16 Multiple Integration

Chapter Preview We have now generalized limits and derivatives to functions of several variables. The next step is to carry out a similar process with respect to integration. As you know, single (one-variable) integrals are developed from Riemann sums and are used to compute areas of regions in $\mathbb{R}^2$. In an analogous way, we use Riemann sums to develop double (two-variable) and triple (three-variable) integrals, which are used to compute volumes of solid regions in $\mathbb{R}^3$. These multiple integrals have many applications in statistics, science, and engineering, including calculating the mass, the center of mass, and moments of inertia of solids with a variable density. Another significant development in this chapter is the appearance of cylindrical and spherical coordinates. These alternative coordinate systems often simplify the evaluation of integrals in three-dimensional space. The chapter closes with the two- and three-dimensional versions of the substitution (change of variables) rule. The overall lesson of the chapter is that we can integrate functions over most geometrical objects, from intervals on the x-axis to regions in the plane bounded by curves to complicated three-dimensional solids.

16.1 Double Integrals over Rectangular Regions

In Chapter 15 the concept of differentiation was extended to functions of several variables. In this chapter we extend integration to multivariable functions. By the close of the chapter, we will have completed Table 16.1, which is a basic road map for calculus.

Table 16.1

	Derivatives	**Integrals**
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ and $f(x, y, z)$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA$, $\iiint_D f(x, y, z)\,dV$

Volumes of Solids »

The problem of finding the net area of a region bounded by a curve led to the definite integral in Chapter 5. Recall that we began that discussion by approximating the region with a collection of rectangles and then formed a Riemann sum of the areas of the rectangles. Under appropriate conditions, as the number of rectangles increases, the sum approaches the value of the definite integral, which is the net area of the region.

We now carry out an analogous procedure with surfaces defined by functions of the form $z = f(x, y)$, where, for the moment, we assume $f(x, y) \geq 0$ on a region R in the xy-plane (**Figure 16.1**). The goal is to determine the volume of the solid bounded by the surface and R. In general terms, the solid is first approximated by boxes. The sum of the volumes of these boxes, which is a Riemann sum, approximates the volume of the solid. Under appropriate conditions, as the number of boxes increases, the approximations converge to the value of a *double integral*, which is the volume of the solid.

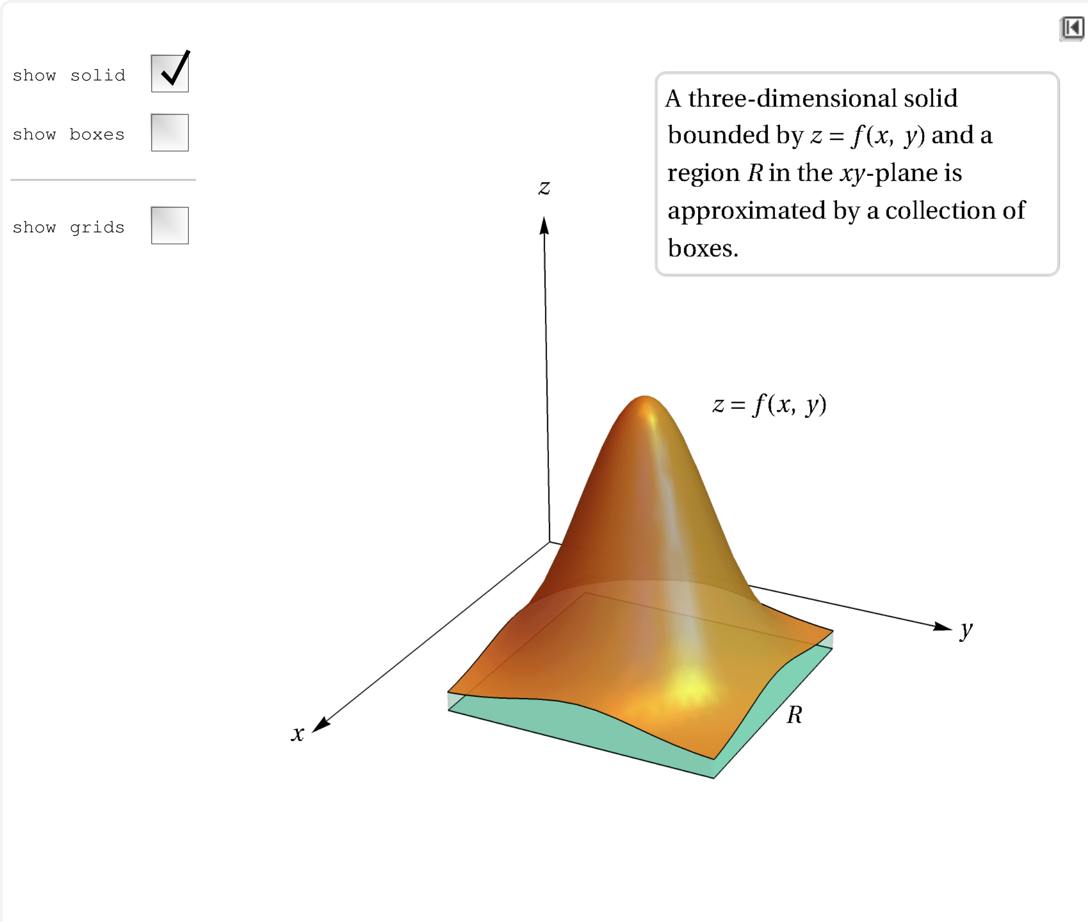


Figure 16.1

We assume $z = f(x, y)$ is a nonnegative continuous function on a *rectangular* region $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. A **partition** of R is formed by dividing R into n rectangular regions using lines running parallel to the x- and y-axes (not necessarily uniformly spaced). The subregions may be numbered in any systematic way; for example, left to right, and then bottom to top. The side lengths of the kth rectangle are denoted Δx_k and Δy_k, so the area of the kth subregion is $\Delta A_k = \Delta x_k \, \Delta y_k$. We also let (x_k^*, y_k^*) be any point in the kth subregion, for $1 \le k \le n$ (**Figure 16.2**).

Note »

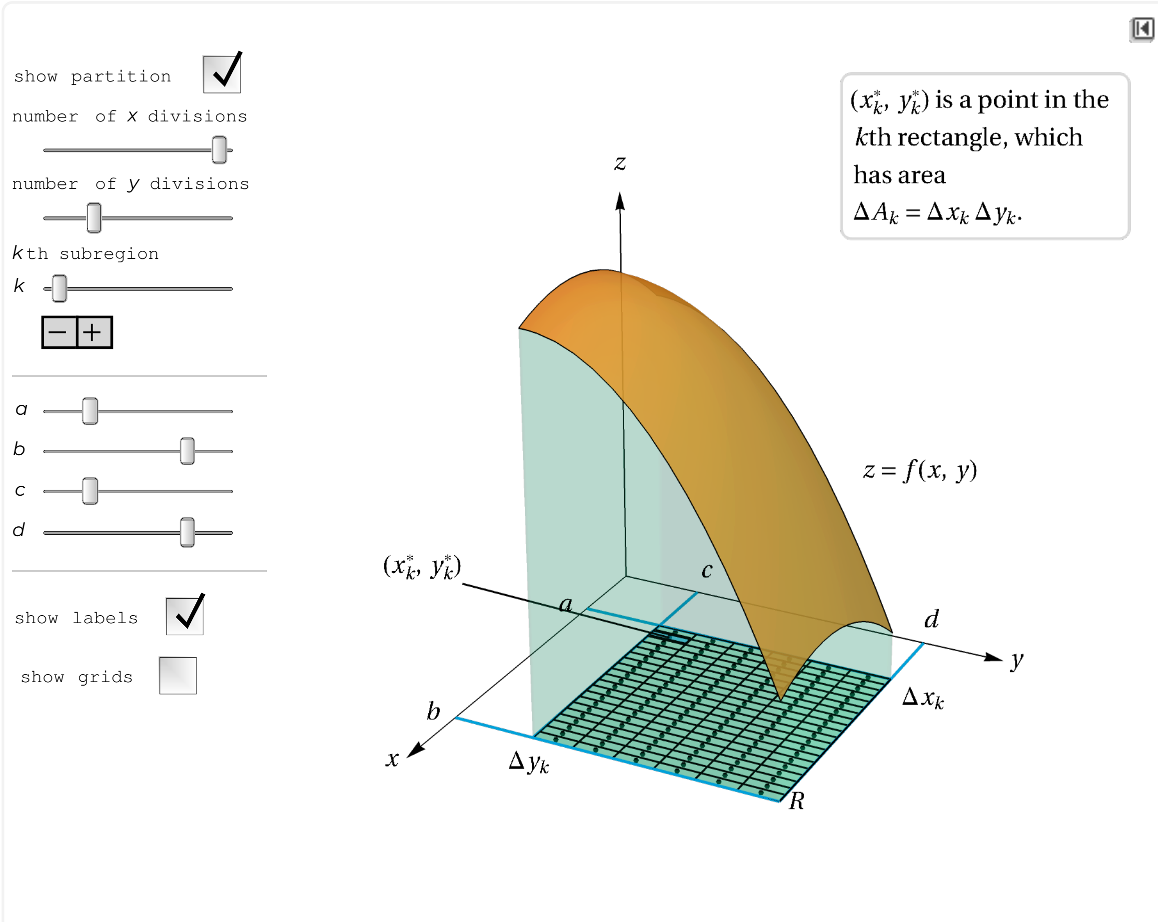


Figure 16.2

To approximate the volume of the solid bounded by the surface $z = f(x, y)$ and the region R, we construct boxes on each of the n subregions; each box has a height of $f(x_k^*, y_k^*)$ and a base with area ΔA_k, for $1 \le k \le n$ (**Figure 16.3**). Therefore, the volume of the kth box is

$$f(x_k^*, y_k^*)\,\Delta A_k = f(x_k^*, y_k^*)\,\Delta x_k\,\Delta y_k.$$

The sum of the volumes of the n boxes gives an approximation to the volume of the solid:

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

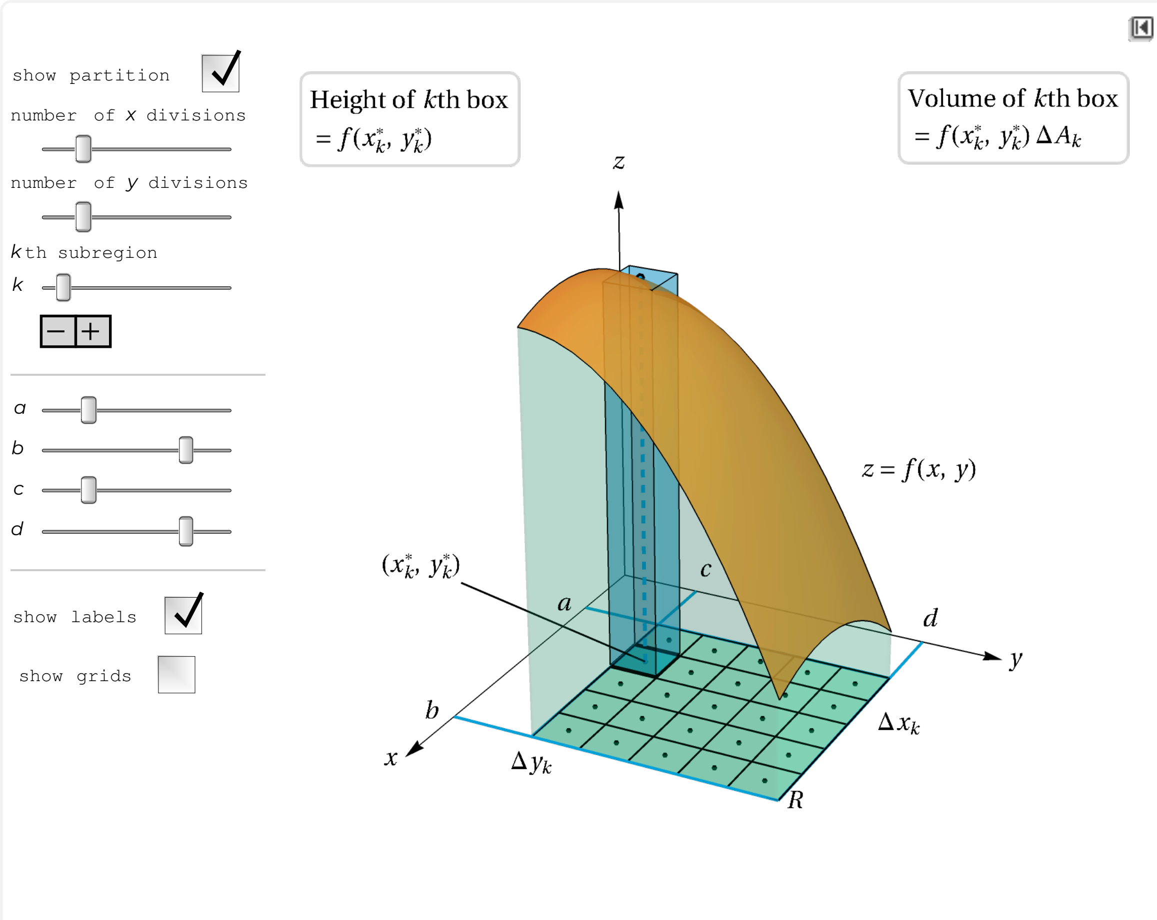


Figure 16.3

Quick Check 1 Explain why the sum for the volume is an approximation. How can the approximation be improved? ◆

Answer »

> The sum gives the volume of a collection of rectangular boxes and these boxes do not exactly fill the solid region under the surface. The approximation is improved by using more boxes.

We now let Δ be the maximum length of the diagonals of the rectangular subregions in the partition. As $\Delta \to 0$, the areas of *all* subregions approach zero ($\Delta A_k \to 0$) and the number of subregions increases ($n \to \infty$). If the approximations given by these Riemann sums have a limit as $\Delta \to 0$, then we define the volume of the solid to be that limit (**Figure 16.4**).

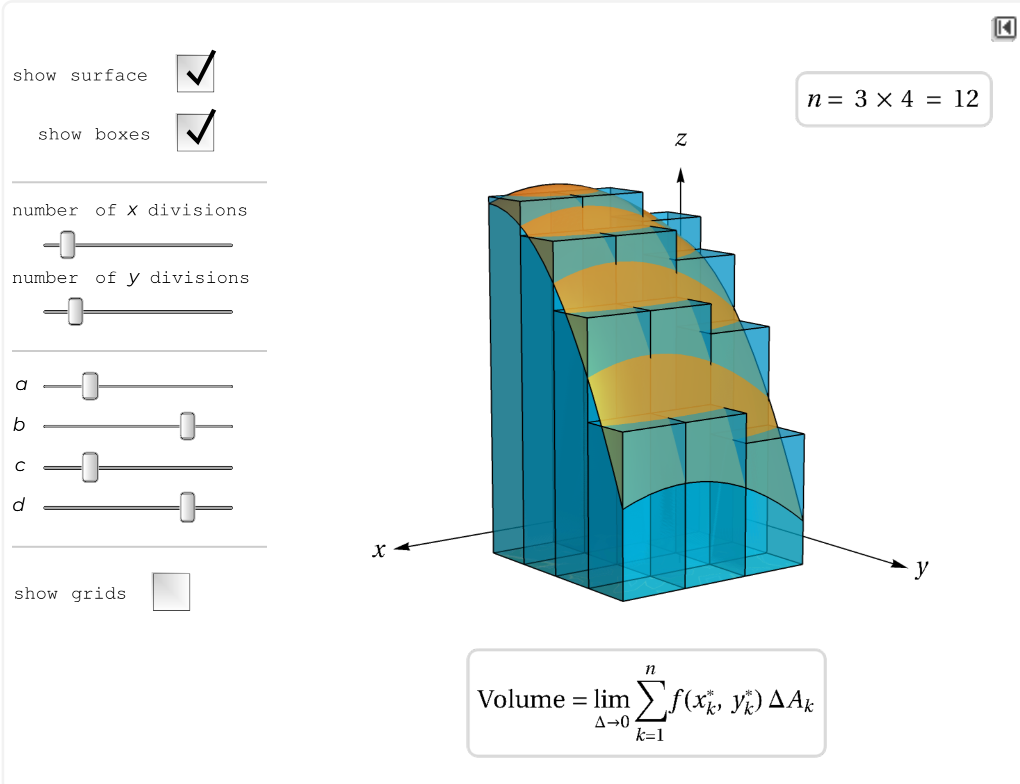


Figure 16.4

DEFINITION **Double Integrals**

A function f defined on a rectangular region R in the xy-plane is **integrable** on R if

$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint_R f(x, y)\,dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

Note »

The functions that we encounter in this book are integrable. Advanced methods are needed to prove that continuous functions and many functions with finite discontinuities are also integrable.

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R. If f is negative on parts of R, the value of the double integral may be zero or negative, and the result is interpreted as a *net volume* (in analogy with *net area* for single variable integrals).

Iterated Integrals »

Evaluating double integrals using limits of Riemann sums is tedious and rarely done. Fortunately, there is a practical method for evaluating double integrals that is based on the general slicing method (Section 6.3). An example illustrates the technique.

Note »

Suppose we wish to compute the volume of the solid region bounded by the plane $z = f(x, y) = 6 - 2x - y$ over the rectangular region $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 2\}$ (**Figure 16.5**).

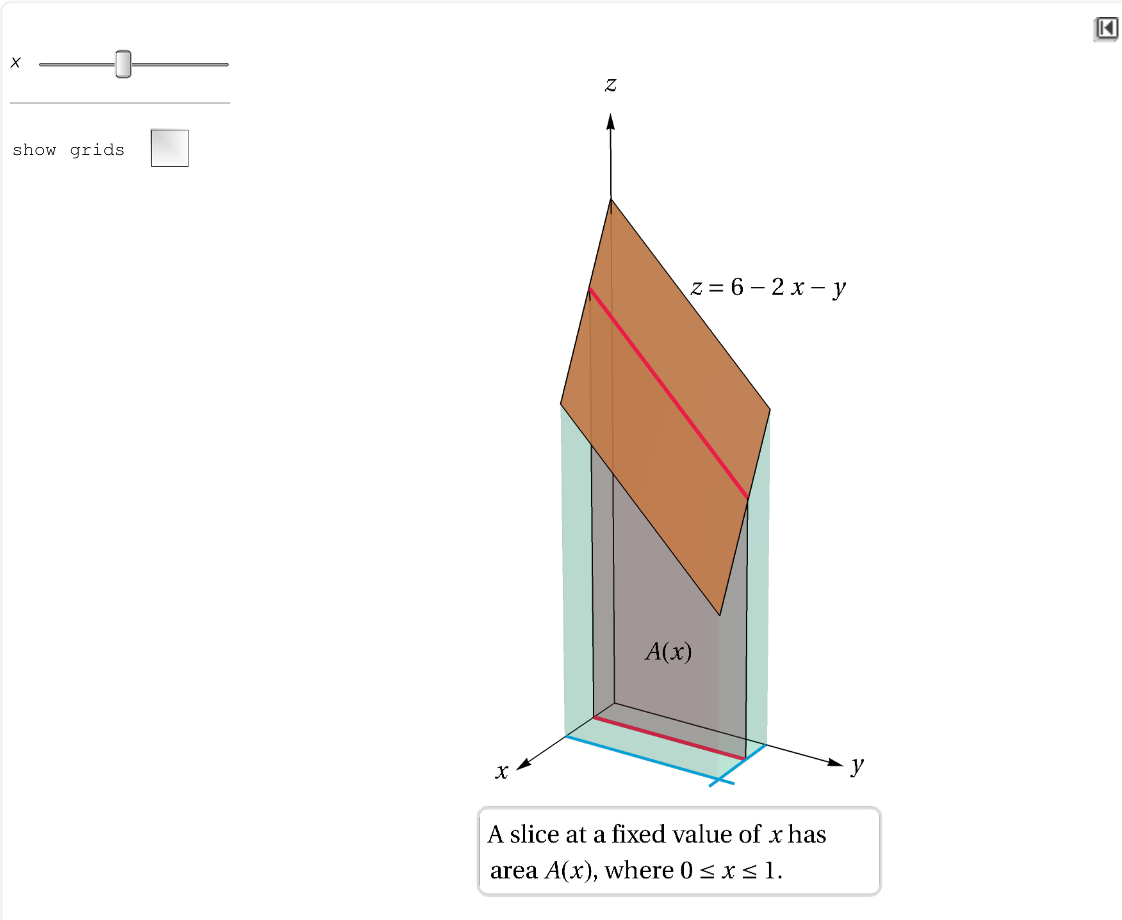


Figure 16.5

By definition, the volume is given by the double integral

$$V = \iint_R f(x, y)\, dA = \iint_R (6 - 2x - y)\, dA.$$

According to the General Slicing Method (see Note above), we can compute this volume by taking slices through the solid parallel to the yz-plane (Figure 16.5). The slice at the point x has a cross-sectional area denoted $A(x)$. In general, as x varies, the area $A(x)$ also changes, so we integrate these cross-sectional areas from $x = 0$ to $x = 1$ to obtain the volume

$$V = \int_0^1 A(x)\, dx.$$

The important observation is that for a fixed value of x, $A(x)$ is the area of the plane region under the curve $z = 6 - 2x - y$. This area is computed by integrating f with respect to y from $y = 0$ to $y = 2$, holding x fixed; that is,

$$A(x) = \int_0^2 (6 - 2x - y)\,dy,$$

where $0 \le x \le 1$, and x is treated as a constant in the integration. Substituting for $A(x)$, we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \underbrace{\left(\int_0^2 (6 - 2x - y)\,dy\right)}_{A(x)} dx.$$

The expression that appears on the right side of this equation is called an **iterated integral** (meaning repeated integral). We first evaluate the inner integral with respect to y holding x fixed; the result is a function of x. Then, the outer integral is evaluated with respect to x; the result is a real number, which is the volume of the solid in Figure 16.5. Both of these integrals are ordinary one-variable integrals.

EXAMPLE 1 Evaluating an iterated integral

Evaluate $V = \int_0^1 A(x)\,dx$, where $A(x) = \int_0^2 (6 - 2x - y)\,dy$.

SOLUTION »

Using the Fundamental Theorem of Calculus, holding x constant, we have

$$\begin{aligned} A(x) &= \int_0^2 (6 - 2x - y)\,dy \\ &= \left(6y - 2xy - \frac{y^2}{2}\right)\Bigg|_0^2 && \text{Evaluate integral with respect to } y;\ x \text{ is constant.} \\ &= (12 - 4x - 2) - 0 && \text{Simplify; limits are in } y. \\ &= 10 - 4x. && \text{Simplify.} \end{aligned}$$

Substituting $A(x) = 10 - 4x$ into the volume integral, we have

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 (10 - 4x)\,dx && \text{Substitute for } A(x). \\ &= \left(10x - 2x^2\right)\Big|_0^1 && \text{Evaluate integral with respect to } x. \\ &= 8. && \text{Simplify.} \end{aligned}$$

Related Exercises 10, 25 ◆

EXAMPLE 2 Same double integral, different order

Example 1 used slices through the solid parallel to the yz-plane. Compute the volume of the same solid using slices through the solid parallel to the xz-plane, for $0 \le y \le 2$ (**Figure 16.6**).

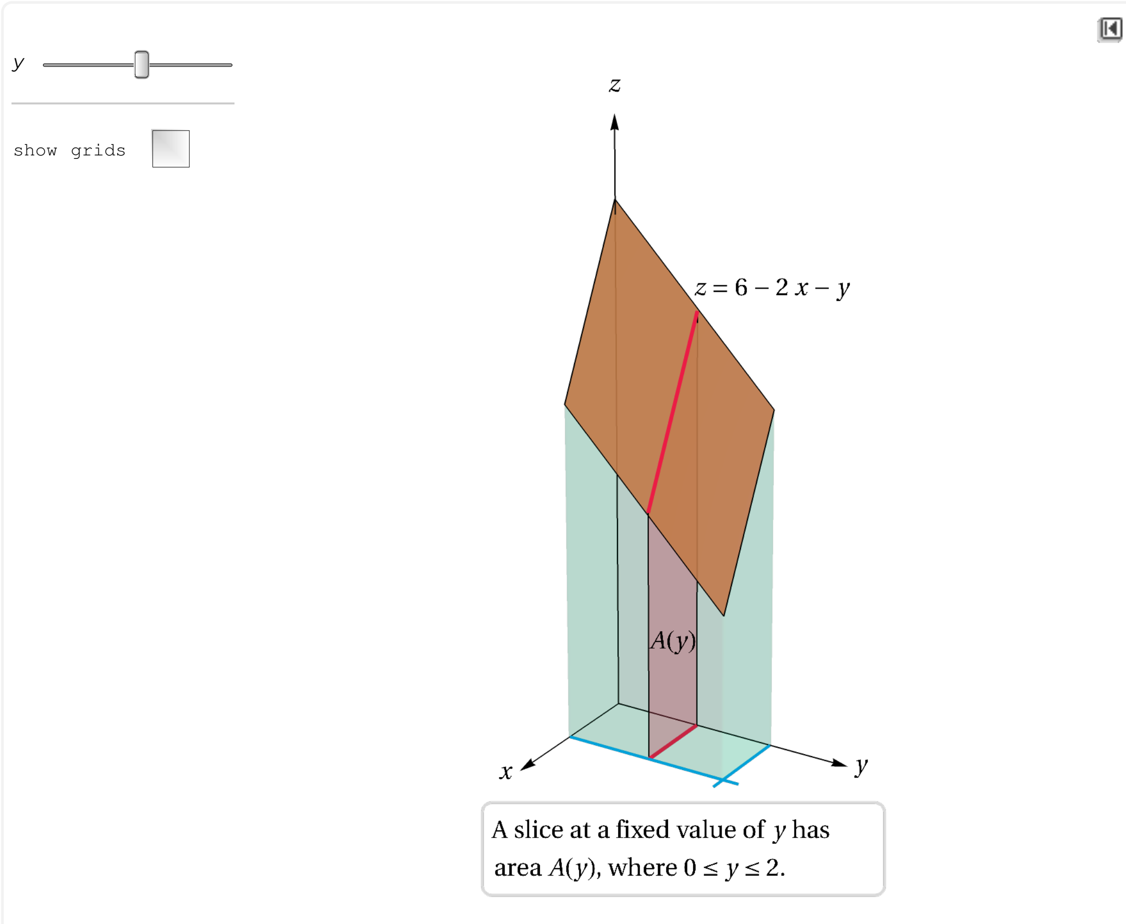


Figure 16.6

SOLUTION »

In this case, $A(y)$ is the area of a slice through the solid for a fixed value of y in the interval $0 \le y \le 2$. This area is computed by integrating $z = 6 - 2x - y$ from $x = 0$ to $x = 1$, holding y fixed; that is,

$$A(y) = \int_0^1 (6 - 2x - y)\, dx,$$

where $0 \le y \le 2$.

Using the General Slicing Method again, the volume is

$$\begin{aligned}
V &= \int_0^2 A(y)\,dy && \text{General slicing method}\\
&= \int_0^2 \underbrace{\left(\int_0^1 (6-2x-y)\,dx\right)}_{A(y)} dy && \text{Substitute for } A(y).\\
&= \int_0^2 \left[\left(6x - x^2 - yx\right)\Big|_0^1\right] dy && \text{Evaluate inner integral with respect to } x;\ y \text{ is constant.}\\
&= \int_0^2 (5-y)\,dy && \text{Simplify; limits are in } x.\\
&= \left(5y - \frac{y^2}{2}\right)\Big|_0^2 && \text{Evaluate outer integral with respect to } y.\\
&= 8. && \text{Simplify.}
\end{aligned}$$

Related Exercise 37 ◆

Several important comments are in order. First, the two iterated integrals give the same value for the double integral. Second, the notation of the iterated integral must be used carefully. When we write $\int_c^d \int_a^b f(x, y)\,dx\,dy$, it means $\int_c^d \left(\int_a^b f(x, y)\,dx\right) dy$. The *inner* integral with respect to x is evaluated first, holding y fixed, and the variable runs from $x = a$ to $x = b$. The result of that integration is a constant or a function of y, which is then integrated in the *outer* integral, with the variable running from $y = c$ to $y = d$. The order of integration is signified by the order of dx and dy.

Similarly, $\int_a^b \int_c^d f(x, y)\,dy\,dx$ means $\int_a^b \left(\int_c^d f(x, y)\,dy\right) dx$. The inner integral with respect to y is evaluated first, holding x fixed. The result is then integrated with respect to x. In both cases, the limits of integration in the iterated integrals determine the boundaries of the rectangular *region of integration*.

Examples 1 and 2 illustrate one version of *Fubini's Theorem*, a deep result that relates double integrals to iterated integrals. The first version of the theorem applies to double integrals over rectangular regions.

Quick Check 2 Consider the integral $\int_3^4 \int_1^2 f(x, y)\,dx\,dy$. Give the limits of integration and the variable of integration for the first (inner) integral and the second (outer) integral. Sketch the region of integration. ◆

Answer »

Inner integral: x runs from $x = 1$ to $x = 2$; outer integral: y runs from $y = 3$ to $y = 4$. The region is the rectangle $\{(x, y) : 1 \le x \le 2,\ 3 \le y \le 4\}$.

THEOREM 16.1 **(Fubini) Double Integrals over Rectangular Regions**

Let f be continuous on the rectangular region $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

Note »

> The area of the kth rectangular subregion in the partition is $\Delta A_k = \Delta x_k \, \Delta y_k$, where Δx_k and Δy_k are the lengths of the sides of that rectangle. Accordingly, the *element of area* in the double integral dA becomes $dx\,dy$ or $dy\,dx$ in the iterated integral.

The importance of Fubini's Theorem is twofold: It says that double integrals may be evaluated by iterated integrals. It *also* says that the order of integration in the iterated integrals does not matter (although in practice, one order of integration is often easier to use than the other).

EXAMPLE 3 A double integral

Find the volume of the solid bounded by the surface $z = 4 + 9x^2y^2$ over the region $R = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 2\}$. Use both possible orders of integration.

SOLUTION »

Because $f(x, y) > 0$ on R, the volume of the region is given by the double integral $\iint_R \left(4 + 9x^2y^2\right) dA$. By Fubini's Theorem, the double integral is evaluated as an iterated integral. If we first integrate with respect to x, the area of a cross section of the solid for a fixed value of y is given by $A(y)$ (**Figure 16.7**). The volume of the region is

$$\iint_R \left(4 + 9x^2y^2\right) dA = \int_0^2 \underbrace{\int_{-1}^1 \left(4 + 9x^2y^2\right) dx}_{A(y)}\, dy \quad \text{Convert to an iterated integral.}$$

$$= \int_0^2 \left(4x + 3x^3y^2\right)\Big|_{-1}^{1} dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \int_0^2 \left(8 + 6y^2\right) dy \quad \text{Simplify.}$$

$$= \left(8y + 2y^3\right)\Big|_0^2 \quad \text{Evaluate outer integral with respect to } y.$$

$$= 32. \quad \text{Simplify.}$$

Alternatively, if we integrate first with respect to y, the area of a cross section of the solid for a fixed value of x is given by $A(x)$ (Figure 16.7). The volume of the region is

$$\iint_R \left(4 + 9x^2y^2\right) dA = \int_{-1}^1 \underbrace{\int_0^2 \left(4 + 9x^2y^2\right) dy}_{A(x)}\, dx \quad \text{Convert to an iterated integral.}$$

$$= \int_{-1}^1 \left(4y + 3x^2y^3\right)\Big|_0^2 dx \quad \text{Evaluate inner integral with respect to } y.$$

$$= \int_{-1}^1 \left(8 + 24x^2\right) dx \quad \text{Simplify.}$$

$$= \left(8x + 8x^3\right)\Big|_{-1}^1 = 32. \quad \text{Evaluate outer integral with respect to } x.$$

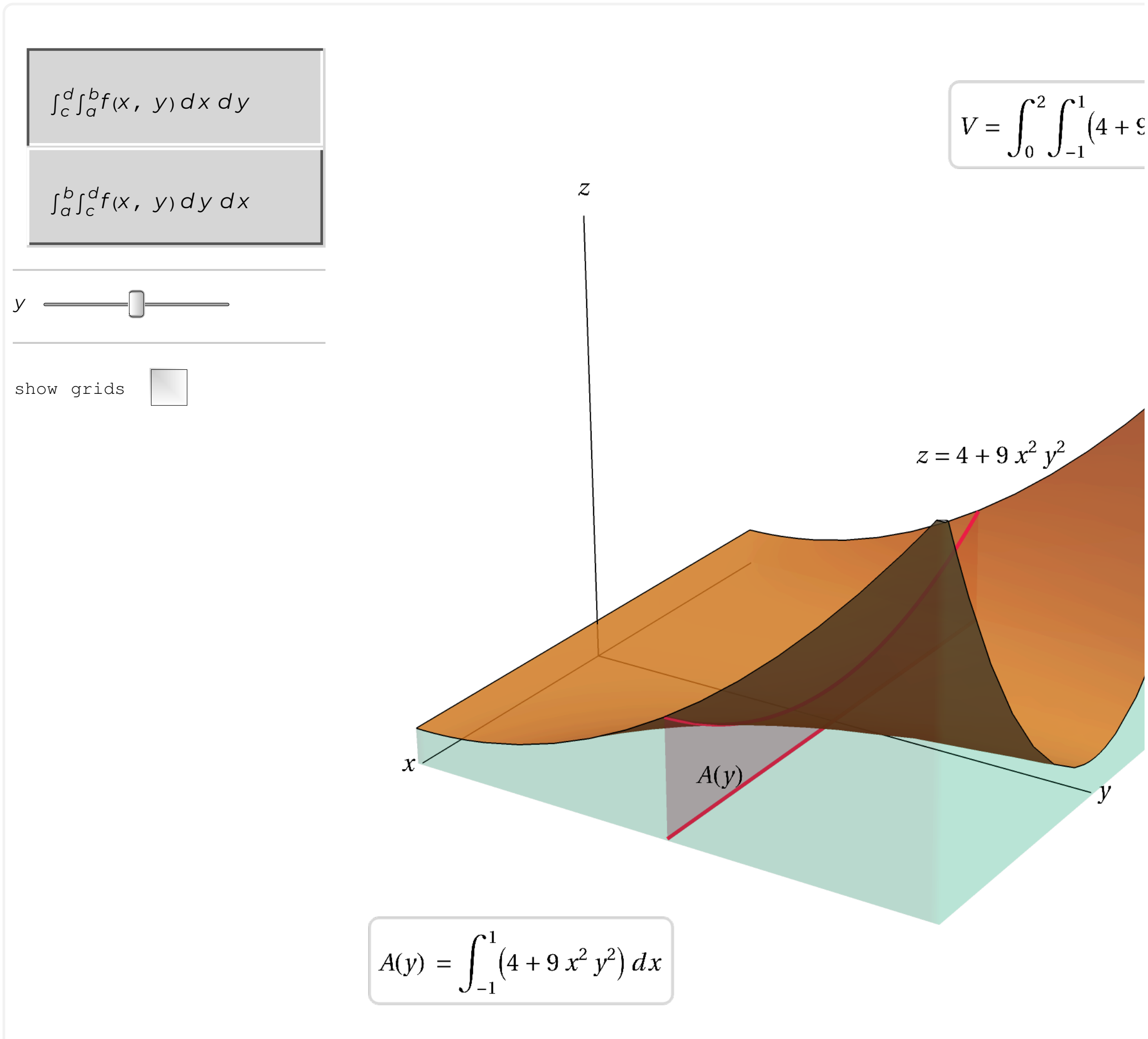


Figure 16.7

As guaranteed by Fubini's Theorem, the two iterated integrals agree, both giving the value of the double integral and the volume of the solid.

Related Exercises 26, 39 ◆

Quick Check 3 Write the iterated integral $\int_{-10}^{10}\int_0^{20}(x^2y + 2xy^3)\,dy\,dx$ with the order of integration reversed. ◆

Answer »

$$\int_0^{20}\int_{-10}^{10}(x^2y + 2xy^3)\,dx\,dy$$

The following example shows that sometimes the order of integration must be chosen carefully either to save work or to make the integration possible.

EXAMPLE 4 Choosing a convenient order of integration

Evaluate $\iint\limits_R y\,e^{xy}\,dA$, where $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le \ln 2\}$.

SOLUTION »

The iterated integral $\int_0^1 \int_0^{\ln 2} y\,e^{xy}\,dy\,dx$ requires first integrating $y\,e^{xy}$ with respect to y, which entails integration by parts. An easier approach is to integrate first with respect to x:

$$\int_0^{\ln 2} \int_0^1 y\,e^{xy}\,dx\,dy = \int_0^{\ln 2} e^{xy}\Big|_0^1\,dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \int_0^{\ln 2} (e^y - 1)\,dy \quad \text{Simplify.}$$

$$= (e^y - y)\Big|_0^{\ln 2} \quad \text{Evaluate outer integral with respect to } y.$$

$$= 1 - \ln 2 \quad \text{Simplify.}$$

Related Exercises 41, 43 ◆

Average Value »

The concept of the average value of a function (Section 5.4) extends naturally to functions of two variables. Recall that the average value of the integrable function f over the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

To find the average value of an integrable function f over a region R, we integrate f over R and divide the result by the "size" of R, which is the area of R in the two-variable case.

DEFINITION **Average Value of a Function over a Plane Region**

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\,dA.$$

Note »

EXAMPLE 5 **Average value**

Find the average value of the quantity $2 - x - y$ over the square $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 2\}$ (**Figure 16.8**).

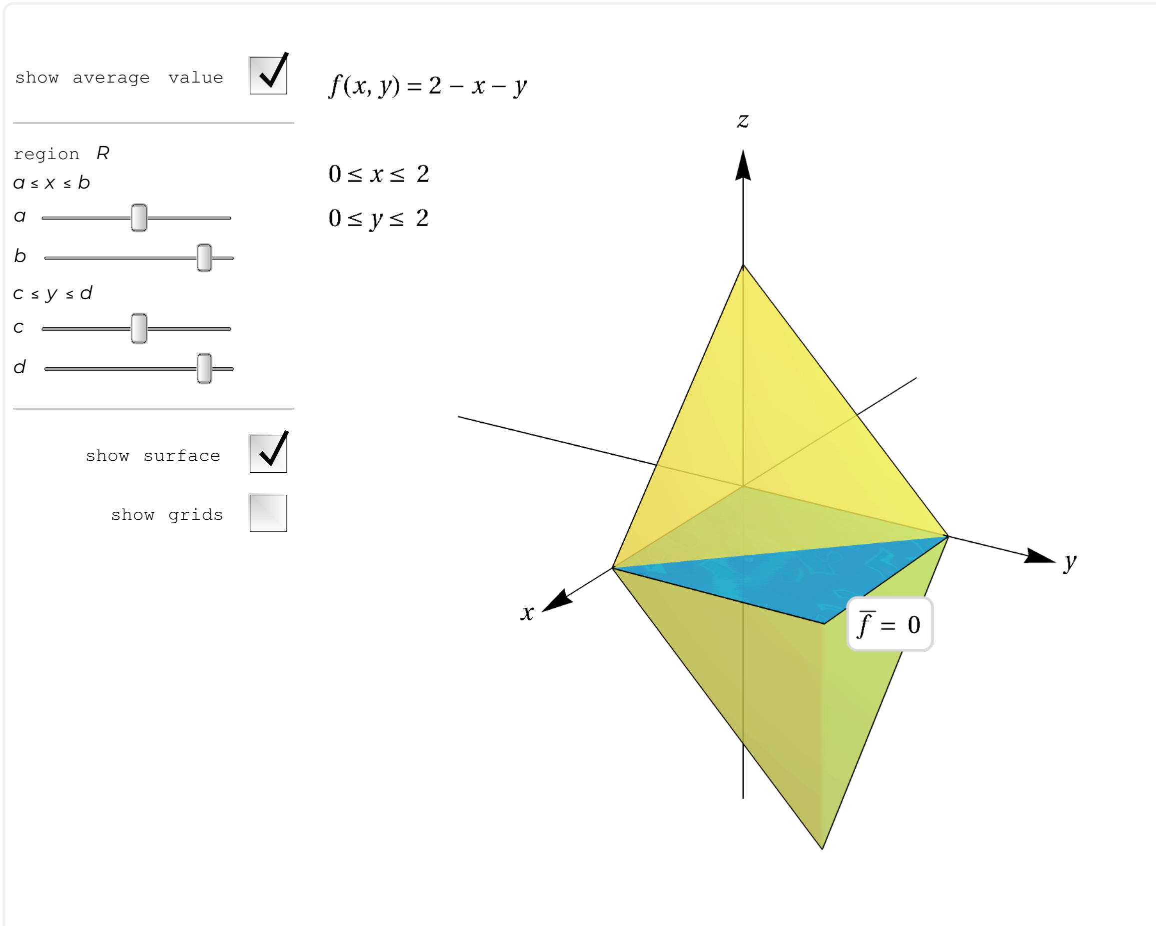


Figure 16.8

SOLUTION »

The area of the region R is 4. Letting $f(x, y) = 2 - x - y$, the average value of f is

$$
\begin{aligned}
\frac{1}{\text{area of } R} \iint_R f(x, y)\, dA &= \frac{1}{4} \iint_R (2 - x - y)\, dA \\
&= \frac{1}{4} \int_0^2 \int_0^2 (2 - x - y)\, dx\, dy && \text{Convert to an iterated integral.} \\
&= \frac{1}{4} \int_0^2 \left(2x - \frac{x^2}{2} - xy \right)\Big|_0^2\, dy && \text{Evaluate inner integral with respect to } x. \\
&= \frac{1}{4} \int_0^2 (2 - 2y)\, dy && \text{Simplify.} \\
&= 0. && \text{Evaluate outer integral with respect to } y.
\end{aligned}
$$

Note »

> An average value of 0 means that over the region R, the volume of the solid above the xy-plane and below the surface equals the volume of the solid below the xy-plane and above the surface.

Related Exercise 46 ◆

Exercises »

Getting Started »

Practice Exercises »

7–24. Iterated integrals *Evaluate the following iterated integrals.*

7. $\int_0^2 \int_0^1 4\,x\,y\,dx\,dy$

8. $\int_1^2 \int_0^1 \left(3\,x^2 + 4\,y^3\right) dy\,dx$

9. $\int_1^3 \int_0^2 x^2\,y\,dx\,dy$

10. $\int_0^3 \int_{-2}^1 (2\,x + 3\,y)\,dx\,dy$

11. $\int_1^3 \int_0^{\pi/2} x \sin y\,dy\,dx$

12. $\int_1^3 \int_1^2 \left(y^2 + y\right) dx\,dy$

13. $\int_1^4 \int_0^4 \sqrt{u\,v}\;du\,dv$

14. $\int_0^{\pi/4} \int_0^3 r \sec\theta\,dr\,d\theta$

15. $\int_1^{\ln 5} \int_0^{\ln 3} e^{x+y}\,dx\,dy$

16. $\int_0^{\pi/2} \int_0^1 u\,v \cos\left(u^2\,v\right) du\,dv$

17. $\int_0^1 \int_0^1 t^2\,e^{s\,t}\,ds\,dt$

18. $\int_0^2 \int_0^1 \frac{8\,x\,y}{1 + x^4}\,dx\,dy$

19. $\int_1^e \int_0^1 4\,(p + q) \ln q\,dp\,dq$

20. $\int_0^1 \int_0^{\pi} y^2 \cos xy\,dx\,dy$

21. $\int_1^2 \int_1^2 \frac{x}{x + y}\,dy\,dx$

22. $\int_0^2 \int_0^1 x^5\, y^2\, e^{x^3\, y^3}\, dy\, dx$

23. $\int_0^1 \int_1^4 \frac{3\, y}{\sqrt{x + y^2}}\, dx\, dy$

24. $\int_0^1 \int_0^1 x^2\, y^2\, e^{x^3\, y}\, dx\, dy$

25–35. Double integrals *Evaluate each double integral over the region R by converting it to an iterated integral.*

25. $\iint_R (x + 2\, y)\, dA;\ R = \{(x,\ y) : 0 \le x \le 3,\ 1 \le y \le 4\}$

26. $\iint_R \left(x^2 + x\, y\right) dA;\ R = \{(x,\ y) : 1 \le x \le 2,\ -1 \le y \le 1\}$

27. $\iint_R s^2\, t \sin\left(s\, t^2\right) dA;\ R = \{(s,\ t) : 0 \le s \le \pi,\ 0 \le t \le 1\}$

28. $\iint_R \frac{x}{1 + xy}\, dA;\ R = \{(x,\ y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

29. $\iint_R \sqrt{\frac{x}{y}}\, dA;\ R = \{(x,\ y) : 0 \le x \le 1,\ 1 \le y \le 4\}$

30. $\iint_R x\, y \sin x^2\, dA;\ R = \left\{(x,\ y) : 0 \le x \le \sqrt{\pi/2}\,,\ 0 \le y \le 1\right\}$

31. $\iint_R e^{x+2\, y}\, dA;\ R = \{(x,\ y) : 0 \le x \le \ln 2,\ 1 \le y \le \ln 3\}$

32. $\iint_R \left(x^2 - y^2\right)^2 dA;\ R = \{(x,\ y) : -1 \le x \le 2,\ 0 \le y \le 1\}$

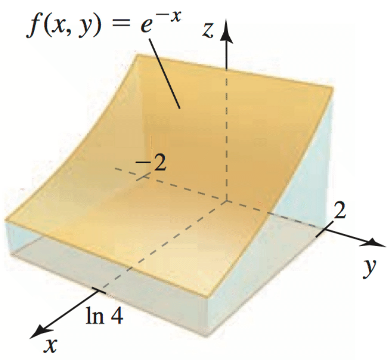
33. $\iint_R \left(x^5 - y^5\right)^2 dA;\ R = \{(x,\ y) : 0 \le x \le 1,\ -1 \le y \le 1\}$

34. $\iint_R \cos\left(x\, \sqrt{y}\right) dA;\ R = \left\{(x,\ y) : 0 \le x \le 1,\ 0 \le y \le \pi^2 / 4\right\}$

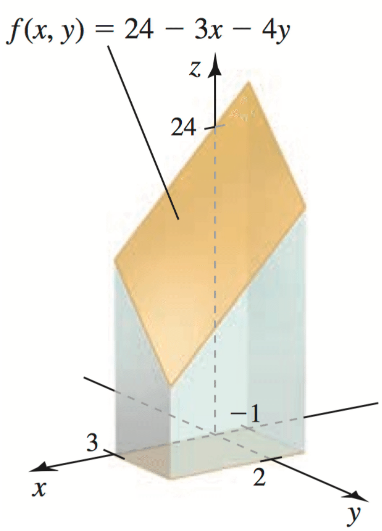
35. $\iint_R x^3\, y \cos\left(x^2\, y^2\right) dA;\ \ R = \left\{(x,\ y) : 0 \le x \le \sqrt{\pi/2}\,,\ 0 \le y \le 1\right\}$

36–39. Volumes of solids *Find the volume of the following solids.*

36. The solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y) : 0 \le x \le \ln 4, -2 \le y \le 2\}$



37. The solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y) : -1 \le x \le 3, 0 \le y \le 2\}$



38. The solid in the first octant bounded above by the surface $z = 9xy\sqrt{1 - x^2}\sqrt{4 - y^2}$ and below by the xy-plane

39. The solid in the first octant bounded by the surface $z = xy^2\sqrt{1 - x^2}$ and the planes $z = 0$ and $y = 3$

40–45. Choose a convenient order *When converted to an iterated integral, the following double integrals are easier to evaluate in one order than the other. Find the best order and evaluate the integral.*

40. $\displaystyle\iint_R y \cos xy \, dA$; $R = \left\{(x, y) : 0 \le x \le 1, 0 \le y \le \frac{\pi}{3}\right\}$

41. $\displaystyle\iint_R (y + 1)\, e^{x(y+1)} \, dA$; $R = \{(x, y) : 0 \le x \le 1, -1 \le y \le 1\}$

42. $\iint_R x \sec^2 xy \, dA$; $R = \left\{(x, y) : 0 \le x \le \frac{\pi}{3},\ 0 \le y \le 1\right\}$

43. $\iint_R 6x^5 e^{x^3 y} \, dA$; $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 2\}$

44. $\iint_R y^3 \sin\left(xy^2\right) dA$; $R = \left\{(x, y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{\frac{\pi}{2}}\right\}$

45. $\iint_R \frac{x}{(1 + xy)^2} \, dA$; $R = \{(x, y) : 0 \le x \le 4,\ 1 \le y \le 2\}$

46–48. Average value *Compute the average value of the following functions over the region R.*

46. $f(x, y) = 4 - x - y$; $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 2\}$

47. $f(x, y) = e^{-y}$; $R = \{(x, y) : 0 \le x \le 6,\ 0 \le y \le \ln 2\}$

48. $f(x, y) = \sin x \sin y$; $R = \{(x, y) : 0 \le x \le \pi,\ 0 \le y \le \pi\}$

49. **Average value** Find the average squared distance between the points of $R = \{(x, y) : -2 \le x \le 2,\ 0 \le y \le 2\}$ and the origin.

50. **Average value** Find the average squared distance between the points of $R = \{(x, y) : 0 \le x \le 3,\ 0 \le y \le 3\}$ and the point $(3, 3)$.

51. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The region of integration for $\int_4^6 \int_1^3 4 \, dx \, dy$ is a square.

b. If f is continuous on $\mathbb{R}^2$, then $\int_4^6 \int_1^3 f(x, y) \, dx \, dy = \int_4^6 \int_1^3 f(x, y) \, dy \, dx$.

c. If f is continuous on $\mathbb{R}^2$, then $\int_4^6 \int_1^3 f(x, y) \, dx \, dy = \int_1^3 \int_4^6 f(x, y) \, dy \, dx$.

52. **Symmetry** Evaluate the following integrals using symmetry arguments. Let $R = \{(x, y) : -a \le x \le a,\ -b \le y \le b\}$, where a and b are positive real numbers.

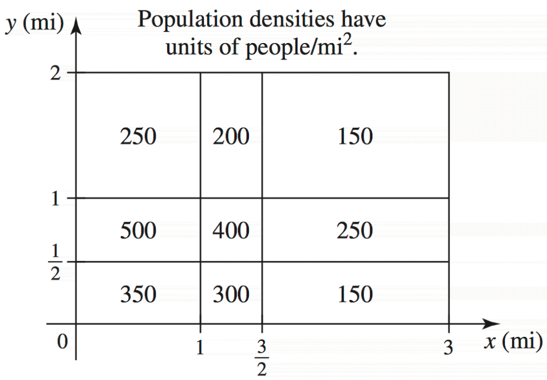
a. $\iint_R x y e^{-(x^2 + y^2)} \, dA$

b. $\iint_R \frac{\sin (x - y)}{x^2 + y^2 + 1} \, dA$

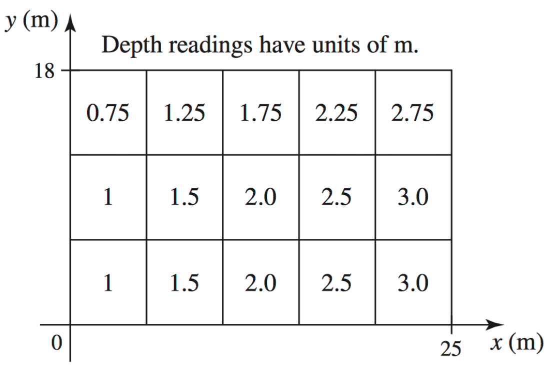
53. **Computing populations** The population densities in nine districts of a rectangular county are shown in the figure.

a. Use the fact that population $=$ (population density) $\times$ (area) to estimate the population of the county.

b. Explain how the calculation of part (a) is related to Riemann sums and double integrals.



T 54. Approximating water volume The varying depth of an 18 m × 25 m swimming pool is measured in 15 different rectangles of equal area (see figure). Approximate the volume of water in the pool.



Explorations and Challenges »

55. Cylinders Let S be the solid in $\mathbb{R}^3$ between the cylinder $z = f(x)$ and the region $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$, where $f(x) \ge 0$ on R. Explain why $\int_c^d \int_a^b f(x)\,dx\,dy$ equals the area of the constant cross section of S multiplied by $(d - c)$, which is the volume of S.

56. Product of integrals Suppose $f(x, y) = g(x)\,h(y)$, where g and h are continuous functions for all real values of x and y.

a. Show that $\int_c^d \int_a^b f(x, y)\,dx\,dy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$. Interpret this result geometrically.

b. Write $\left(\int_a^b g(x)\,dx\right)^2$ as an iterated integral.

c. Use the result of part (a) to evaluate $\int_0^{2\pi} \int_{10}^{30} e^{-4y^2} \cos x\,dy\,dx$.

57. Solving for a parameter Let $R = \{(x, y) : 0 \le x \le \pi,\ 0 \le y \le a\}$. For what values of a, with $0 \le a \le \pi$, is $\iint_R \sin(x + y)\,dA$ equal to 1?

58–59. Zero average value *Find the value of $a > 0$ such that the average value of the following functions over $R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le a\}$ is zero.*

58. $f(x, y) = x + y - 8$

59. $f(x, y) = 4 - x^2 - y^2$

60. Maximum integral Consider the plane $x + 3y + z = 6$ over the rectangle R with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), where the vertex (a, b) lies on the line where the plane intersects the xy-plane (so $a + 3b = 6$). Find the point (a, b) for which the volume of the solid between the plane and R is a maximum.

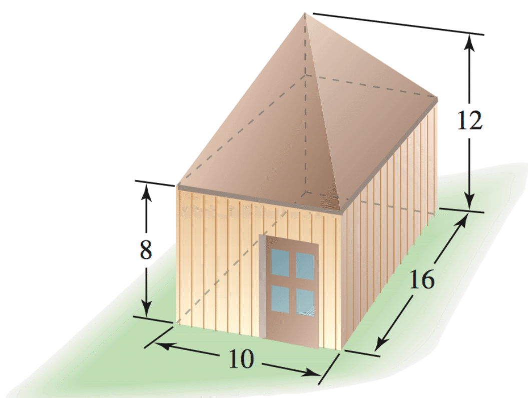
61. Density and mass Suppose a thin rectangular plate, represented by a region R in the xy-plane, has a density given by the function $\rho(x, y)$; this function gives the *area density* in units such as grams per square centimeter $\left(g/\text{cm}^2\right)$. The mass of the plate is $\iint_R \rho(x, y)\, dA$. Assume $R = \left\{(x, y) : 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \pi\right\}$ and find the mass of the plates with the following density functions.

a. $\rho(x, y) = 1 + \sin x$

b. $\rho(x, y) = 1 + \sin y$

c. $\rho(x, y) = 1 + \sin x \sin y$

62. Approximating volume Propose a method based on Riemann sums to approximate the volume of the shed shown in the figure (the peak of the roof is directly above the rear corner of the shed). Carry out the method and provide an estimate of the volume.



63. An identity Suppose the second partial derivatives of f are continuous on $R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le b\}$. Simplify $\iint_R \frac{\partial^2 f}{\partial x\, \partial y}\, dA$.