15.8 Lagrange Multipliers

One of many challenges in economics and marketing is predicting the behavior of consumers. Basic models of consumer behavior often involve a *utility function* that expresses consumers' combined preference for several different amenities. For example, a simple utility function might have the form $U = f(\ell, g)$, where ℓ represents the amount of leisure time and g represents the number of consumable goods. The model assumes consumers try to maximize their utility function, but they do so under certain constraints on the variables of the problem. For example, increasing leisure time may increase utility, but leisure time produces no income for consumable goods. Similarly, consumable goods may also increase utility, but they require income, which reduces leisure time. We first develop a general method for solving such constrained optimization problems and then return to economics problems later in the section.

The Basic Idea »

We start with a typical constrained optimization problem with two independent variables and give its method of solution; a generalization to more variables then follows. We seek maximum and/or minimum values of a differentiable **objective function** f with the restriction that x and y must lie on a **constraint** curve C in the xy-plane given by $g(x, y) = 0$ (**Figure 15.79**).

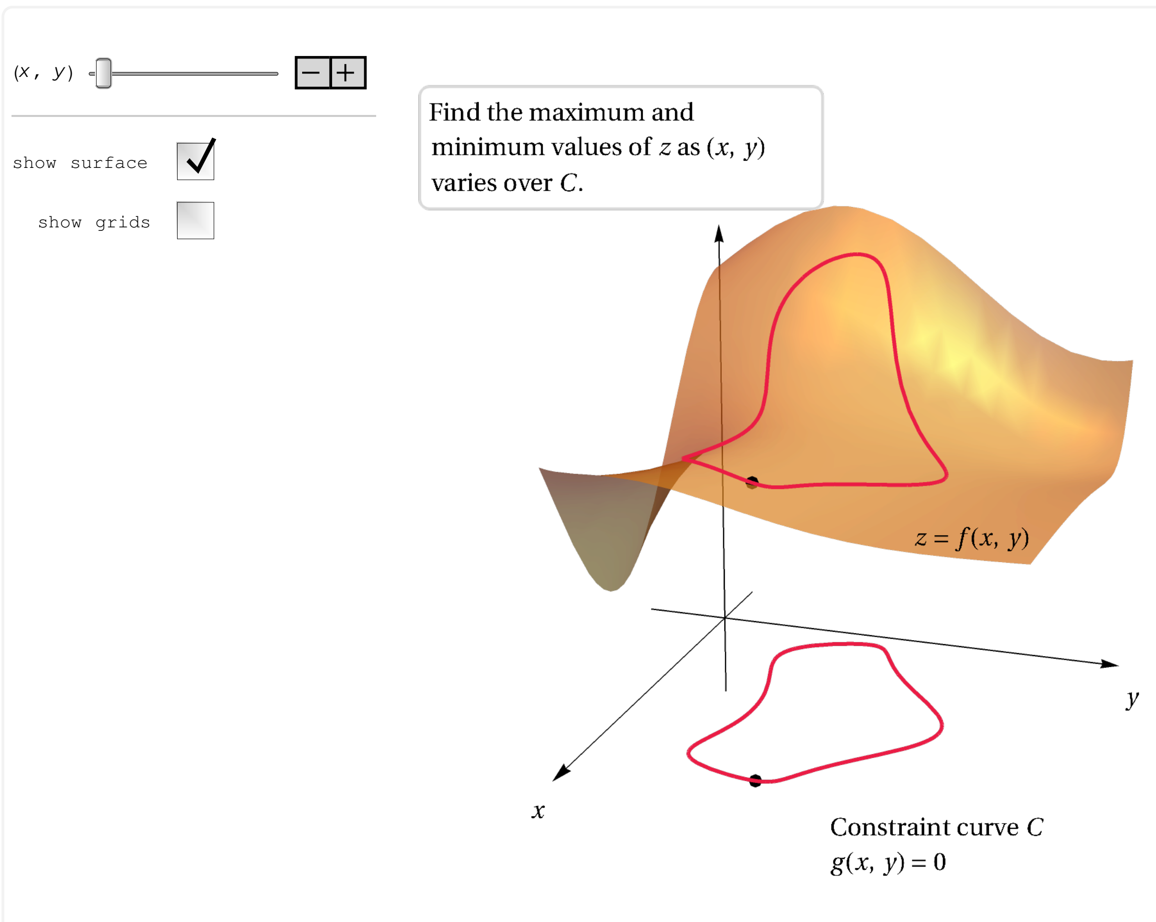


Figure 15.79

Figure 15.80 shows the detail of a typical situation in which we assume the (green) level curves of f have increasing z-values moving away from the origin. Now imagine moving along the (black) constraint curve

$C: g(x, y) = 0$ toward the point $P(a, b)$. As we approach P (from either side), the values of f evaluated on C increase, and as we move past P along C, the values of f decrease.

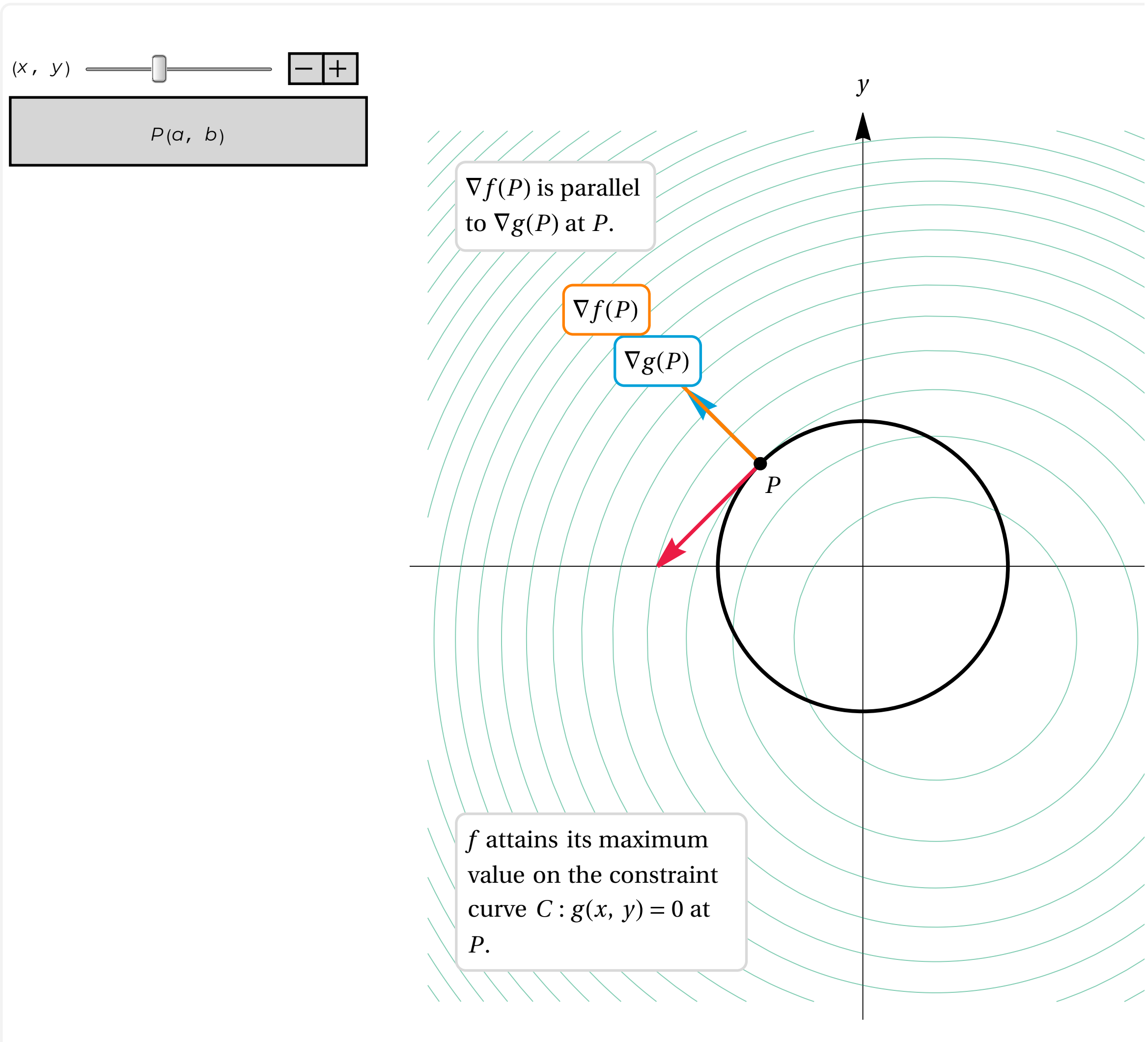


Figure 15.80

What is special about the point P at which f appears to have a local maximum value on C? From Theorem 15.12 (Section 15.5), we know that at any point $P(a, b)$ on a level curve of f, the line tangent to the level curve at P is orthogonal to $\nabla f(a, b)$. Figure 15.80 also suggests that the line tangent to the level curve of f at P is tangent to the constraint curve C at P. We prove this fact shortly. This observation implies that $\nabla f(a, b)$ is also orthogonal to the line tangent to C at $P(a, b)$.

We need one more observation. The constraint curve C is just one level curve of the function $z = g(x, y)$. Using Theorem 15.12 again, the line tangent to C at $P(a, b)$ is orthogonal to $\nabla g(a, b)$. We have now found two vectors $\nabla f(a, b)$ and $\nabla g(a, b)$ that are both orthogonal to the line tangent to the level curve C at $P(a, b)$. Therefore, these two gradient vectors are parallel. These properties characterize the point P at which f has a local extremum on the constraint curve. They are the basis of the method of *Lagrange multipliers* that we now formalize.

Lagrange Multipliers with Two Independent Variables »

The major step in establishing the method of Lagrange multipliers is to prove that Figure 15.80 is drawn correctly; that is, at the point on the constraint curve C where f has a local extreme value, the line tangent to C is orthogonal to $\nabla f(a, b)$ and $\nabla g(a, b)$.

THEOREM 15.16 **Parallel Gradients**

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume f has a local extreme value on C at a point $P(a, b)$. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Note »

The Greek lowercase ℓ is λ; it is read *lambda*.

Proof: Because C is smooth it can be expressed parametrically in the form $C : \mathbf{r}(t) = \langle x(t), y(t)\rangle$, where x and y are differentiable functions on an interval in t that contains t_0 with $P(a, b) = (x(t_0), y(t_0))$. As we vary t and follow C, the rate of change of f is given by the Chain Rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \mathbf{r}'(t).$$

At the point $(x(t_0), y(t_0)) = (a, b)$ at which f has a local maximum or minimum value, we have $\left.\dfrac{df}{dt}\right|_{t=t_0} = 0$, which implies that $\nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0$. Because $\mathbf{r}'(t)$ is tangent to C, the gradient $\nabla f(a, b)$ is orthogonal to the line tangent to C at P.

To prove the second assertion, note that the constraint curve C given by $g(x, y) = 0$ is also a level curve of the surface $z = g(x, y)$. Recall that gradients are orthogonal to level curves. Therefore, at the point $P(a, b)$, $\nabla g(a, b)$ is orthogonal to C at (a, b). Because both $\nabla f(a, b)$ and $\nabla g(a, b)$ are orthogonal to C, the two gradients are parallel, so there is a real number λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$. ◆

Theorem 15.16 leads directly to the method of Lagrange Multipliers, which produces candidates for *local* maxima and minima of f on the constraint curve. In many problems, however, the goal is to find *absolute* maxima and minima of f on the constraint curve. Much as we did with optimization problems in one variable, we find absolute extrema by examining both local extrema and endpoints. Several different cases arise:

- If the constraint curve is bounded (it lies within a circle of finite radius) and it closes on itself (for example, an ellipse), then we know that the absolute extrema of f exist. In this case, there are no endpoints to consider, and the absolute extrema are found among the local extrema.
- If the constraint curve is bounded and includes its endpoints but does not close on itself (for example, a closed line segment), then the absolute extrema of f exist, and we find them by examining the local extrema and the endpoints.
- In the case that the constraint curve is unbounded (for example, a line or a parabola) or the curve excludes one or both of its endpoints, we have no guarantee that absolute extrema exist. We can find local extrema, but they must be examined carefully to determine whether they are, in fact, absolute extrema (see Example 2 and Exercise 65).

We deal first with the case of finding absolute extrema on closed and bounded constraint curves.

Quick Check 1 It can be shown that the function $f(x, y) = x^2 + y^2$ attains its minimum value on the curve $C: g(x, y) = \frac{1}{4}(x-3)^2 - y = 0$ at the point $(1, 1)$. Verify that $\nabla f(1, 1)$ and $\nabla g(1, 1)$ are parallel, and that both vectors are orthogonal to the line tangent to C at $(1, 1)$, thereby confirming Theorem 15.16. ◆

Answer »

Note that $\nabla f(1, 1) = \langle 2x, 2y \rangle|_{(1,1)} = \langle 2, 2 \rangle$ and $\nabla g(1, 1) = \left\langle \frac{1}{2}(x-3), -1 \right\rangle\Big|_{(1,1)} = \langle -1, -1 \rangle$, which implies the gradients are multiples of one another, and therefore parallel. The equation of the line tangent to C at $(1, 1)$ is $y = -x + 2$; therefore the vector $\mathbf{v} = \langle 1, -1 \rangle$ is parallel to this tangent line. Because $\nabla f(1, 1) \cdot \mathbf{v} = 0$ and $\nabla g(1, 1) \cdot \mathbf{v} = 0$, both gradients are orthogonal to the tangent line.

PROCEDURE **Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves**

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$

2. Evaluate f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values, which are the absolute maximum and minimum values of f subject to the constraint.

Notice that $\nabla f = \lambda \nabla g$ is a vector equation: $\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$. It is satisfied provided $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Therefore, the crux of the method is solving the three equations

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \quad \text{and} \quad g(x, y) = 0$$

for the three variables x, y, and λ.

Note »

In principle, it is possible to solve a constrained optimization problem by solving the constraint equation for one of the variables and eliminating that variable in the objective function. In practice, this method is often prohibitive, particularly with three or more variables or two or more constraints.

EXAMPLE 1 Lagrange multipliers with two variables

Find the absolute maximum and minimum values of the objective function $f(x, y) = x^2 + y^2 + 2$, where x and y lie on the ellipse C given by $g(x, y) = x^2 + xy + y^2 - 4 = 0$.

SOLUTION »

Because C is closed and bounded, the absolute maximum and minimum values of f exist. **Figure 15.81a** shows the paraboloid $z = f(x, y)$ above the ellipse C in the xy-plane. As the ellipse is traversed, the corresponding function values on the surface vary. The goal is to find the maximum and minimum of these function values. An alternative view is given in **Figure 15.81b**, where we see the level curves of f and the constraint curve C. As the ellipse is traversed, the values of f vary, reaching maximum and minimum values along the way.

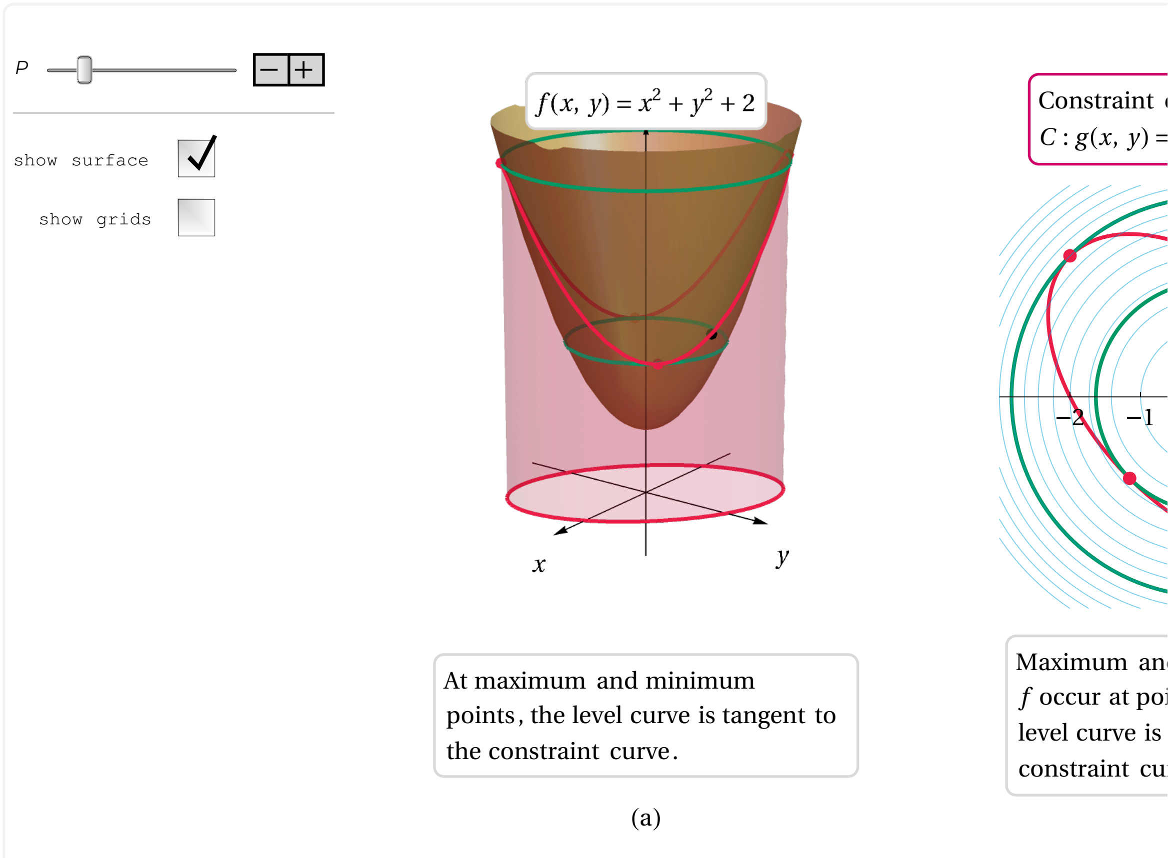


Figure 15.81

Noting that $\nabla f(x, y) = \langle 2x, 2y \rangle$ and $\nabla g(x, y) = \langle 2x + y, x + 2y \rangle$, the equations that result from $\nabla f = \lambda \nabla g$ and the constraint are

$$\underbrace{4x = \lambda(2x + y)}_{f_x = \lambda g_x}, \quad \underbrace{2y = \lambda(x + 2y)}_{f_y = \lambda g_y}, \quad \text{and} \quad \underbrace{x^2 + xy + y^2 - 4 = 0}_{\text{constraint } g(x,y) = 0}.$$

Subtracting the second equation from the first leads to

$$(x - y)(2 - \lambda) = 0,$$

which implies that $y = x$, or $\lambda = 2$. In the case that $y = x$, the constraint equation simplifies to $3x^2 - 4 = 0$, or $x = \pm \frac{2}{\sqrt{3}}$. Therefore, two candidates for locations of extreme values are $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$.

Substituting $\lambda = 2$ into the first equation leads to $y = -x$, and then the constraint equation simplifies to $x^2 - 4 = 0$, or $x = \pm 2$. These values give two additional points of interest, $(2, -2)$ and $(-2, 2)$. Evaluating f at each of these points, we find that $f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{14}{3}$ and $f(2, -2) = f(-2, 2) = 10$. Therefore, the absolute maximum of f on C is 10, which occurs at $(2, -2)$ and $(-2, 2)$, and the absolute minimum value of f on

C is $\frac{14}{3}$, which occurs at $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$. Notice that the value of λ is not used in final result.

Related Exercises 9–10 ◆

Quick Check 2 Choose any point on the constraint curve in Figure 15.81b other than a solution point. Draw ∇f and ∇g at that point and show that they are not parallel. ◆

Lagrange Multipliers with Three Independent Variables »

The technique just outlined extends to three or more independent variables. With three variables, suppose an objective function $w = f(x, y, z)$ is given; its level surfaces are surfaces in $\mathbb{R}^3$ (**Figure 15.82a**). The constraint equation takes the form $g(x, y, z) = 0$, which is another surface S in $\mathbb{R}^3$ (**Figure 15.82b**). To find the maximum and minimum values of f on S (assuming they exist), we must find the points (a, b, c) on S at which $\nabla f(a, b, c)$ is parallel to $\nabla g(a, b, c)$, assuming $\nabla g(a, b, c) \neq \mathbf{0}$ (**Figure 15.82c, d**). In the case where the surface $g(x, y, z) = 0$ is closed and bounded, the procedure for finding the absolute maximum and minimum values of $f(x, y, z)$, where the point (x, y, z) is constrained to lie on S, is similar to the procedure for two variables.

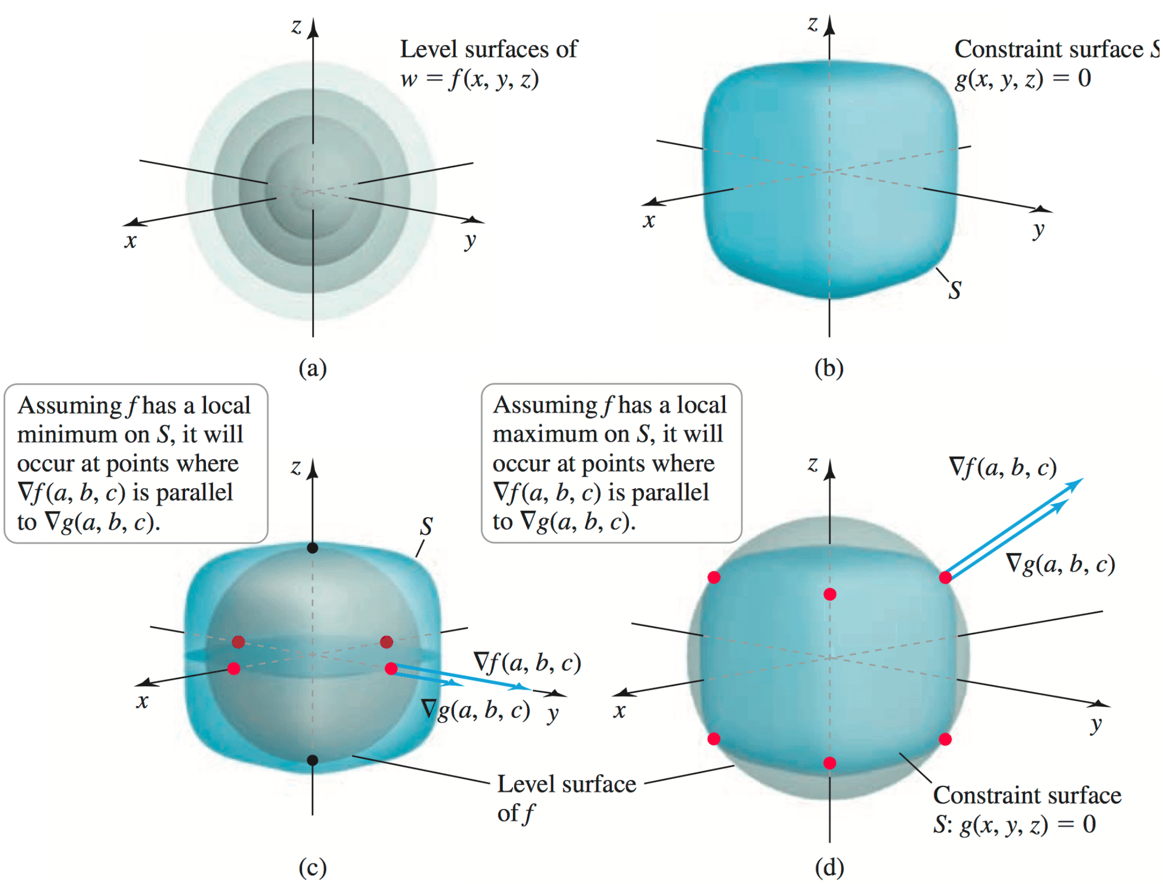


Figure 15.82

PROCEDURE **Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces**

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, carry out the following steps.

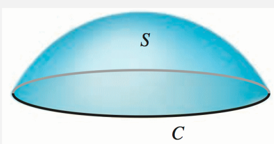
1. Find the values of x, y, z, and λ that satisfy the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0.$$

2. Among the points (x, y, z) found in Step 1, select the largest and smallest corresponding values of the objective function. These values are the absolute maximum and minimum values of f subject to the constraint.

Note »

If the constraint surface $S : g(x, y, z) = 0$ has a boundary curve C (see figure), then each point on C is a candidate for the location of an absolute maximum or minimum value of f, and these points must be analyzed in Step 2 of the procedure. We avoid this case in the exercise set.



Now, there are four equations to be solved for x, y, z, and λ:

$$f_x(x, y, z) = \lambda\, g_x(x, y, z), \qquad f_y(x, y, z) = \lambda\, g_y(x, y, z),$$
$$f_z(x, y, z) = \lambda\, g_z(x, y, z), \text{ and } \quad g(x, y, z) = 0.$$

As in the two-variable case, special care must be given to constraint surfaces that are not closed and bounded. We examine one such case in Example 2.

EXAMPLE 2 A geometry problem

Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

Note »

Problems similar to Example 2 were solved in Section 15.7 using ordinary optimization techniques. These methods may or may not be easier to apply than Lagrange multipliers.

SOLUTION »

The cone is not bounded, so we begin our calculations recognizing that solutions are only candidates for local extrema. **Figure 15.83** shows both sheets of the cone and the point $P(3, 4, 0)$.

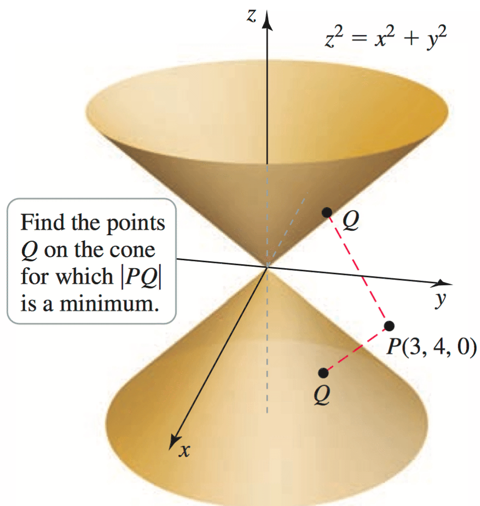


Figure 15.83

Because P is in the xy-plane, we anticipate two solutions, one for each sheet of the cone. The distance between P and any point $Q(x, y, z)$ on the cone is

$$d(x, y, z) = \sqrt{(x-3)^2 + (y-4)^2 + z^2}.$$

In many distance problems it is easier to work with the *square* of the distance to avoid dealing with square roots. This maneuver is allowable because if a point minimizes $(d(x, y, z))^2$, it also minimizes $d(x, y, z)$. Therefore, we define

$$f(x, y, z) = (d(x, y, z))^2 = (x-3)^2 + (y-4)^2 + z^2.$$

The constraint is the condition that the point (x, y, z) must lie on the cone, which implies $z^2 = x^2 + y^2$, or $g(x, y, z) = z^2 - x^2 - y^2 = 0$.

Now we proceed with Lagrange multipliers; the conditions are

$$f_x(x, y, z) = \lambda\, g_x(x, y, z), \text{ or } 2(x-3) = \lambda(-2x), \text{ or } x(1+\lambda) = 3, \quad (1)$$
$$f_y(x, y, z) = \lambda\, g_y(x, y, z), \text{ or } 2(y-4) = \lambda(-2y), \text{ or } y(1+\lambda) = 4, \quad (2)$$
$$f_z(x, y, z) = \lambda\, g_z(x, y, z), \text{ or } 2z = \lambda(2z), \text{ or } z = \lambda z, \text{ and} \quad (3)$$
$$g(x, y, z) = z^2 - x^2 - y^2 = 0. \quad (4)$$

The solutions of equation (3) (the simplest of the four equations) are either $z = 0$, or $\lambda = 1$ and $z \neq 0$. In the first case, if $z = 0$, then by equation (4), $x = y = 0$; however, $x = 0$ and $y = 0$ do not satisfy (1) and (2). So no solution results from this case.

On the other hand, if $\lambda = 1$ in equation (3), then by (1) and (2), we find that $x = \frac{3}{2}$ and $y = 2$. Using (4), the corresponding values of z are $\pm\frac{5}{2}$. Therefore, the two solutions and the values of f are

$$x = \frac{3}{2}, \quad y = 2, \quad z = \frac{5}{2} \quad \text{with } f\left(\frac{3}{2}, 2, \frac{5}{2}\right) = \frac{25}{2}, \text{ and}$$
$$x = \frac{3}{2}, \quad y = 2, \quad z = -\frac{5}{2} \quad \text{with } f\left(\frac{3}{2}, 2, -\frac{5}{2}\right) = \frac{25}{2}.$$

You can check that moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$ in any direction on the cone has the effect of increasing the values of f. Therefore, the points correspond to *local* minima of the distance function. Do these points also correspond to *absolute* minima? The domain of this problem is unbounded; however, one can argue geometrically that f increases without bound moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$ with $|x| \to \infty$ and $|y| \to \infty$. Therefore, these points correspond to absolute minimum values and the points on the cone nearest to $(3, 4, 0)$ are $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$, at a distance of $\sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$. (Recall that $f = d^2$.)

Note »

With three independent variables, it is possible to impose two constraints. These problems are explored in Exercises 61–64.

Related Exercises 32–34 ◆

Quick Check 3 In Example 2, is there a point that *maximizes* the distance between (3, 4, 0) and the cone? If the point (3, 4, 0) were replaced by (3, 4, 1), how many minimizing solutions would there be? ◆

Answer »

The distance between (3, 4, 0) and the cone can be arbitrarily large, so there is no maximizing solution. If the point of interest is not in the *xy*-plane, there is one minimizing solution.

Economic Models

In the opening of this section, we briefly described how utility functions are used to model consumer behavior. We now look in more detail at some specific— admittedly simple—utility functions and the constraints that are imposed upon them.

As described earlier, a prototype model for consumer behavior uses two independent variables: leisure time ℓ and consumable goods g. A utility function $U = f(\ell, g)$ measures consumer preferences for various combinations of leisure time and consumable goods. The following assumptions about utility functions are commonly made.

1. Utility increases if any variable increases (essentially, *more is better*).
2. Various combinations of leisure time and consumable goods have the same utility; that is, giving up some leisure time for additional consumable goods (or vice versa) results in the same utility.

The level curves of a typical utility function are shown in **Figure 15.84**. Assumption 1 is reflected by the fact that the utility values on the level curves increase as either ℓ or g increases. Consistent with Assumption 2, a single level curve shows the combinations of ℓ and g that have the same utility; for this reason, economists call the level curves *indifference curves*. Notice that if ℓ increases, then g must decrease on a level curve to maintain the same utility, and vice versa.

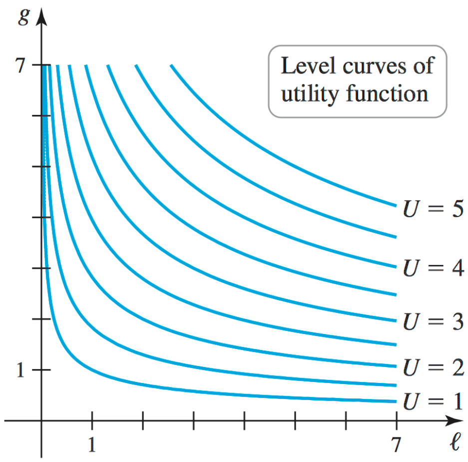


Figure 15.84

Economic models assert that consumers maximize utility subject to constraints on leisure time and consumable goods. One assumption that leads to a reasonable constraint is that an increase in leisure time implies a linear decrease in consumable goods. Therefore, the constraint curve is a line with negative slope (**Figure 15.85**). When such a constraint is superimposed on the level curves of the utility function, the optimization problem becomes evident. Among all points on the constraint line, which one maximizes utility? A solution is marked in the figure; at this point the utility has a maximum value (between 2.5 and 3.0).

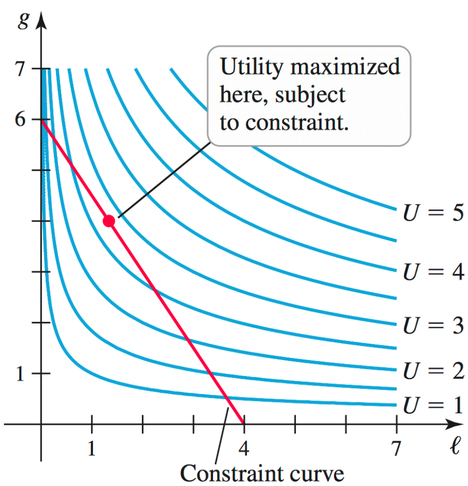


Figure 15.85

EXAMPLE 3 Constrained optimization of utility

Find the absolute maximum value of the utility function $U = f(\ell, g) = \ell^{1/3} g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \ge 0$ and $g \ge 0$.

SOLUTION »

The constraint is closed and bounded, so we expect to find an absolute maximum value of f. The level curves of the utility function and the linear constraint are shown in Figure 15.85. The solution follows the Lagrange multiplier method with two variables. The gradient of the utility function is

$$\nabla f(\ell, g) = \left\langle \frac{\ell^{-2/3}\, g^{2/3}}{3}, \frac{2\,\ell^{1/3}\, g^{-1/3}}{3} \right\rangle = \frac{1}{3}\left\langle \left(\frac{g}{\ell}\right)^{2/3}, 2\left(\frac{\ell}{g}\right)^{1/3} \right\rangle.$$

The gradient of the constraint function is $\nabla G(\ell, g) = \langle 3, 2\rangle$. Therefore, the equations that must be solved are

$$\frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3} = 3\,\lambda, \quad \frac{2}{3}\left(\frac{\ell}{g}\right)^{1/3} = 2\,\lambda, \quad \text{and} \quad G(\ell, g) = 3\,\ell + 2\,g - 12 = 0.$$

Eliminating λ from the first two equations leads to the condition $g = 3\,\ell$, which, when substituted into the constraint equation, gives the solution $\ell = \frac{4}{3}$ and $g = 4$. This point is a candidate for the location of the absolute maximum; the other candidates are the endpoints of the constraint curve, $(4, 0)$ and $(0, 6)$. The actual value of the utility function at these point are $U = f\left(\frac{4}{3}, 4\right) = \frac{4}{\sqrt[3]{3}} \approx 2.8$ and $f(4, 0) = f(0, 6) = 0$. We conclude that the maximum value of f is 2.8; this solution occurs at $\ell = \frac{4}{3}$ and $g = 4$, and it is consistent with Figure 15.85.

Related Exercise 38 ◆

Quick Check 4 In Figure 15.85, explain why, if you move away from the optimal point along the constraint line, the utility decreases. ◆

Answer »

If you move along the constraint line away from the optimal solution in either direction, you cross level curves of the utility function with decreasing values.

Exercises »

Getting Started »

Practice Exercises »

7–26. Lagrange multipliers *Each function f has an absolute maximum value and absolute minimum value subject to the given constraint. Use Lagrange multipliers to find these values.*

7. $f(x, y) = x + 2\,y$ subject to $x^2 + y^2 = 4$

8. $f(x, y) = x\,y^2$ subject to $x^2 + y^2 = 1$

9. $f(x, y) = x + y$ subject to $x^2 - x\,y + y^2 = 1$

10. $f(x, y) = x^2 + y^2$ subject to $2x^2 + 3\,x\,y + 2\,y^2 = 7$

11. $f(x, y) = x\,y$ subject to $x^2 + y^2 - x\,y = 9$

12. $f(x, y) = x - y$ subject to $x^2 + y^2 - 3\,x\,y = 20$

13. $f(x, y) = e^{x\,y}$ subject to $x^2 + x\,y + y^2 = 9$

14. $f(x, y) = x^2\,y$ subject to $x^2 + y^2 = 9$

15. $f(x, y) = 2x^2 + y^2$ subject to $x^2 + 2y + y^2 = 15$

16. $f(x, y) = x^2$ subject to $x^2 + xy + y^2 = 3$

17. $f(x, y, z) = x + 3y - z$ subject to $x^2 + y^2 + z^2 = 4$

18. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

19. $f(x, y, z) = x$ subject to $x^2 + y^2 + z^2 - z = 1$

20. $f(x, y, z) = x - z$ subject to $x^2 + y^2 + z^2 - y = 2$

21. $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 - xy = 5$

22. $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 - 2x - 2y = 1$

23. $f(x, y, z) = 2x + z^2$ subject to $x^2 + y^2 + 2z^2 = 25$

24. $f(x, y, z) = xy - z$ subject to $x^2 + y^2 + z^2 - xy = 1$

25. $f(x, y, z) = x^2 + y + z$ subject to $2x^2 + 2y^2 + z^2 = 2$

26. $f(x, y, z) = (xyz)^{1/2}$ subject to $x + y + z = 1$ with $x \geq 0$, $y \geq 0$, $z \geq 0$

27–36. Applications of Lagrange multipliers *Use Lagrange multipliers in the following problems. When the constraint curve is unbounded, explain why you have found an absolute maximum or minimum value.*

27. Shipping regulations A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (The girth is the perimeter of the smallest side of the box.)

28. Box with minimum surface area Find the dimensions of the rectangular box with a volume of 16 ft^3 that has minimum surface area.

T **29. Extreme distances to an ellipse** Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

30. Maximum area rectangle in an ellipse Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$.

31. Maximum perimeter rectangle in an ellipse Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$.

32. Minimum distance to a plane Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$.

33. Minimum distance to a surface Find the point on the surface $4x + y - 1 = 0$ closest to the point $(1, 2, -3)$.

34. Minimum distance to a cone Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(1, 2, 0)$.

35. **Extreme distances to a sphere** Find the minimum and maximum distances between the sphere $x^2 + y^2 + z^2 = 9$ and the point $(2, 3, 4)$.

36. **Maximum volume cylinder in a sphere** Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 16.

37–40. Maximizing utility functions *Find the values of ℓ and g with $\ell \geq 0$ and $g \geq 0$ that maximize the following utility functions subject to the given constraints. Give the value of the utility function at the optimal point.*

37. $U = f(\ell, g) = 10\,\ell^{1/2}\,g^{1/2}$ subject to $3\,\ell + 6\,g = 18$

38. $U = f(\ell, g) = 32\,\ell^{2/3}\,g^{1/3}$ subject to $4\,\ell + 2\,g = 12$

39. $U = f(\ell, g) = 8\,\ell^{4/5}\,g^{1/5}$ subject to $10\,\ell + 8\,g = 40$

40. $U = f(\ell, g) = \ell^{1/6}\,g^{5/6}$ subject to $4\,\ell + 5\,g = 20$

41. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

- **a.** Suppose you are standing at the center of a sphere looking at a point P on the surface of the sphere. Your line of sight to P is orthogonal to the plane tangent to the sphere at P.
- **b.** At a point that maximizes f on the curve $g(x, y) = 0$, the dot product $\nabla f \cdot \nabla g$ is zero.

42–47. Alternative method *Solve the following problems from Section 15.7 using Lagrange multipliers.*

42. Exercise 43

43. Exercise 44

44. Exercise 45

45. Exercise 46

46. Exercise 70

47. Exercise 63

48–51. Absolute maximum and minimum values *Find the absolute maximum and minimum values of the following functions over the given regions R. Use Lagrange multipliers to check for extreme points on the boundary.*

48. $f(x, y) = x^2 + 4\,y^2 + 1$; $R = \{(x, y) : x^2 + 4\,y^2 \leq 1\}$

49. $f(x, y) = x^2 + y^2 - 2\,y + 1$; $R = \{(x, y) : x^2 + y^2 \leq 4\}$ (This is Exercise 47, Section 15.7.)

50. $f(x, y) = 2\,x^2 + y^2$; $R = \{(x, y) : x^2 + y^2 \leq 16\}$ (This is Exercise 48, Section 15.7.)

51. $f(x, y) = 2\,x^2 - 4\,x + 3\,y^2 + 2$; $R = \{(x, y) : (x-1)^2 + y^2 \leq 1\}$ (This is Exercise 51, Section 15.7.)

52. **Extreme points on flattened spheres** The equation $x^{2n} + y^{2n} + z^{2n} = 1$, where n is a positive integer, describes a flattened sphere. Define the extreme points to be the points on the flattened sphere with a maximum distance from the origin.

a. Find all the extreme points on the flattened sphere with $n = 2$. What is the distance between the extreme points and the origin?

b. Find all the extreme points on the flattened sphere for integers $n > 2$. What is the distance between the extreme points and the origin?

c. Give the location of the extreme points in the limit as $n \to \infty$. What is the limiting distance between the extreme points and the origin as $n \to \infty$?

53–55. Production functions *Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = C K^a L^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $p K + q L = B$. Find the values of K and L that maximize the following production functions subject to the given constraint, assuming $K \ge 0$ and $L \ge 0$.*

53. $P = f(K, L) = K^{1/2} L^{1/2}$ for $20 K + 30 L = 300$

54. $P = f(K, L) = 10 K^{1/3} L^{2/3}$ for $30 K + 60 L = 360$

55. Given the production function $P = f(K, L) = K^a L^{1-a}$ and the budget constraint $p K + q L = B$, where a, p, q, and B are given, show that P is maximized when $K = \dfrac{a B}{p}$ and $L = \dfrac{(1-a) B}{q}$.

56. Temperature of an elliptical plate The temperature of points on an elliptical plate $x^2 + y^2 + x y \le 1$ is given by $T(x, y) = 25 \left(x^2 + y^2\right)$. Find the hottest and coldest temperatures on the edge of the plate.

Explorations and Challenges »

57–59. Maximizing a sum

57. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 16$.

58. Generalize Exercise 57 and find the maximum value of $x_1 + x_2 + \cdots + x_n$ subject to the condition that ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 = c^2$ for a real number c and a positive integer n.

59. Generalize Exercise 57 and find the maximum value of $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ subject to the condition that ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 = 1$, for given positive real numbers $a_1, \ldots, a_n$ and a positive integer n.

60. Geometric and arithmetic means Given positive numbers $x_1, \ldots, x_n$, prove that the geometric mean $(x_1 x_2 \cdots x_n)^{1/n}$ is no greater than the arithmetic mean $\dfrac{x_1 + \cdots + x_n}{n}$ in the following cases.

a. Find the maximum value of xyz, subject to $x + y + z = k$, where k is a positive real number and $x > 0$, $y > 0$, and $z > 0$. Use the result to prove that

$$(xyz)^{1/3} \le \frac{x + y + z}{3}.$$

b. Generalize part (a) and show that

$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + \cdots + x_n}{n}.$$

61. Problems with two constraints Given a differentiable function $w = f(x, y, z)$, the goal is to find its absolute maximum and minimum values (assuming they exist) subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, where g and h are also differentiable.

a. Imagine a level surface of the function f and the constraint surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Note that g and h intersect (in general) in a curve C on which maximum and minimum values of f must be found. Explain why ∇g and ∇h are orthogonal to their respective surfaces.

b. Explain why ∇f lies in the plane formed by ∇g and ∇h at a point of C where f has a maximum or minimum value.

c. Explain why part (b) implies that $\nabla f = \lambda \nabla g + \mu \nabla h$ at a point of C where f has a maximum or minimum value, where λ and μ (the Lagrange multipliers) are real numbers.

d. Conclude from part (c) that the equations that must be solved for maximum or minimum values of f subject to two constraints are $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$.

62–64. Two-constraint problems *Use the result of Exercise 61 to solve the following problems.*

62. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

63. Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the conditions that $x^2 + y^2 = 4$ and $x + y + z = 1$.

64. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

65. Check assumptions Consider the function $f(x, y) = xy + x + y + 100$ subject to the constraint $xy = 4$.

a. Use the method of Lagrange multipliers to write a system of three equations with three variables x, y, and λ.

b. Solve the system in part (a) to verify that $(x, y) = (-2, -2)$ and $(x, y) = (2, 2)$ are solutions.

c. Let the curve C_1 be the branch of the constraint curve corresponding to $x > 0$. Calculate $f(2, 2)$ and determine whether this value is an absolute maximum or minimum value of f over C_1. (*Hint:* Let $h_1(x)$, for $x > 0$, equal the values of f over the curve C_1 and determine whether h_1 attains an absolute maximum or minimum value at $x = 2$.)

d. Let the curve C_2 be the branch of the constraint curve corresponding to $x < 0$. Calculate $f(-2, -2)$ and determine whether this value is an absolute maximum or minimum value of f over C_2. (*Hint:* Let $h_2(x)$, for $x < 0$, equal the values of f over the curve C_2 and determine whether h_2 attains an absolute maximum or minimum value at $x = -2$.)

e. Show that the method of Lagrange multipliers fails to find the absolute maximum and minimum values of f over the constraint curve $xy = 4$. Reconcile your explanation with the method of Lagrange multipliers.