15.7 Maximum/Minimum Problems

In Chapter 4 we showed how to use derivatives to find maximum and minimum values of functions of a single variable. When those techniques are extended to functions of two variables, we discover both similarities and differences. The landscape of a surface is far more complicated than the profile of a curve in the plane, so we see more interesting features when working with several variables. In addition to peaks (maximum values) and hollows (minimum values), we encounter winding ridges, long valleys, and mountain passes. Yet despite these complications, many of the ideas used for single-variable functions reappear in higher dimensions. For example, the Second Derivative Test, suitably adapted for two variables, plays a central role. As with single-variable functions, the techniques developed here are useful for solving practical optimization problems.

Local Maximum/Minimum Values »

The concepts of local maximum and minimum values encountered in Chapter 4 extend readily to functions of two variables of the form $z = f(x, y)$. **Figure 15.67** shows a general surface defined on a domain D, which is a subset of $\mathbb{R}^2$. The surface has peaks (local high points) and hollows (local low points) at points in the interior of D. The goal is to locate and classify these extreme points.

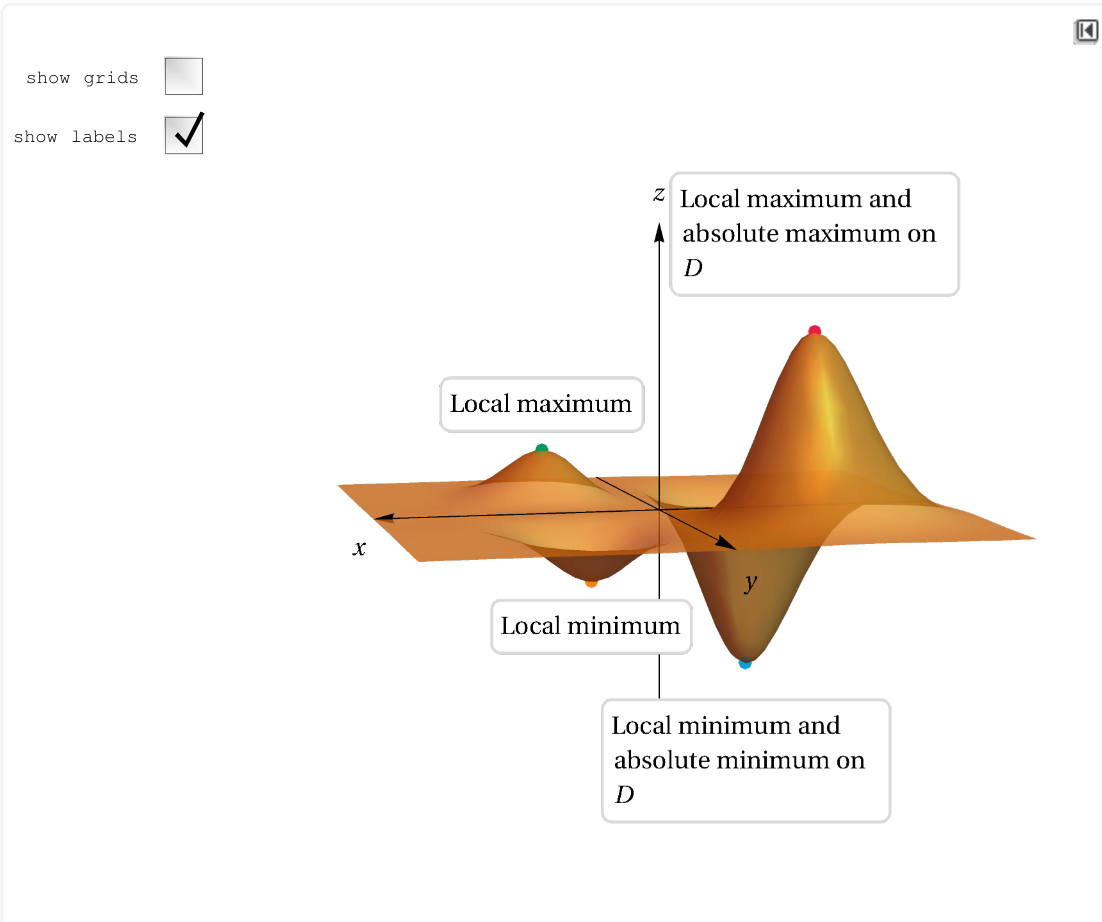


Figure 15.67

DEFINITION **Local Maximum/Minimum Values**

Suppose (a, b) is a point in a region R on which f is defined. If $f(x, y) \le f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local maximum value** of f. If $f(x, y) \ge f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local minimum value** of f. Local maximum or local minimum values are also called **local extreme values** or **local extrema**.

Note »

We maintain the convention adopted in Chapter 4 that local maxima or minima occur at interior points of the domain. Recall that an open disk centered at (a, b) is the set of points within a circle centered at (a, b).

In familiar terms, a local maximum is a point on a surface from which you cannot walk uphill. A local minimum is a point from which you cannot walk downhill. The following theorem is the analog of Theorem 4.2.

THEOREM 15.14 **Derivatives and Local Maximum/Minimum Values**

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Proof: Suppose f has a local maximum value at (a, b). The function of one variable $g(x) = f(x, b)$, obtained by holding $y = b$ fixed, also has a local maximum at (a, b). By Theorem 4.2, $g'(a) = 0$. However, $g'(a) = f_x(a, b)$; therefore, $f_x(a, b) = 0$. Similarly, the function $h(y) = f(a, y)$, obtained by holding $x = a$ fixed, has a local maximum at (a, b), which implies that $f_y(a, b) = h'(b) = 0$. An analogous argument is used for the local minimum case. ◆

Suppose f is differentiable at (a, b) (ensuring the existence of a tangent plane) and f has a local extremum at (a, b). Then, $f_x(a, b) = f_y(a, b) = 0$, which, when substituted into the equation of the tangent plane, gives the equation $z = f(a, b)$ (a constant). Therefore, if the tangent plane exists at a local extremum, then it is horizontal there.

Quick Check 1 The paraboloid $z = x^2 + y^2 - 4x + 2y + 5$ has a local minimum at $(2, -1)$. Verify the conclusion of Theorem 15.14 for this function. ◆

Answer »

$f_x(2, -1) = f_y(2, -1) = 0$.

Recall that for a function of one variable the condition $f'(a) = 0$ does not guarantee a local extremum at a. A similar precaution must be taken with Theorem 15.14. The conditions $f_x(a, b) = f_y(a, b) = 0$ do not imply that f has a local extremum at (a, b), as we show momentarily. Theorem 15.14 provides *candidates* for local extrema. We call these candidates *critical points*, as we did for functions of one variable. Therefore, the procedure for locating local maximum and minimum values is to find the critical points and then determine whether these candidates correspond to genuine local maximum and minimum values.

DEFINITION **Critical Point**

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives f_x or f_y does not exist at (a, b).

EXAMPLE 1 **Finding critical points**

Find the critical points of $f(x, y) = x\,y\,(x-2)\,(y+3)$.

SOLUTION »

This function is differentiable at all points of $\mathbb{R}^2$, so the critical points occur only at points where $f_x(x, y) = f_y(x, y) = 0$. Computing and simplifying the partial derivatives, these conditions become

$$f_x(x, y) = 2\,y\,(x-1)\,(y+3) = 0$$
$$f_y(x, y) = x\,(x-2)\,(2\,y+3) = 0.$$

We must now identify all (x, y) pairs that satisfy both equations. The first equation is satisfied if and only if $y = 0$, $x = 1$, or $y = -3$. We consider each of these cases.

- Substituting $y = 0$, the second equation is $3\,x\,(x-2) = 0$, which has solutions $x = 0$ and $x = 2$. So, $(0, 0)$ and $(2, 0)$ are critical points.

We find that there are five critical points: $(0, 0)$, $(2, 0)$, $\left(1, -\frac{3}{2}\right)$, $(0, -3)$, and $(2, -3)$. Some of these critical points may correspond to local maximum or minimum values. We return to this example and a complete analysis shortly.

Related Exercises 15, 18 ◆

Second Derivative Test »

Critical points are candidates for local extreme values. With functions of one variable, the Second Derivative Test may be used to determine whether critical points correspond to local maxima or minima (the test can also be inconclusive). The analogous test for functions of two variables not only detects local maxima and minima, but also identifies another type of point known as a *saddle point*.

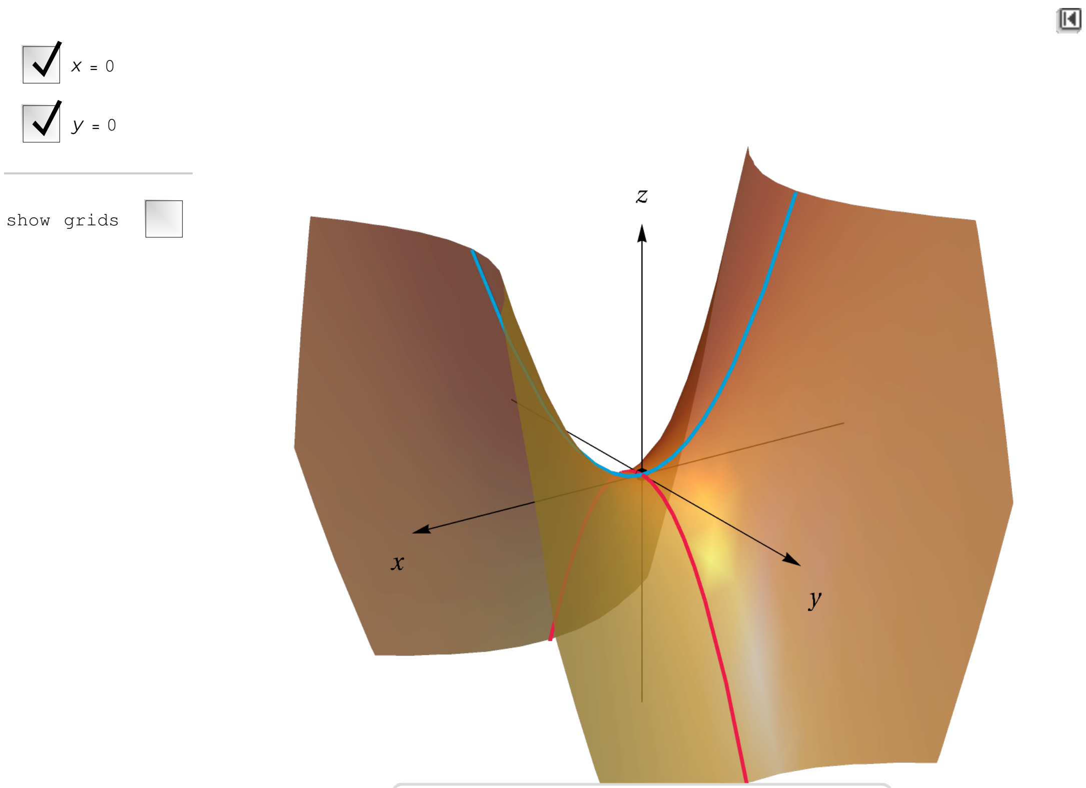
DEFINITION **Saddle Point**

Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Note »

The usual image of a saddle point is that of a mountain pass (or a horse saddle), where you can walk upward in some directions and downward in other directions. The definition of a saddle point we have given includes other less common situations. For example, with this definition, the cylinder $z = x^3$ has a line of saddle points along the y-axis.

If (a, b) is a critical point of f and f has a saddle point at (a, b), then from the point $(a, b, f(a, b))$, it is possible to walk uphill in some directions and downhill in other directions. The function $f(x, y) = x^2 - y^2$ (a hyperbolic paraboloid) is a good example to remember. The surface *rises* from (0, 0) along the x-axis and *falls* from (0, 0) along the y-axis (**Figure 15.68**). We can easily check that $f_x(0, 0) = f_y(0, 0) = 0$, demonstrating that critical points do not necessarily correspond to local maxima or minima.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

Figure 15.68

Quick Check 2 Consider the plane tangent to a surface at a saddle point. In what direction does the normal to the plane point? ◆

Answer »

Vertically, in the directions $\langle 0, 0, \pm 1 \rangle$

THEOREM 15.15 Second Derivative Test

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - \left(f_{xy}(x, y)\right)^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).

2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).

3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

4. If $D(a, b) = 0$, then the test is inconclusive.

Note »

The Second Derivative Test for functions of a single variable states that if a is a critical point with $f'(a) = 0$, then $f''(a) > 0$ implies that f has a local minimum at a, $f''(a) < 0$ implies that f has a local maximum at a, and if $f''(a) = 0$, the test is inconclusive. Theorem 15.15 is easier to remember if you notice the parallels between the two second derivative tests.

The proof of this theorem is given in Appendix A, but a few comments are in order. The test relies on the quantity $D(x, y) = f_{xx} f_{yy} - \left(f_{xy}\right)^2$, which is called the **discriminant** of f. It can be remembered as the 2×2 determinant of the **Hessian** matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, where $f_{xy} = f_{yx}$, provided these derivatives are continuous (Theorem 15.4). The condition $D(x, y) > 0$ means that the surface has the same general behavior in all directions near (a, b); either the surface rises in all directions, or it falls in all directions. In the case that $D(a, b) = 0$, the test is inconclusive: (a, b) could correspond to a local maximum, a local minimum, or a saddle point.

Finally, another useful characterization of a saddle point can be derived from Theorem 15.15: The tangent plane at a saddle point lies both above and below the surface.

Quick Check 3 Compute the discriminant $D(x, y)$ of $f(x, y) = x^2 y^2$. ◆

Answer »

$D(x, y) = -12 x^2 y^2$

EXAMPLE 2 Analyzing critical points

Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.

SOLUTION »

We begin with the following derivative calculations:

$$f_x = 2x - 4, \quad f_y = 4y + 4,$$
$$f_{xx} = 2, \quad f_{xy} = f_{yx} = 0, \text{ and } \quad f_{yy} = 4.$$

Setting both f_x and f_y equal to zero yields the single critical point $(2, -1)$. The value of the discriminant at the critical point is $D(2, -1) = f_{xx} f_{yy} - \left(f_{xy}\right)^2 = 8 > 0$. Furthermore, $f_{xx}(2, -1) = 2 > 0$. By the Second Derivative Test, f has a local minimum at $(2, -1)$; the value of the function at that point is $f(2, -1) = 0$ (**Figure 15.69**).

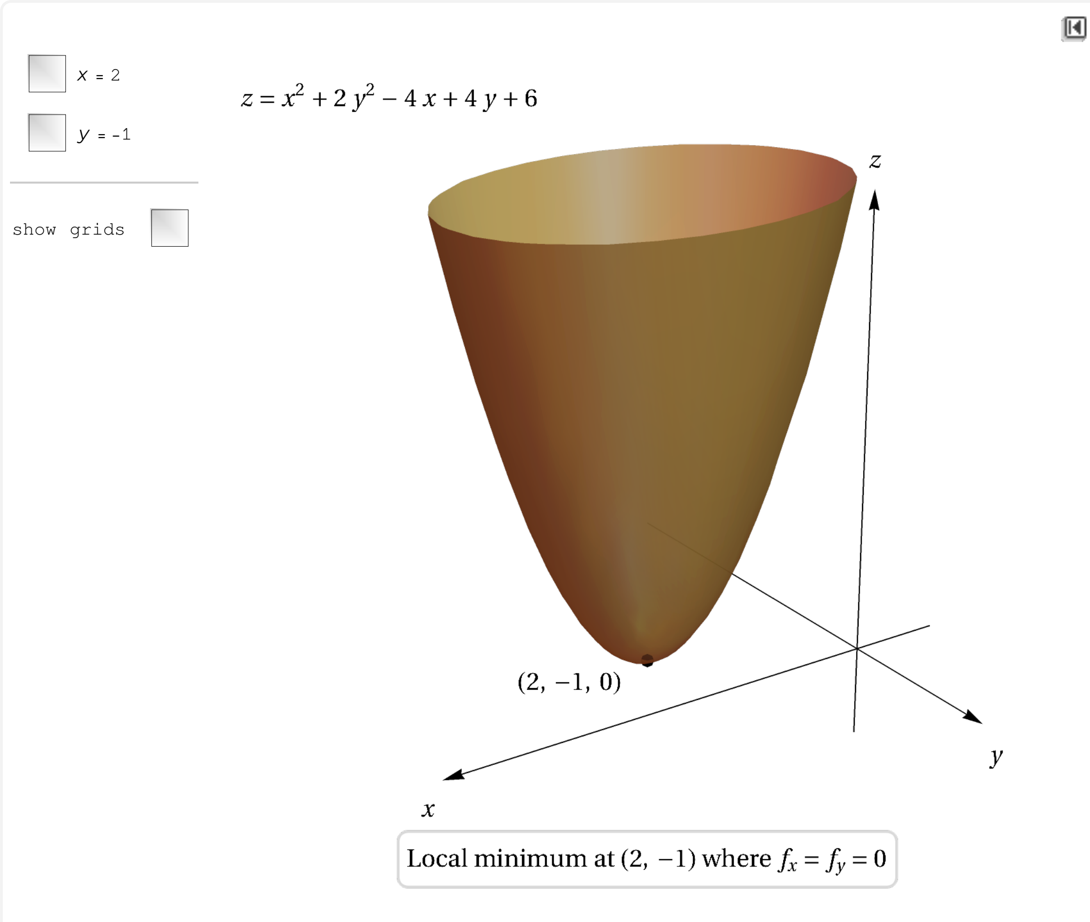


Figure 15.69

Related Exercise 24 ◆

EXAMPLE 3 Analyzing critical points

Use the Second Derivative Test to classify the critical points of $f(x, y) = x\,y\,(x - 2)\,(y + 3)$.

SOLUTION »

In Example 1, we determined that the critical points of f are $(0, 0)$, $(2, 0)$, $\left(1, -\frac{3}{2}\right)$, $(0, -3)$, and $(2, -3)$. The derivatives needed to evaluate the discriminant are

$$f_x = 2\,y\,(x - 1)\,(y + 3), \quad f_y = x\,(x - 2)\,(2\,y + 3),$$
$$f_{xx} = 2\,y\,(y + 3), \quad f_{xy} = 2\,(2\,y + 3)\,(x - 1), \text{ and } f_{yy} = 2\,x\,(x - 2).$$

The values of the discriminant at the critical points and the conclusions of the Second Derivative Test are shown in Table 15.4.

Table 15.4

(x, y)	$D(x, y)$	f_{xx}	Conclusion
$(0, 0)$	-36	0	Saddle Point
$(2, 0)$	-36	0	Saddle Point
$\left(1, -\frac{3}{2}\right)$	9	$-\frac{9}{2}$	Local maximum
$(0, -3)$	-36	0	Saddle Point
$(2, -3)$	-36	0	Saddle Point

The surface described by f has one local maximum at $\left(1, -\frac{3}{2}\right)$, surrounded by four saddle points (**Figure 15.70**). The structure of the surface may also be visualized by plotting the level curves of f.

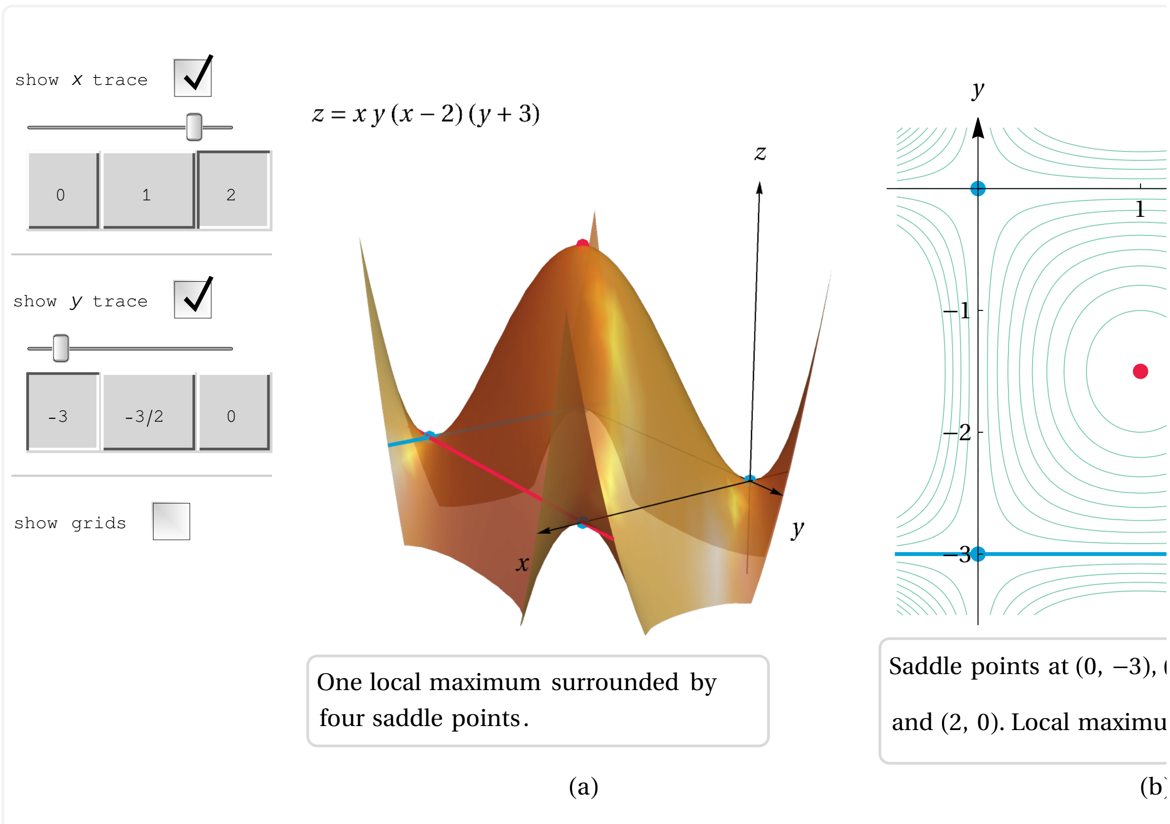


Figure 15.70

Related Exercise 27 ◆

EXAMPLE 4 Inconclusive tests

Apply the Second Derivative Test to the following functions and interpret the results.

a. $f(x, y) = 2\,x^4 + y^4$

b. $f(x, y) = 2 - x\,y^2$

SOLUTION »

a. The critical points of f satisfy the conditions

$$f_x = 8\,x^3 = 0 \quad \text{and} \quad f_y = 4\,y^3 = 0,$$

so the sole critical point is $(0, 0)$. The second partial derivatives evaluated at $(0, 0)$ are

$$f_{xx}(0, 0) = f_{xy}(0, 0) = f_{yy}(0, 0) = 0.$$

We see that $D(0, 0) = 0$, and the Second Derivative Test is inconclusive. While the bowl-shaped surface (**Figure 15.71**) described by f has a local minimum at $(0, 0)$, the surface also has a broad flat bottom, which makes the local minimum "invisible" to the Second Derivative Test.

Note »

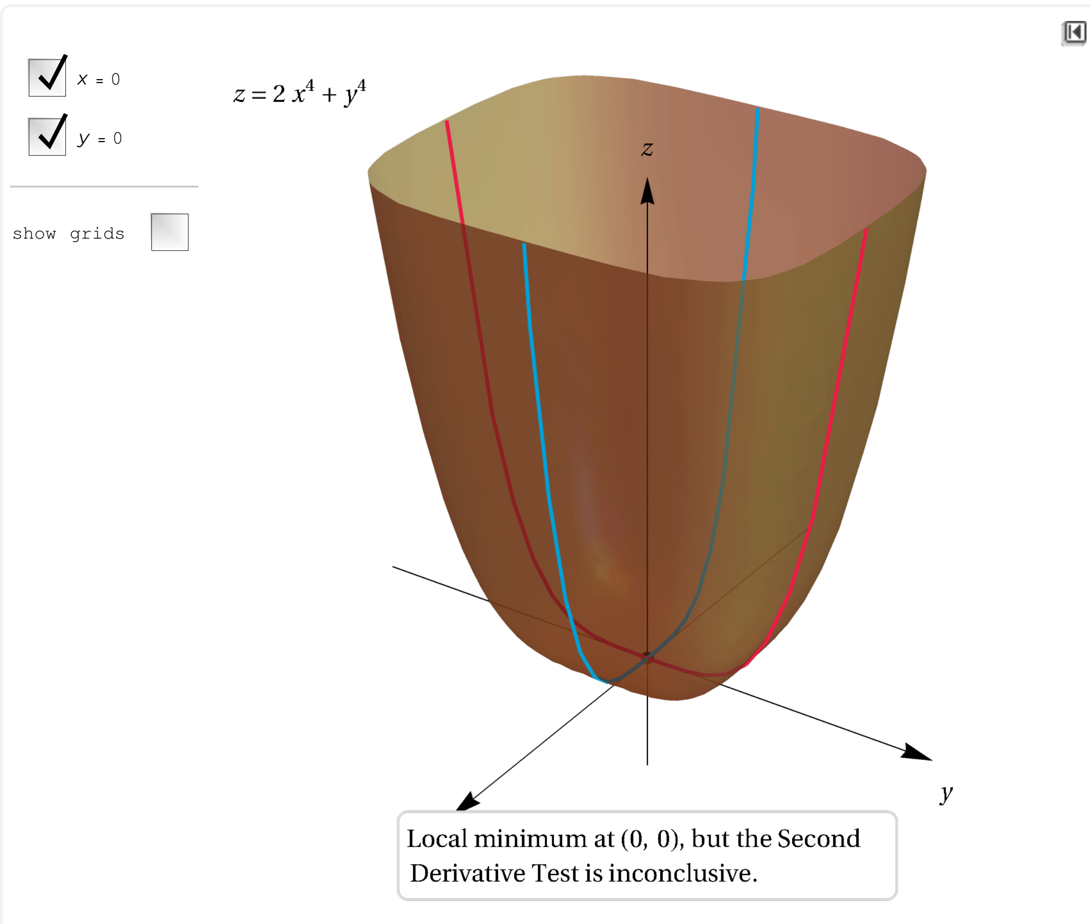


Figure 15.71

b. The critical points of this function satisfy

$$f_x(x, y) = -y^2 = 0 \quad \text{and} \quad f_y(x, y) = -2\,x\,y = 0.$$

The solutions of these equations have the form $(a, 0)$, where a is a real number. It is easy to check that the second partial derivatives evaluated at $(a, 0)$ are

$$f_{xx}(a, 0) = f_{xy}(a, 0) = 0 \quad \text{and} \quad f_{yy}(a, 0) = -2a.$$

Therefore, the discriminant is $D(a, 0) = 0$, and the Second Derivative Test is inconclusive. **Figure 15.72** shows that f has a flat ridge above the x-axis that the Second Derivative Test is unable to classify.

Note »

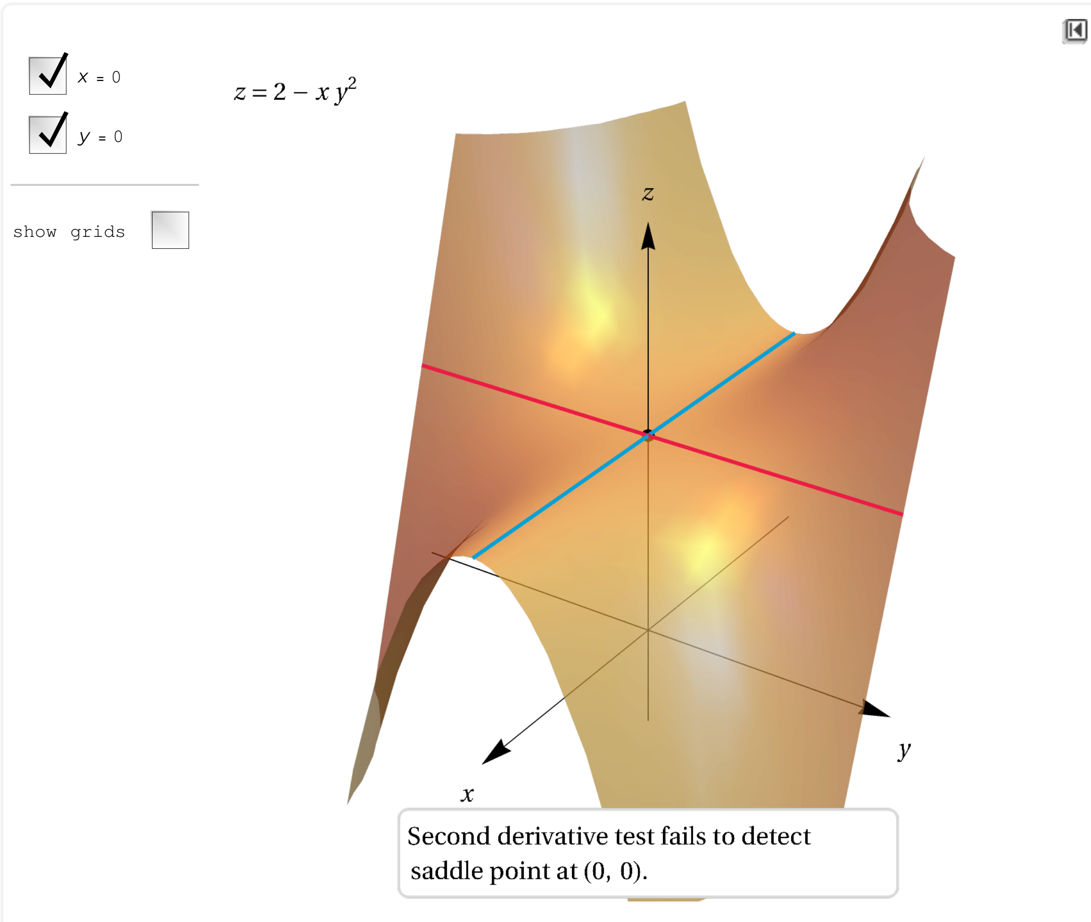


Figure 15.72

Related Exercises 29–30 ◆

Absolute Maximum and Minimum Values »

As in the one-variable case, we are often interested in knowing where a function of two or more variables attains its extreme values over its domain (or a subset of its domain).

DEFINITION **Absolute Maximum/Minimum Values**

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

It should be noted that the Extreme Value Theorem of Chapter 4 has an analog in $\mathbb{R}^2$ (or in higher dimensions): A function that is continuous on a closed bounded set in $\mathbb{R}^2$ attains its absolute maximum and absolute minimum values on that set. Absolute maximum and minimum values on a closed bounded set R occur in two ways.

Note »

- They may be local maximum or minimum values at interior points of R, where they are associated with critical points.
- They may occur on the boundary of R.

Therefore, the search for absolute maximum and minimum values on a closed bounded set amounts to examining the behavior of the function on the boundary of R and at the interior points of R.

EXAMPLE 5 Shipping regulations

A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

Note »

> Example 5 is a *constrained optimization problem*, in which the goal is to maximize the volume subject to an additional condition called a *constraint*. We return to such problems in the next section and present another method of solution.

SOLUTION »

Let x, y, and z be the dimensions of the box; its volume is $V = xyz$. The box with the maximum volume must also satisfy the condition $x + y + z = 96$, which is used to eliminate any one of the variables from the volume function. Noting that $z = 96 - x - y$, the volume function becomes

$$V(x, y) = xy(96 - x - y).$$

Notice that because x, y, and $96 - x - y$ are dimensions of the box, they must be nonnegative. The condition $96 - x - y \geq 0$ implies that $x + y \leq 96$. Therefore, among points in the xy-plane, the constraint is met only if (x, y) lies in the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 96$ (**Figure 15.73**).

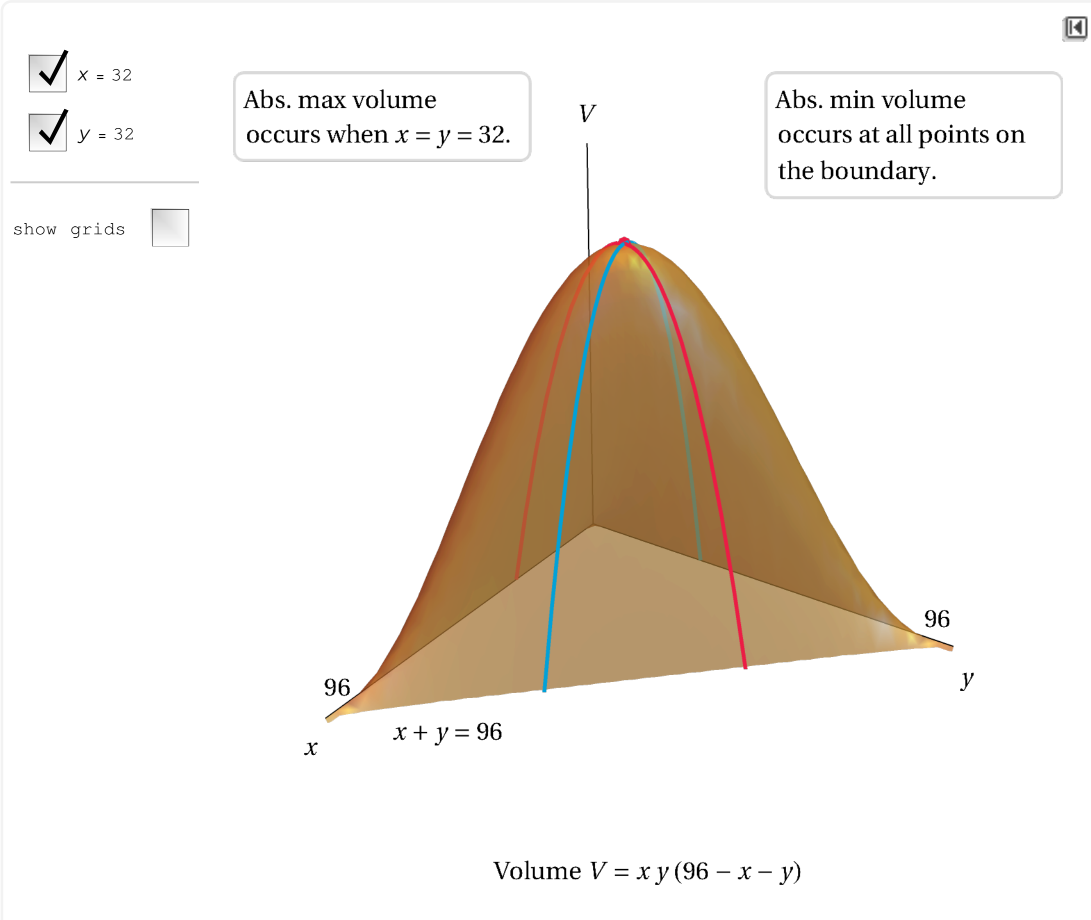


Figure 15.73

At this stage, we have reduced the original problem to a related problem: Find the absolute maximum value of $V(x, y) = x\,y\,(96 - x - y)$ over the triangular region

$$R = \{(x, y) : 0 \le x \le 96,\ 0 \le y \le 96 - x\}.$$

The boundaries of R consist of the line segments $x = 0, 0 \le y \le 96$; $y = 0, 0 \le x \le 96$; and $x + y = 96$, $0 \le x \le 96$. We find that on these boundary segments, $V = 0$. To determine the behavior of V at interior points of R, we need to find critical points. The critical points of V satisfy

$$V_x = 96\,y - 2\,x\,y - y^2 = y\,(96 - 2\,x - y) = 0$$
$$V_y = 96\,x - 2\,x\,y - x^2 = x\,(96 - 2\,y - x) = 0.$$

You can check that these two equations have four solutions: (0, 0), (96, 0), (0, 96), and (32, 32). The first three solutions lie on the boundary of the domain, where $V = 0$. At the fourth point, we have $V(32, 32) = 32{,}768\text{ in}^3$, which is the absolute maximum volume of the box. The dimensions of the box with maximum volume are $x = 32$, $y = 32$, and $z = 96 - x - y = 32$ (it is a cube). We also find that V has an absolute minimum of 0 at every point on the boundary of R.

Related Exercise 43 ◆

We summarize the method of solution given in Example 5 in the following procedure box.

PROCEDURE **Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets**

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.

2. Find the maximum and minimum values of f on the boundary of R.

3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

The techniques for carrying out Step 1 of this process have been presented. The challenge often lies in locating extreme values on the boundary. Examples 6 and 7 illustrate two approaches to handling the boundary of R. The first expresses the boundary using functions of a single variable, and the second describes the boundary parametrically. In both cases, finding extreme values on the boundary becomes a one-variable problem. In the next section, we discuss an alternative method for finding extreme values on boundaries.

EXAMPLE 6 Extreme values over a region

Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \{(x, y) : 0 \le x \le 15, 0 \le y \le 15 - x\}$.

SOLUTION »

Figure 15.74 shows the graph of f over the region R. The goal is to determine the absolute maximum and minimum values of f over R—including the boundary of R.

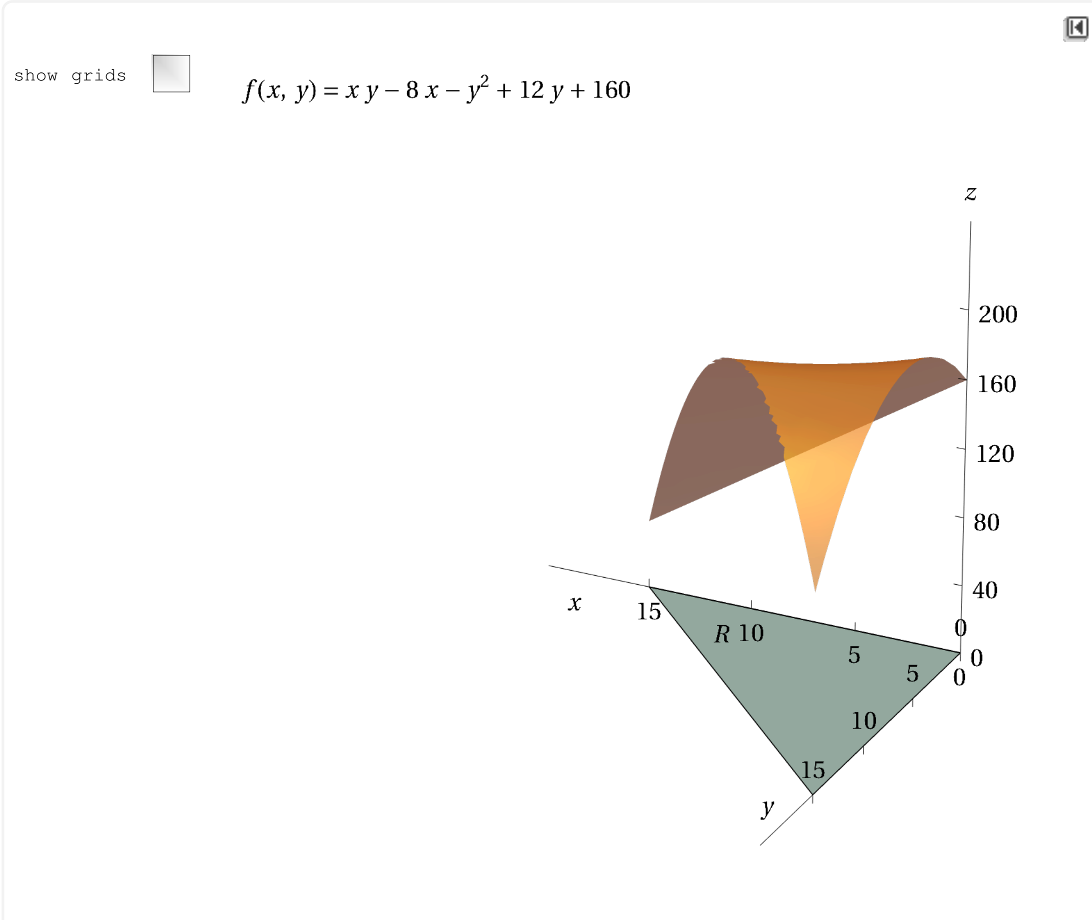


Figure 15.74

We begin by finding the critical points of f on the interior of R. The partial derivatives of f are

$$f_x(x, y) = y - 8 \text{ and } f_y(x, y) = x - 2\,y + 12.$$

The conditions $f_x(x, y) = f_y(x, y) = 0$ are satisfied only when $(x, y) = (4, 8)$, which is a point in the interior of R. This critical point is a candidate for the location of an extreme value of f, and the value of the function at this point is $f(4, 8) = 192$.

To search for extrema on the boundary of R, we consider each edge of R separately. Let C_1 be the line segment $\{(x, y) : y = 0, \text{ for } 0 \le x \le 15\}$ on the x-axis and define the single-variable function g_1 to equal f at all points along C_1 (**Figure 15.75**). We substitute $y = 0$ and find that g_1 has the form

$$g_1(x) = f(x, 0) = 160 - 8\,x.$$

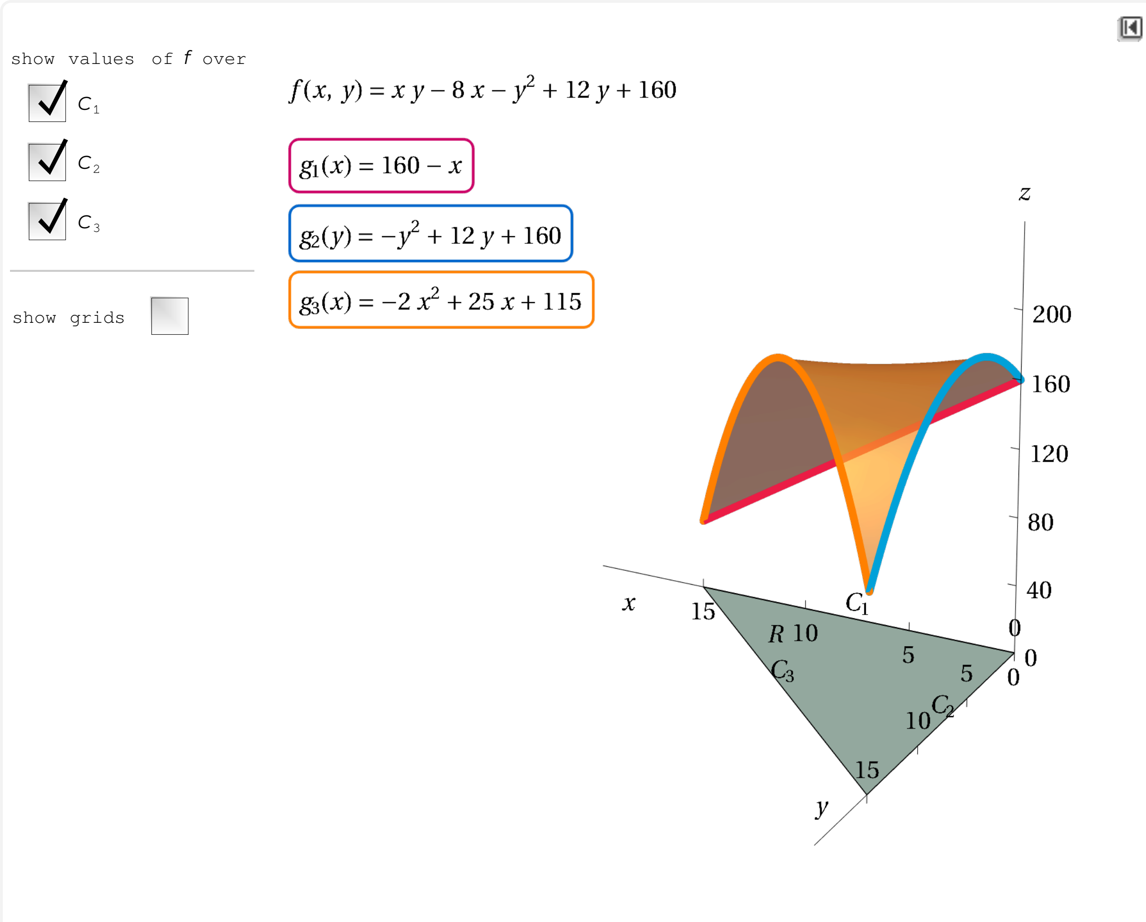


Figure 15.75

Using what we learned in Chapter 4, the candidates for absolute extreme values of g_1 on $0 \le x \le 15$ occur at critical points and endpoints. Specifically, the critical points of g_1 correspond to values where its derivative is zero, but in this case $g_1\,'(x) = -8$. So there is no critical point, which implies that the extreme values of g_1 occur at the endpoints of the interval $[0, 15]$. As the endpoints, we find that

$$g_1(0) = f(0, 0) = 160 \text{ and } g_1(15) = f(15, 0) = 40.$$

Let's set aside this information while we do a similar analysis on the other two edges of the boundary of R.

Let C_2 be the line segment $\{(x, y) : x = 0, \text{ for } 0 \le y \le 15\}$ and define g_2 to equal f on C_2 (Figure 15.75). Substituting $x = 0$, we see that

$$g_2(y) = f(0, y) = -y^2 + 12 y + 160.$$

The critical points of g_2 satisfy

$$g_2\,'(y) = -2 y + 12 = 0,$$

which has the single root $y = 6$. Evaluating g_2 at this point and the endpoints, we have

$$g_2(6) = f(0, 6) = 196, \; g_2(0) = f(0, 0) = 160, \text{ and } g_2(15) = f(0, 15) = 115.$$

Observe that $g_1(0) = g_2(0)$ because C_1 and C_2 intersect at the origin.

Finally, we let C_3 be the line segment $\{(x, y) : y = 15 - x,\ 0 \leq x \leq 15\}$ and define g_3 to equal f on C_3 (Figure 15.75). Substituting $y = 15 - x$ and simplifying, we find that

$$g_3(x) = f(x, 15 - x) = -2x^2 + 25x + 115.$$

The critical points of g_3 satisfy

$$g_3'(x) = -4x + 25,$$

whose only root on the interval $0 \leq x \leq 15$ is $x = 6.25$. Evaluating g_3 at this critical point and the endpoints, we have

$$g_3(6.25) = f(6.25, 8.75) = 193.125,\ \ g_3(15) = f(15, 0) = 40,\ \text{ and } g_3(0) = f(0, 15) = 115.$$

Observe that $g_3(15) = g_1(15)$ and $g_3(0) = g_2(15)$; the only new candidate for the location of an extreme value is the point $(6.25, 8.75)$.

Collecting and summarizing our work, we have 6 candidates for absolute extreme values:

$$f(4, 8) = 192,\ f(0, 0) = 160,\ f(15, 0) = 40,\ f(0, 6) = 196,\ f(0, 15) = 115,\ \text{ and } f(6.25, 8.75) = 193.125.$$

We see that f has an absolute minimum value of 40 at $(15, 0)$ and an absolute maximum value of 196 at $(0, 6)$. These findings are illustrated in **Figure 15.76**.

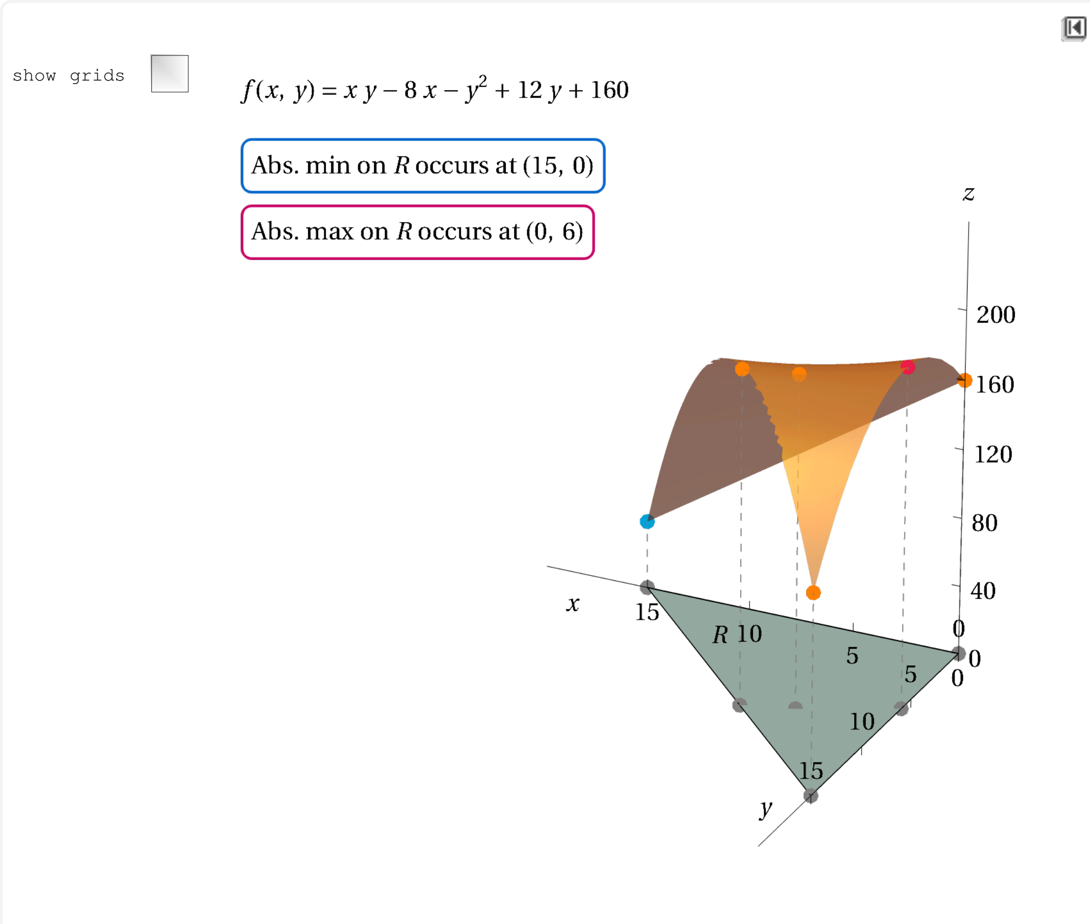


Figure 15.76

Related Exercise 52 ◆

EXAMPLE 7 Absolute maximum and minimum values

Find the absolute maximum and minimum values of $f(x, y) = \frac{1}{2}\left(x^3 - x - y^2\right) + 3$ on the region $R = \left\{(x, y) : x^2 + y^2 \le 1\right\}$ (the closed disk centered at (0, 0) with radius 1).

SOLUTION »

We begin by locating the critical points of f on the interior of R. The critical points satisfy the equations

$$f_x(x, y) = \frac{1}{2}\left(3x^2 - 1\right) = 0 \text{ and } f_y(x, y) = -y = 0,$$

which have the solutions $x = \pm\frac{1}{\sqrt{3}}$ and $y = 0$. The values of the function at these points are

$f\left(\frac{1}{\sqrt{3}}, 0\right) = 3 - \frac{1}{3\sqrt{3}}$ and $f\left(-\frac{1}{\sqrt{3}}, 0\right) = 3 + \frac{1}{3\sqrt{3}}$.

Note »

We now determine the maximum and minimum values of f on the boundary of R, which is a circle of radius 1 described by the parametric equations

$$x = \cos\theta, \quad y = \sin\theta, \quad \text{for } 0 \le \theta \le 2\pi.$$

Note »

Substituting x and y in terms of θ into the function f, we obtain a new function $g(\theta)$ that gives the values of f on the boundary of R:

$$g(\theta) = \frac{1}{2}\left(\cos^3\theta - \cos\theta - \sin^2\theta\right) + 3.$$

Finding the maximum and minimum boundary values is now a one-variable problem. The critical points of g satisfy

$$\begin{aligned} g'(\theta) &= \frac{1}{2}\left(-3\cos^2\theta\sin\theta + \sin\theta - 2\sin\theta\cos\theta\right) \\ &= -\frac{1}{2}\sin\theta\left(3\cos^2\theta + 2\cos\theta - 1\right) \\ &= -\frac{1}{2}\sin\theta\,(3\cos\theta - 1)(\cos\theta + 1) = 0. \end{aligned}$$

This condition is satisfied when $\sin\theta = 0$, $\cos\theta = \frac{1}{3}$, or $\cos\theta = -1$. The solutions of these equations on the interval $(0, 2\pi)$ are $\theta = \pi$, $\theta = \cos^{-1}\frac{1}{3}$, and $\theta = 2\pi - \cos^{-1}\frac{1}{3}$, which correspond to the points $(-1, 0)$, $\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$, and $\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ in the xy-plane, respectively. Notice that the endpoints of the interval ($\theta = 0$ and $\theta = 2\pi$) correspond to the same point on the boundary of R, namely $(1, 0)$.

Note »

Having completed the first two steps of the procedure, we have six function values to consider:

- $f\left(\frac{1}{\sqrt{3}}, 0\right) = 3 - \frac{1}{3\sqrt{3}} \approx 2.81$ and $f\left(-\frac{1}{\sqrt{3}}, 0\right) = 3 + \frac{1}{3\sqrt{3}} \approx 3.19$ (critical points),

The greatest value of f on R, $f\left(-\frac{1}{\sqrt{3}}, 0\right) = 3 + \frac{1}{3\sqrt{3}}$, is the absolute maximum value, and it occurs at an interior point (**Figure 15.77**). The least value, $f\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right) = f\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right) = \frac{65}{27}$, is the absolute minimum value, and it occurs at two symmetric boundary points. Also revealing is the plot of the level curves of the surface with the boundary of R superimposed. As the boundary of R is traversed, the values of f vary, reaching a maximum value of 3 at $(1, 0)$ and $(-1, 0)$, and a minimum value of $\frac{65}{27}$ at $\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ and $\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$.

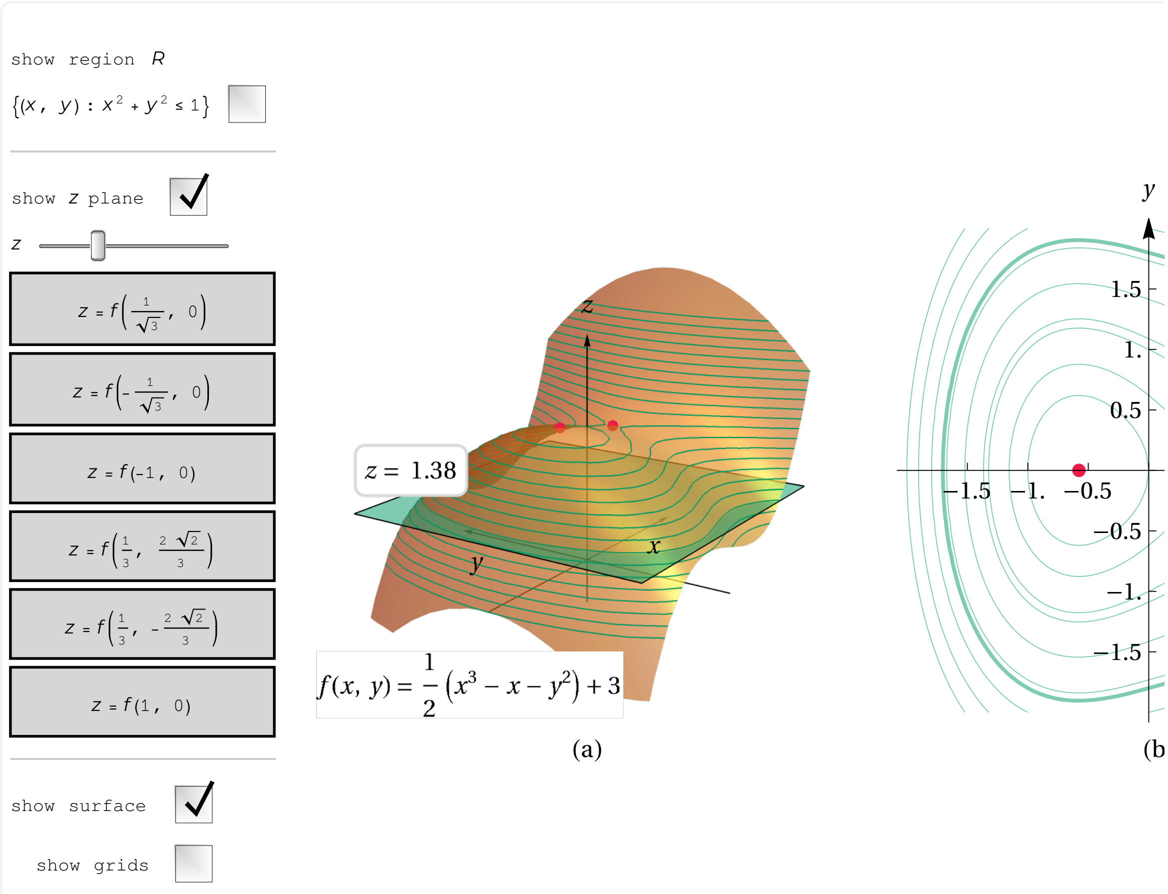


Figure 15.77

Note »

Observe that the level curves of f in Figure 15.77b appear to be tangent to the curve $x^2 + y^2 = 1$ (the boundary of the region R) at the points corresponding to the maximum and minimum values of f on this boundary. The significance of this observation is explained in Section 15.8.

Related Exercises 47–48 ◆

Open and/or Unbounded Regions

Finding absolute maximum and minimum values of a function on an open region (for example, $R = \{(x, y) : x^2 + y^2 < 9\}$) or an unbounded region (for example, $R = \{(x, y) : x > 0, y > 0\}$) presents additional challenges. Because there is no systematic procedure for dealing with such problems, some ingenuity is generally needed. Notice that absolute extrema may not exist on such domains.

EXAMPLE 8 Absolute extreme values on an open region

Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disk $R = \{(x, y) : x^2 + y^2 < 1\}$ (if they exist).

SOLUTION »

You should verify that f has a critical point at $(0, 0)$ and it corresponds to a local maximum (on an inverted paraboloid). Moving away from $(0, 0)$ in all directions, the function values decrease, so f also has an absolute maximum value of 4 at $(0, 0)$. The boundary of R is the unit circle $\{(x, y) : x^2 + y^2 = 1\}$, which is not contained in R. As (x, y) approaches any point on the unit circle along any path in R, the function values $f(x, y) = 4 - (x^2 + y^2)$ decrease and approach 3 but never reach 3. Therefore, f does not have an absolute minimum on R.

Related Exercise 59 ◆

Quick Check 4 Does the linear function $f(x, y) = 2x + 3y$ have an absolute maximum or minimum value on the open unit square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$? ◆

Answer »

It has neither an absolute maximum nor an absolute minimum value on this set.

EXAMPLE 9 Absolute extreme values on an open region

Find the point(s) on the plane $x + 2y + z = 2$ closest to the point $P(2, 0, 4)$.

SOLUTION »

Suppose (x, y, z) is a point on the plane, which means that $z = 2 - x - 2y$. The distance between $P(2, 0, 4)$ and (x, y, z) that we seek to minimize is

$$d(x, y, z) = \sqrt{(x-2)^2 + y^2 + (z-4)^2}.$$

It is easier to minimize d^2, which has the same critical points as d. Squaring d and eliminating z using $z = 2 - x - 2y$, we have

$$\begin{aligned} f(x, y) &= (d(x, y, z))^2 = (x-2)^2 + y^2 + (\underbrace{-x - 2y - 2}_{z-4})^2 \\ &= 2x^2 + 5y^2 + 4xy + 8y + 8. \end{aligned}$$

Note »

Notice that $\frac{\partial}{\partial x}(d^2) = 2d\frac{\partial d}{\partial x}$ and $\frac{\partial}{\partial y}(d^2) = 2d\frac{\partial d}{\partial y}$. Because $d \ge 0$, d^2 and d have the same critical points.

The critical points of f satisfy the equations

$$f_x = 4x + 4y = 0 \quad \text{and} \quad f_y = 4x + 10y + 8 = 0,$$

whose only solution is $x = \frac{4}{3}$, $y = -\frac{4}{3}$. The Second Derivative Test confirms that this point corresponds to a local minimum of f. We now ask: Does $\left(\frac{4}{3}, -\frac{4}{3}\right)$ correspond to the *absolute* minimum value of f over the entire domain of the problem, which is $\mathbb{R}^2$? Because the domain has no boundary, we cannot check values of f on the boundary. Instead, we argue geometrically that there is exactly one point on the plane that is closest to P. We have found a point that is closest to P among nearby points on the plane. As we move away from this point, the values of f increase without bound. Therefore, $\left(\frac{4}{3}, -\frac{4}{3}\right)$ corresponds to the absolute minimum value of f. A graph of f (**Figure 15.78**) confirms this reasoning, and we conclude that the point $\left(\frac{4}{3}, -\frac{4}{3}, \frac{10}{3}\right)$ is the point on the plane nearest P.

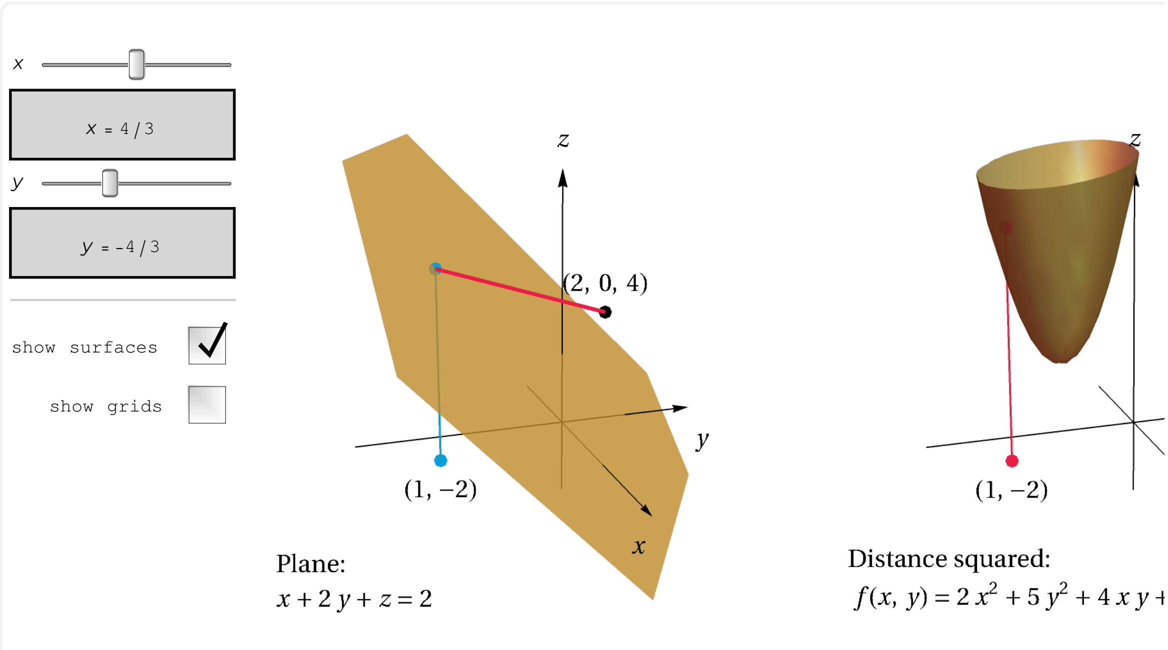


Figure 15.78

Related Exercises 62–63 ◆

Exercises »

Getting Started »

Practice Exercises »

13–22. Critical points *Find all critical points of the following functions.*

13. $f(x, y) = 3x^2 - 4y^2$

14. $f(x, y) = x^2 - 6x + y^2 + 8y$

15. $f(x, y) = 3x^2 + 3y - y^3$

16. $f(x, y) = x^3 - 12x + 6y^2$

17. $f(x, y) = x^4 + y^4 - 16xy$

18. $f(x, y) = \frac{x^3}{3} - \frac{y^3}{3} + 3xy$

19. $f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5$

20. $f(x, y) = x^3 + 6xy - 6x + y^2 - 2y$

21. $f(x,\ y) = y^3 + 6\,x\,y + x^2 - 18\,y - 6\,x$

22. $f(x,\ y) = e^{8\,x^2\,y^2 - 24\,x^2 - 8\,x\,y^4}$

23–40. Analyzing critical points *Find the critical points of the following functions. Use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point. If the Second Derivative Test is inconclusive, determine the behavior of the function at the critical points.*

23. $f(x,\ y) = -4\,x^2 + 8\,y^2 - 3$

24. $f(x,\ y) = x^4 + y^4 - 4\,x - 32\,y + 10$

25. $f(x,\ y) = 4 + 2\,x^2 + 3\,y^2$

26. $f(x,\ y) = x\,y\,e^{-x-y}$

27. $f(x,\ y) = x^4 + 2\,y^2 - 4\,x\,y$

28. $f(x,\ y) = (4\,x - 1)^2 + (2\,y + 4)^2 + 1$

29. $f(x,\ y) = 4 + x^4 + 3\,y^4$

30. $f(x,\ y) = x^4\,y^2$

31. $f(x,\ y) = \sqrt{x^2 + y^2 - 4\,x + 5}$

32. $f(x,\ y) = \tan^{-1} x\,y$

33. $f(x,\ y) = 2\,x\,y\,e^{-x^2-y^2}$

34. $f(x,\ y) = x^2 + x\,y^2 - 2\,x + 1$

35. $f(x,\ y) = \dfrac{x}{1 + x^2 + y^2}$

36. $f(x,\ y) = \dfrac{x - 1}{x^2 + y^2}$

37. $f(x,\ y) = x^4 + 4\,x^2(y - 2) + 8\,(y - 1)^2$

38. $f(x,\ y) = x\,e^{-x-y}\sin y$, for $|x| \le 2,\ 0 \le y \le \pi$

39. $f(x,\ y) = y\,e^x - e^y$

40. $f(x,\ y) = \sin(2\,\pi\,x)\cos(\pi\,y)$, for $|x| \le \dfrac{1}{2}$ and $|y| \le \dfrac{1}{2}$

41–42. Inconclusive tests *Show that the Second Derivative Test is inconclusive when applied to the following functions at* $(0,\ 0)$. *Describe the behavior of the function at* $(0,\ 0)$.

41. $f(x, y) = x^2 y - 3$

42. $f(x, y) = \sin\left(x^2 y^2\right)$

43. **Shipping regulations** A shipping company handles rectangular boxes provided the sum of the height and the girth of the box does not exceed 96 in. (The girth is the perimeter of the smallest side of the box.) Find the dimensions of the box that meets this condition and has the largest volume.

44. **Cardboard boxes** A lidless box is to be made using 2 m^2 of cardboard. Find the dimensions of the box with the largest possible volume.

45. **Cardboard boxes** A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that requires the least amount of cardboard.

46. **Optimal box** Find the dimensions of the largest rectangular box in the first octant of the *xyz*-coordinate system that has one vertex at the origin and the opposite vertex on the plane $x + 2y + 3z = 6$.

47–52. Absolute maxima and minima *Find the absolute maximum and minimum values of the following functions on the given region R.*

47. $f(x, y) = x^2 + y^2 - 2y + 1$; $R = \left\{(x, y) : x^2 + y^2 \le 4\right\}$

48. $f(x, y) = 2x^2 + y^2$; $R = \left\{(x, y) : x^2 + y^2 \le 16\right\}$

49. $f(x, y) = 4 + 2x^2 + y^2$; $R = \{(x, y) : -1 \le x \le 1, -1 \le y \le 1\}$

50. $f(x, y) = 6 - x^2 - 4y^2$; $R = \{(x, y) : -2 \le x \le 2, -1 \le y \le 1\}$

51. $f(x, y) = 2x^2 - 4x + 3y^2 + 2$; $R = \left\{(x, y) : (x - 1)^2 + y^2 \le 1\right\}$

52. $f(x, y) = x^2 + y^2 - 2x - 2y$; R is the closed region bounded by the triangle with vertices (0, 0), (2, 0), and (0, 2).

53. $f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1$; $R = \left\{(x, y) : (x - 1)^2 + (y + 1)^2 \le 1\right\}$

54. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}$; $R = \left\{(x, y) : x^2 + y^2 \le 4, y \ge 0\right\}$

55. $f(x, y) = \dfrac{2y^2 - x^2}{2 + 2x^2 y^2}$; R is the closed region bounded by the lines $y = x$, $y = 2x$, and $y = 2$.

56. $f(x, y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$.

T **57. Pectin Extraction** An increase in world production of processed fruit has led to an increase in fruit waste. One way of reducing this waste is to find useful waste byproducts. For example, waste from pineapples is reduced by extracting pectin from pineapple peels (pectin is commonly used as a thickening agent in jam and jellies and it is also widely used in the pharmaceutical industry). Pectin extraction involves heating and drying the peels, then grinding the peels into a fine powder. The powder is then placed in a solution with a particular pH level H, for $1.5 \le H \le 2.5$, and heated to a temperature T (in degrees Celsius), for $70 \le T \le 90$. The percentage of the powder $F(H, T)$ that becomes extracted pectin is

$$F(H, T) = -0.042\,T^2 - 0.213\,TH - 11.219\,H^2 + 7.327\,T + 58.729\,H - 342.684.$$

a. It can be shown that F attains its absolute maximum in the interior of the domain $D = \{(H, T) : 1.5 \le H \le 2.5,\ 70 \le T \le 90\}$. Find the pH level H and temperature T that maximizes the amount of pectin extracted from the powder.

b. What is the maximum percentage of pectin that can be extracted from the powder? Round your answer to the nearest whole number.

(*Source: Carpathian Journal of Food Science and Technology*, Dec 2014)

58–61. Absolute extrema on open and/or unbounded regions *If possible, find the absolute maximum and minimum values of the following functions on the region R.*

58. $f(x, y) = x + 3y$; $R = \{(x, y) : |x| < 1,\ |y| < 2\}$

59. $f(x, y) = x^2 + y^2 - 4$; $R = \{(x, y) : x^2 + y^2 < 4\}$

60. $f(x, y) = x^2 - y^2$; $R = \{(x, y) : |x| < 1,\ |y| < 1\}$

61. $f(x, y) = 2e^{-x-y}$; $R = \{(x, y) : x \ge 0,\ y \ge 0\}$

62–66. Absolute extrema on open and/or unbounded regions

62. Find the point on the plane $x + y + z = 4$ nearest the point $P(5, 4, 4)$.

63. Find the point on the plane $x - y + z = 2$ nearest the point $P(1, 1, 1)$.

64. Find the point on the paraboloid $z = x^2 + y^2$ nearest the point $P(3, 3, 1)$.

65. Find the points on the cone $z^2 = x^2 + y^2$ nearest the point $P(6, 8, 0)$.

66. Rectangular boxes with a volume of $10\ \text{m}^3$ are made of two materials. The material for the top and bottom of the box costs $\$10/\text{m}^2$ and the material for the sides of the box costs $\$1/\text{m}^2$. What are the dimensions of the box that minimize the cost of the box?

67. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume f is differentiable at the points in question.

a. The fact that $f_x(2, 2) = f_y(2, 2) = 0$ implies that f has a local maximum, local minimum, or saddle point at $(2, 2)$.

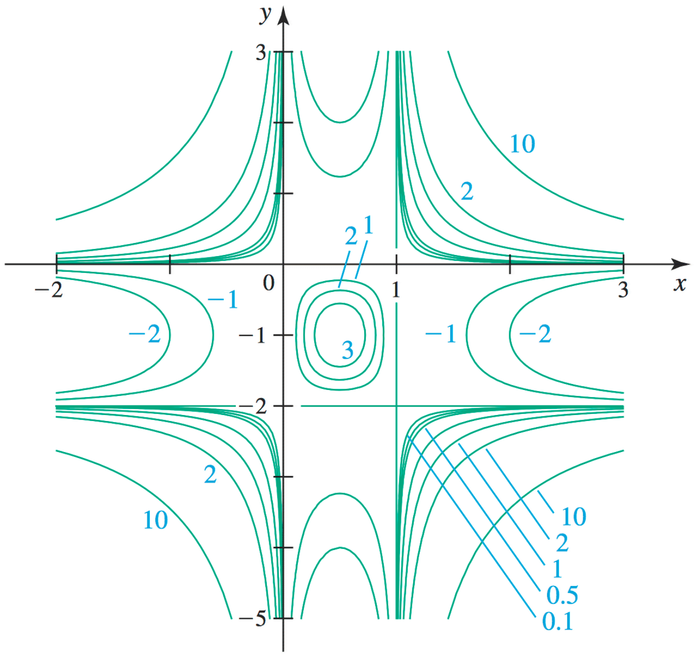
b. The function f could have a local maximum at (a, b) where $f_y(a, b) \ne 0$.

c. The function f could have both an absolute maximum and an absolute minimum at two different points that are not critical points.

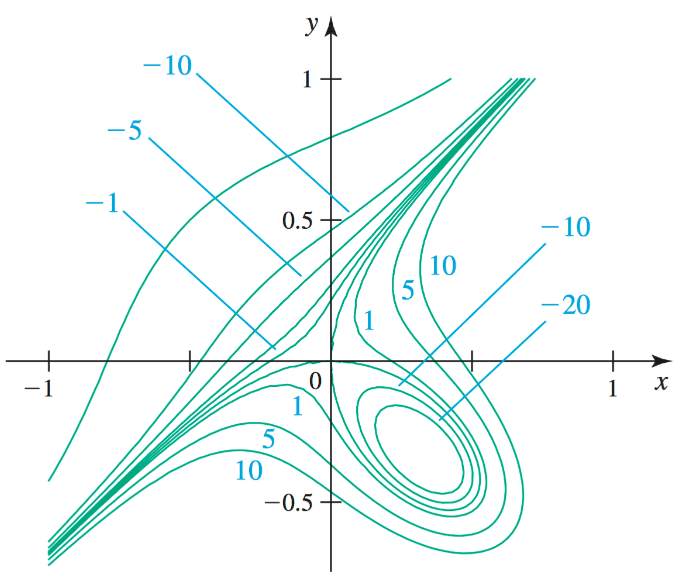
d. The tangent plane is horizontal at a point on a smooth surface corresponding to a critical point.

68–69. Extreme points from contour plots *Based on the level curves that are visible in the following graphs, identify the approximate locations of the local maxima, local minima, and saddle points.*

68.



69.



70. **Optimal box** Find the dimensions of the rectangular box with maximum volume in the first octant with one vertex at the origin and the opposite vertex on the ellipsoid $36x^2 + 4y^2 + 9z^2 = 36$.

Explorations and Challenges »

71. **Magic triples** Let x, y, and z be nonnegative numbers with $x + y + z = 200$.

a. Find the values of x, y, and z that minimize $x^2 + y^2 + z^2$.

b. Find the values of x, y, and z that minimize $\sqrt{x^2 + y^2 + z^2}$.

c. Find the values of x, y, and z that maximize xyz.

d. Find the values of x, y, and z that maximize $x^2 y^2 z^2$.

72. **Maximum/minimum of linear functions** Let R be a closed bounded region in $\mathbb{R}^2$ and let $f(x, y) = ax + by + c$, where a, b, and c are real numbers, with a and b not both zero. Give a geometric argument explaining why the absolute maximum and minimum values of f over R occur on the boundaries of R.

T **73.** **Optimal locations** Suppose n houses are located at the distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. A power substation must be located at a point such that the *sum of the squares* of the distances between the houses and the substation is minimized.

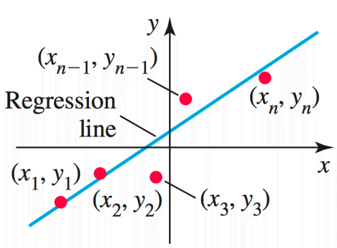
a. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at $(0, 0)$, $(2, 0)$, and $(1, 1)$.

b. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at distinct points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

c. Find the optimal location of the substation in the general case of n houses located at distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

d. You might argue that the locations found in parts (a), (b) and (c) are not optimal because they result from minimizing the sum of the *squares* of the distances, not the sum of the distances themselves. Use the locations in part (a) and write the function that gives the sum of the distances. Note that minimizing this function is much more difficult than in part (a). Then use a graphing utility to determine whether the optimal location is the same in the two cases. (Also see Exercise 81 about Steiner's problem.)

74–75. Least squares approximation *In its many guises, least squares approximation arises in numerous areas of mathematics and statistics. Suppose you collect data for two variables (for example, height and shoe size) in the form of pairs* $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. *The data may be plotted as a scatterplot in the xy-plane, as shown in the figure. The technique known as* linear regression *asks the question: What is the equation of the line that "best fits" the data? The least squares criterion for best fit requires that the sum of the squares of the vertical distances between the line and the data points be a minimum.*



74. Let the equation of the best-fit line be $y = mx + b$, where the slope m and the y-intercept b must be determined using the least squares condition. First assume there are three data points $(1, 2)$, $(3, 5)$, and $(4, 6)$. Show that the function of m and b that gives the sum of the squares of the vertical distances between the line and the three data points is

$$E(m, b) = ((m + b) - 2)^2 + ((3m + b) - 5)^2 + ((4m + b) - 6)^2.$$

Find the critical points of E and find the values of m and b that minimize E. Graph the three data points and the best-fit line.

T **75.** Generalize the procedure in Exercise 74 by assuming n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are given. Write the function $E(m, b)$ (summation notation allows for a more compact calculation). Show that the coefficients of the best-fit line are

$$m = \frac{(\sum x_k)(\sum y_k) - n\sum x_k\, y_k}{(\sum x_k)^2 - n\sum {x_k}^2},$$

$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right),$$

where all sums run from $k = 1$ to $k = n$.

T **76–77. Least squares practice** *Use the results of Exercise 75 to find the best-fit line for the following data sets. Plot the points and the best-fit line.*

76. (0, 0), (2, 3), (4, 5)

77. (−1, 0), (0, 6), (3, 8)

78. Second Derivative Test Suppose the conditions of the Second Derivative Test are satisfied on an open disk containing the point (a, b). Use the test to prove that if (a, b) is a critical point of f at which $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}(a, b) < 0 < f_{yy}(a, b)$ or $f_{yy}(a, b) < 0 < f_{xx}(a, b)$, then f has a saddle point at (a, b).

79. Maximum area triangle Among all triangles with a perimeter of 9 units, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

80. Slicing plane Find an equation of the plane passing through the point (3, 2, 1) that slices off the solid in the first octant with the least volume.

T **81. Steiner's problem for three points** Given three distinct noncollinear points A, B, and C in the plane, find the point P in the plane such that the sum of the distances $|AP| + |BP| + |CP|$ is a minimum. Here is how to proceed with three points, assuming the triangle formed by the three points has no angle greater than $\frac{2\pi}{3}$ (120°).

a. Assume the coordinates of the three given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b. Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of the line between the two points.

c. The goal is to minimize $f(x, y) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d. Explain why part (c) implies that the optimal point P has the property that the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e. What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (just draw a picture)?

f. Estimate the Steiner point for the three points (0, 0), (0, 1), and (2, 0).

T **82. Solitary critical points** A function of *one* variable has the property that a local maximum (or minimum) occurring at the only critical point is also the absolute maximum (or minimum) (for example, $f(x) = x^2$). Does the same result hold for a function of *two* variables? Show that the following functions have the property that they have a single local maximum (or minimum), occurring at the only critical point, but that the local maximum (or minimum) is not an absolute maximum (or minimum) on $\mathbb{R}^2$.

a. $f(x, y) = 3xe^y - x^3 - e^{3y}$

b. $f(x, y) = (2y^2 - y^4)\left(e^x + \dfrac{1}{1 + x^2}\right) - \dfrac{1}{1 + x^2}$

This property has the following interpretation. Suppose a surface has a single local minimum that is not the absolute minimum. Then water can be poured into the basin around the local minimum and the surface never overflows, even though there are points on the surface below the local minimum.

(*Source: Mathematics Magazine*, May 1985, and *Calculus and Analytical Geometry*, 2nd ed., Philip Gillett, 1984)

T **83. Two mountains without a saddle** Show that the following two functions have two local maxima but no other extreme points (therefore, no saddle or basin between the mountains).

a. $f(x, y) = -(x^2 - 1)^2 - (x^2 - e^y)^2$

b. $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$

(Source: Ira Rosenholtz, *Mathematics Magazine*, Feb 1987)

84. Powers and roots Assume $x + y + z = 1$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$.

a. Find the maximum and minimum values of $(1 + x^2)(1 + y^2)(1 + z^2)$.

b. Find the maximum and minimum values of $(1 + \sqrt{x})(1 + \sqrt{y})(1 + \sqrt{z})$.

(*Source: Math Horizons*, Apr 2004)

85. Ellipsoid inside a tetrahedron (1946 Putnam Exam) Let P be a plane tangent to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at a point in the first octant. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x = 0$, $y = 0$, and $z = 0$. Find the minimum volume of T. (The volume of a tetrahedron is one-third the area of the base times the height.)