_0^\infty \int_0^\infty e^{-x-y}\, dy\, dx$

102. $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\, dy\, dx$