15.6 Tangent Planes and Linear Approximation

In Section 4.6, we saw that if we zoom in on a point on a smooth curve (one described by a differentiable function), the curve looks more and more like the tangent line at that point. Once we have the tangent line at a point, it can be used to approximate function values and to estimate changes in the dependent variable. In this section, an analogous story is developed in three dimensions. Now we see that differentiability at a point (as discussed in Section 15.3) implies the existence of a tangent *plane* at that point (**Figure 15.58**).

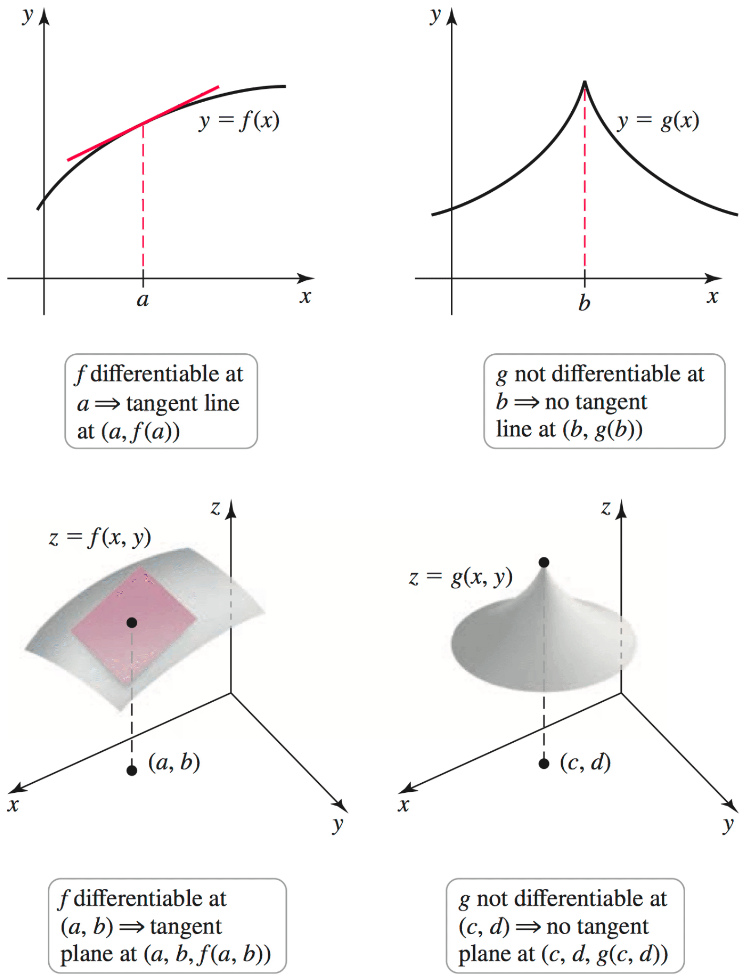


Figure 15.58

Consider a smooth surface described by a differentiable function f, and focus on a single point on the surface. As we zoom in on that point (**Figure 15.59**), the surface appears more and more like a plane. The first step is to define this plane carefully; it is called the *tangent plane*. Once we have the tangent plane, we can use it to approximate function values and to estimate changes in the dependent variable.

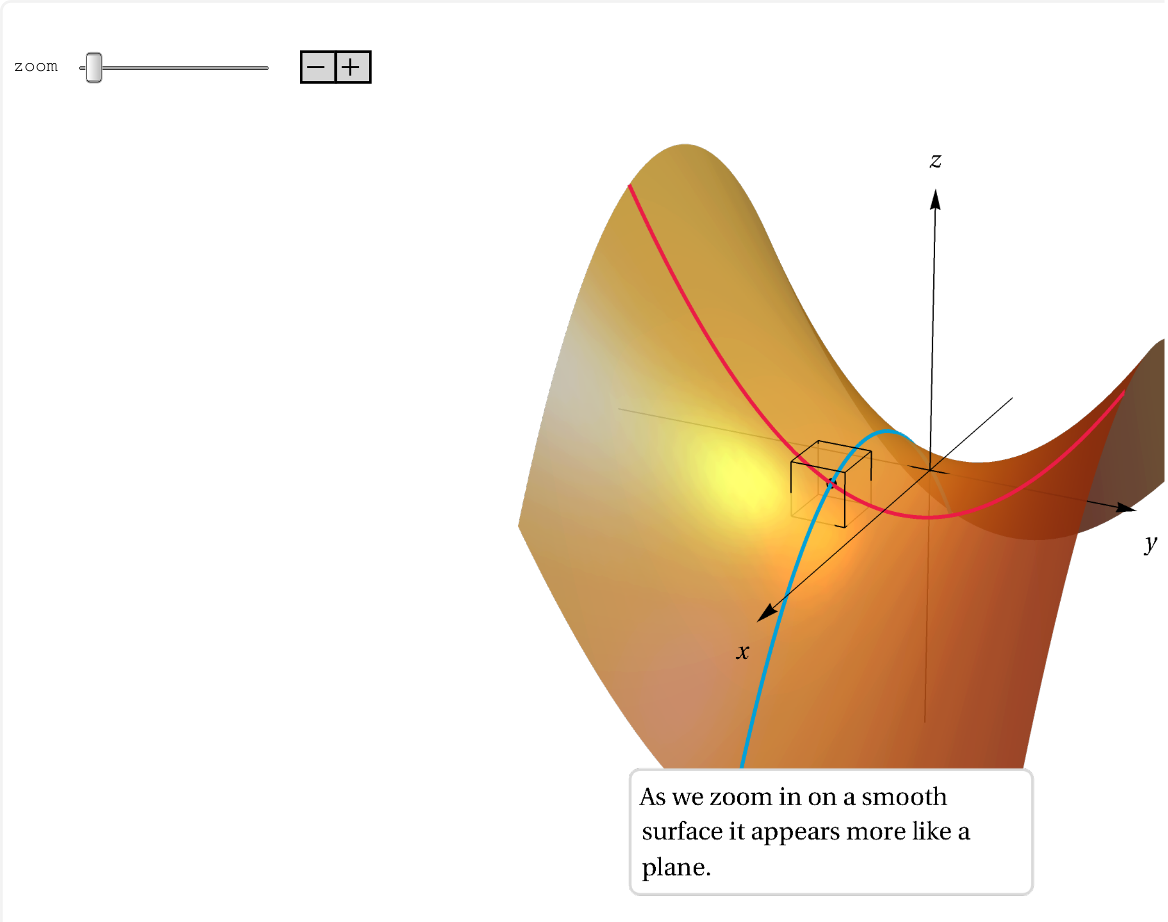


Figure 15.59

Tangent Planes »

Recall that a surface in $\mathbb{R}^3$ may be defined in at least two different ways:

- **Explicitly** in the form $z = f(x, y)$ or
- **Implicitly** in the form $F(x, y, z) = 0$.

It is easiest to begin by considering a surface defined implicitly by $F(x, y, z) = 0$, where F is differentiable at a particular point. Such a surface may be viewed as a level surface of a function $w = F(x, y, z)$; it is the level surface for $w = 0$.

Quick Check 1 Write the function $z = xy + x - y$ in the form $F(x, y, z) = 0$. ◆

Answer »

> $F(x, y, z) = z - xy - x + y = 0$

Tangent Planes for $F(x, y, z) = 0$

To find an equation of the tangent plane, consider a smooth curve $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on the surface $F(x, y, z) = 0$ (**Figure 15.60**). Because the points of C lie on the surface, we have $F(x(t), y(t), z(t)) = 0$. Differentiating both sides of this equation with respect to t, a useful relationship emerges. The derivative of the right side is 0. The Chain Rule applied to the left side yields

$$\begin{aligned}\frac{d}{dt}[F(x(t), y(t), z(t))] &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle}_{\nabla F(x,y,z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle}_{\mathbf{r}'(t)} \\ &= \nabla F(x, y, z) \cdot \mathbf{r}'(t).\end{aligned}$$

Therefore, $\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$ and at any point on the curve, the tangent vector $\mathbf{r}'(t)$ is orthogonal to the gradient.

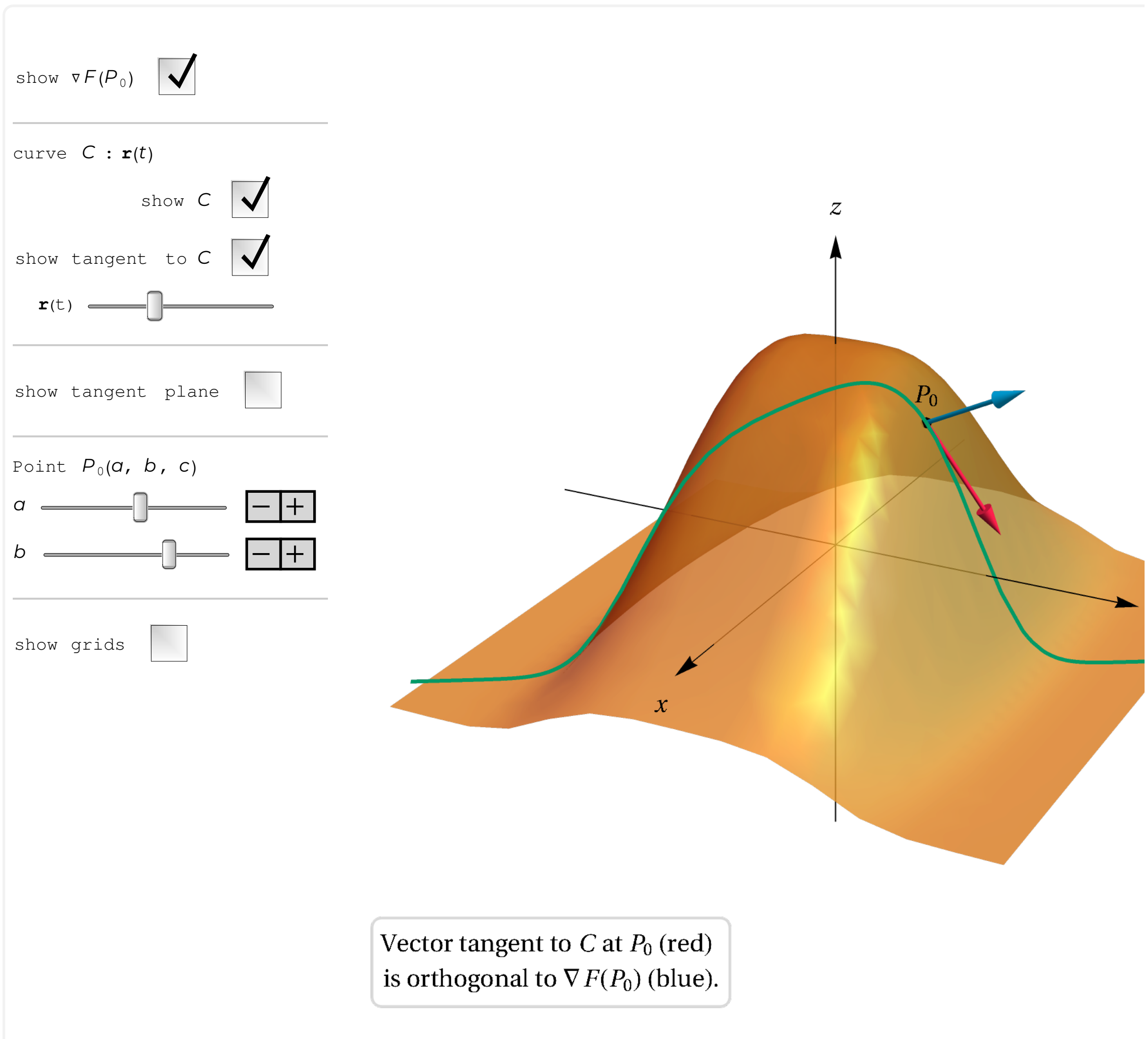


Figure 15.60

Now fix a point $P_0(a, b, c)$ on the surface, assume $\nabla F(a, b, c) \neq \mathbf{0}$, and let C be any smooth curve on the surface passing through P_0. We have shown that the vector tangent to C is orthogonal to $\nabla F(a, b, c)$ at P_0. Because this argument applies to *all* smooth curves on the surface passing through P_0, the tangent vectors for all these curves (with their tails at P_0) are orthogonal to $\nabla F(a, b, c)$, and thus they all lie in the same plane (Figure 15.60). This plane is called the *tangent plane* at P_0. We can easily find an equation of the tangent plane because we know both a point on the plane $P_0(a, b, c)$ and a normal vector $\nabla F(a, b, c)$; an equation is

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0.$$

Note »

DEFINITION **Equation of the Tangent Plane for $F(x, y, z) = 0$**

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

Note »

EXAMPLE 1 Equation of a tangent plane

Consider the ellipsoid $F(x, y, z) = \dfrac{x^2}{9} + \dfrac{y^2}{25} + z^2 - 1 = 0$.

a. Find an equation of the plane tangent to the ellipsoid at $\left(0, 4, \dfrac{3}{5}\right)$.

b. At what points on the ellipsoid is the tangent plane horizontal?

SOLUTION »

a. Notice that we have written the equation of the ellipsoid in the implicit form $F(x, y, z) = 0$. The gradient of F is $\nabla F(x, y, z) = \left\langle \dfrac{2x}{9}, \dfrac{2y}{25}, 2z \right\rangle$. Evaluated at $\left(0, 4, \dfrac{3}{5}\right)$, we have

$$\nabla F\left(0, 4, \frac{3}{5}\right) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle.$$

An equation of the tangent plane at this point is

$$0 \cdot (x - 0) + \frac{8}{25}(y - 4) + \frac{6}{5}\left(z - \frac{3}{5}\right) = 0,$$

or $4y + 15z = 25$. The equation does not involve x, so the tangent plane is parallel to (does not intersect) the x-axis (**Figure 15.61**).

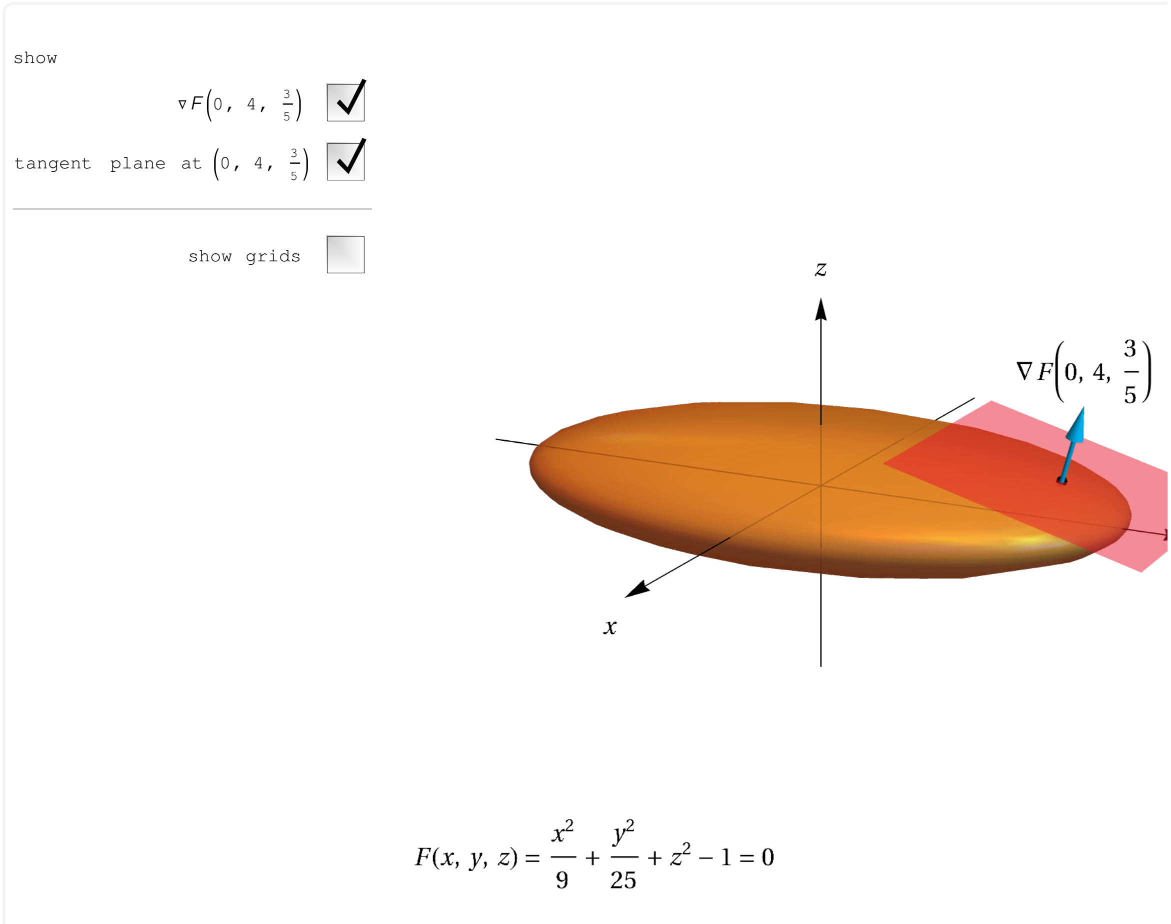


Figure 15.61

b. A horizontal plane has a normal vector of the form $\langle 0, 0, c\rangle$, where $c \neq 0$. A plane tangent to the ellipsoid has a normal vector $\nabla F(x, y, z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$. Therefore, the ellipsoid has a horizontal tangent plane when $F_x = \frac{2x}{9} = 0$ and $F_y = \frac{2y}{25} = 0$, or when $x = 0$ and $y = 0$. Substituting these values into the original equation for the ellipsoid, we find that horizontal planes occur at $(0, 0, 1)$ and $(0, 0, -1)$.

Related Exercises 14, 16 ◆

The preceding discussion allows us to confirm a claim made in Section 15.5. The surface $F(x, y, z) = 0$ is a level surface of the function $w = F(x, y, z)$ (corresponding to $w = 0$). At any point on that surface, the tangent plane has a normal vector $\nabla F(x, y, z)$. Therefore, the gradient $\nabla F(x, y, z)$ is orthogonal to the level surface $F(x, y, z) = 0$ at all points of the domain at which F is differentiable.

Note »

> This result extends Theorem 15.12, which states that for functions $f(x, y) = 0$, the gradient at a point is orthogonal to the level curve that passes through that point.

Tangent Planes for $z = f(x, y)$

Surfaces in $\mathbb{R}^3$ are often defined explicitly in the form $z = f(x, y)$. In this situation, the equation of the tangent plane is a special case of the general equation just derived. The equation $z = f(x, y)$ is written as $F(x, y, z) = z - f(x, y) = 0$, and the gradient of F at the point $(a, b, f(a, b))$ is

$$\begin{aligned}\nabla F(a, b, f(a, b)) &= \langle F_x(a, b, f(a, b)), F_y(a, b, f(a, b)), F_z(a, b, f(a, b))\rangle \\ &= \langle -f_x(a, b), -f_y(a, b), 1\rangle.\end{aligned}$$

Note »

To be clear, when $F(x, y, z) = z - f(x, y)$, we have $F_x = -f_x$, $F_y = -f_y$, and $F_z = 1$.

Using the tangent plane definition, an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + 1(z - f(a, b)) = 0.$$

After some rearranging, we obtain an equation of the tangent plane.

Tangent Plane for $z = f(x, y)$

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

EXAMPLE 2 Tangent plane for $z = f(x, y)$

Find an equation of the plane tangent to the paraboloid $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$.

SOLUTION »

The partial derivatives are $f_x = -6x$ and $f_y = -8y$. Evaluating the partial derivatives at $(2, 1)$, we have $f_x(2, 1) = -12$ and $f_y(2, 1) = -8$. Therefore, an equation of the tangent plane (**Figure 15.62**) is

$$\begin{aligned}z &= f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \\ &= -12(x - 2) - 8(y - 1) + 16 \\ &= -12x - 8y + 48.\end{aligned}$$

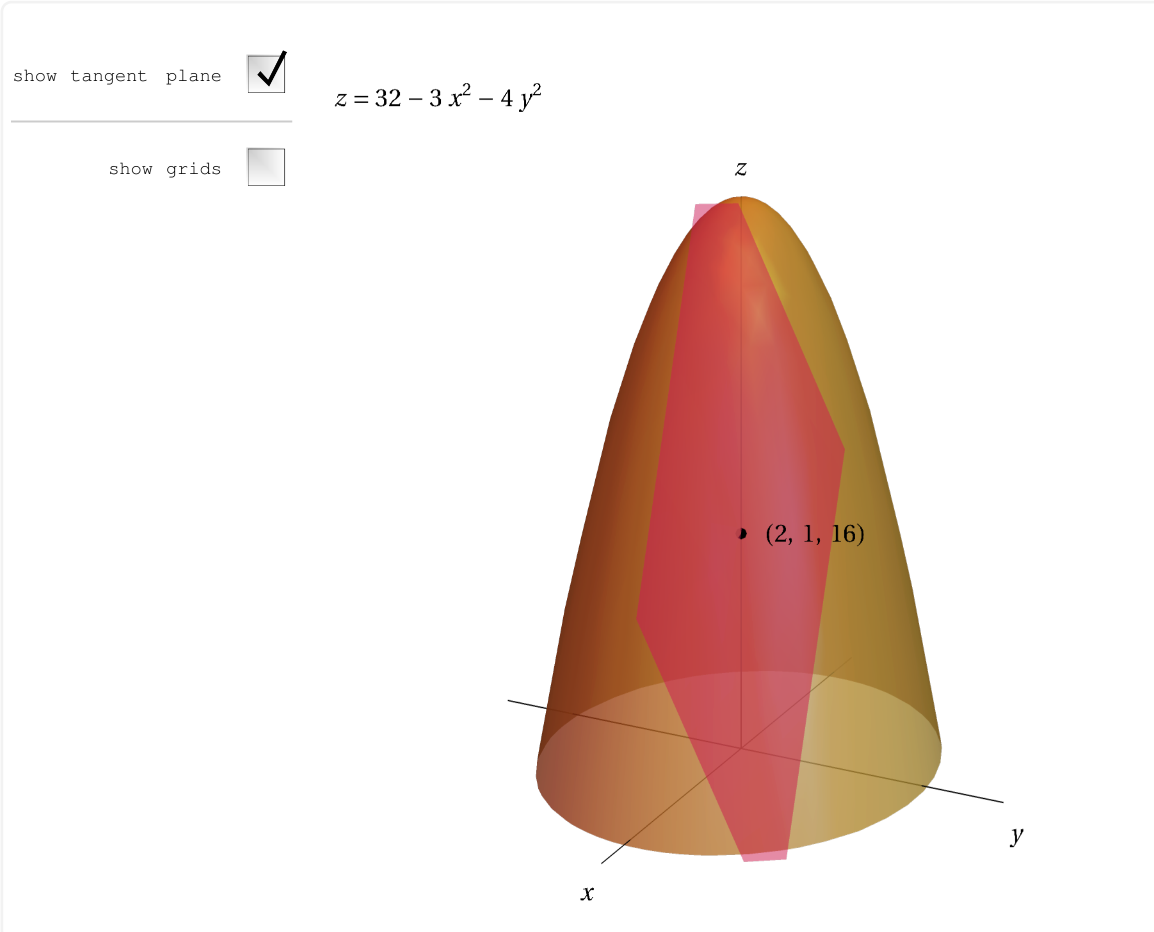


Figure 15.62

Related Exercises 17–18 ◆

Linear Approximation »

With a function of the form $y = f(x)$, the tangent line at a point often gives good approximations to the function near that point. A straightforward extension of this idea applies to approximating functions of two variables with tangent planes. As before, the method is called *linear approximation*.

Note »

Figure 15.63 shows the details of linear approximation in the one- and two-variable cases. In the one-variable case (Section 4.6), if f is differentiable at a, the equation of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$L(x) = f(a) + f'(a)(x - a).$$

The tangent line gives an approximation to the function. At points near a, we have $f(x) \approx L(x)$.

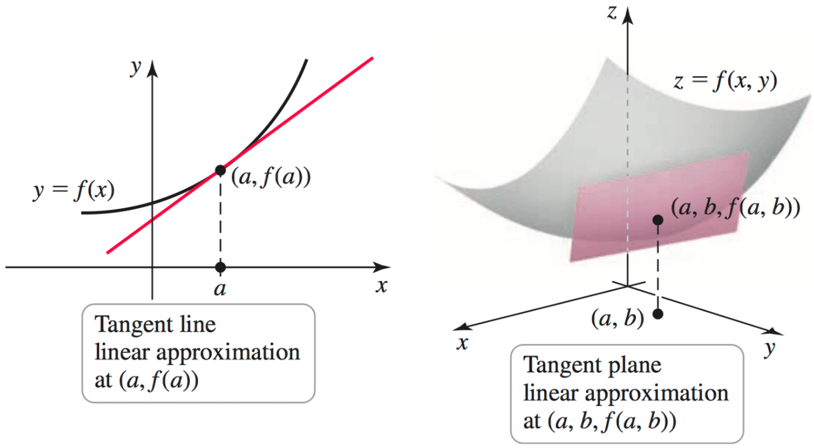


Figure 15.63

The two-variable case is analogous. If f is differentiable at (a, b), an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

This tangent plane is the linear approximation to f at (a, b). At points near (a, b), we have $f(x, y) \approx L(x, y)$. The pattern established here continues for linear approximations in higher dimensions: For each additional variable, a new term is added to the approximation formula.

DEFINITION **Linear Approximation**

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).$$

EXAMPLE 3 **Linear approximation**

Let $f(x, y) = \dfrac{5}{x^2 + y^2}$.

a. Find the linear approximation to the function at the point $(-1, 2, 1)$.

b. Use the linear approximation to estimate the value of $f(-1.05, 2.1)$.

SOLUTION »

a. The partial derivatives of f are

$$f_x = -\frac{10x}{(x^2 + y^2)^2} \quad \text{and} \quad f_y = -\frac{10y}{(x^2 + y^2)^2}.$$

Evaluated at $(-1, 2)$, we have $f_x(-1, 2) = \frac{2}{5} = 0.4$ and $f_y(-1, 2) = -\frac{4}{5} = -0.8$. Therefore, the linear approximation to the function at $(-1, 2, 1)$ is

$$\begin{aligned} L(x, y) &= f_x(-1, 2)\,(x - (-1)) + f_y(-1, 2)\,(y - 2) + f(-1, 2) \\ &= 0.4\,(x + 1) - 0.8\,(y - 2) + 1 \\ &= 0.4\,x - 0.8\,y + 3. \end{aligned}$$

The surface and the tangent plane are shown in **Figure 15.64** .

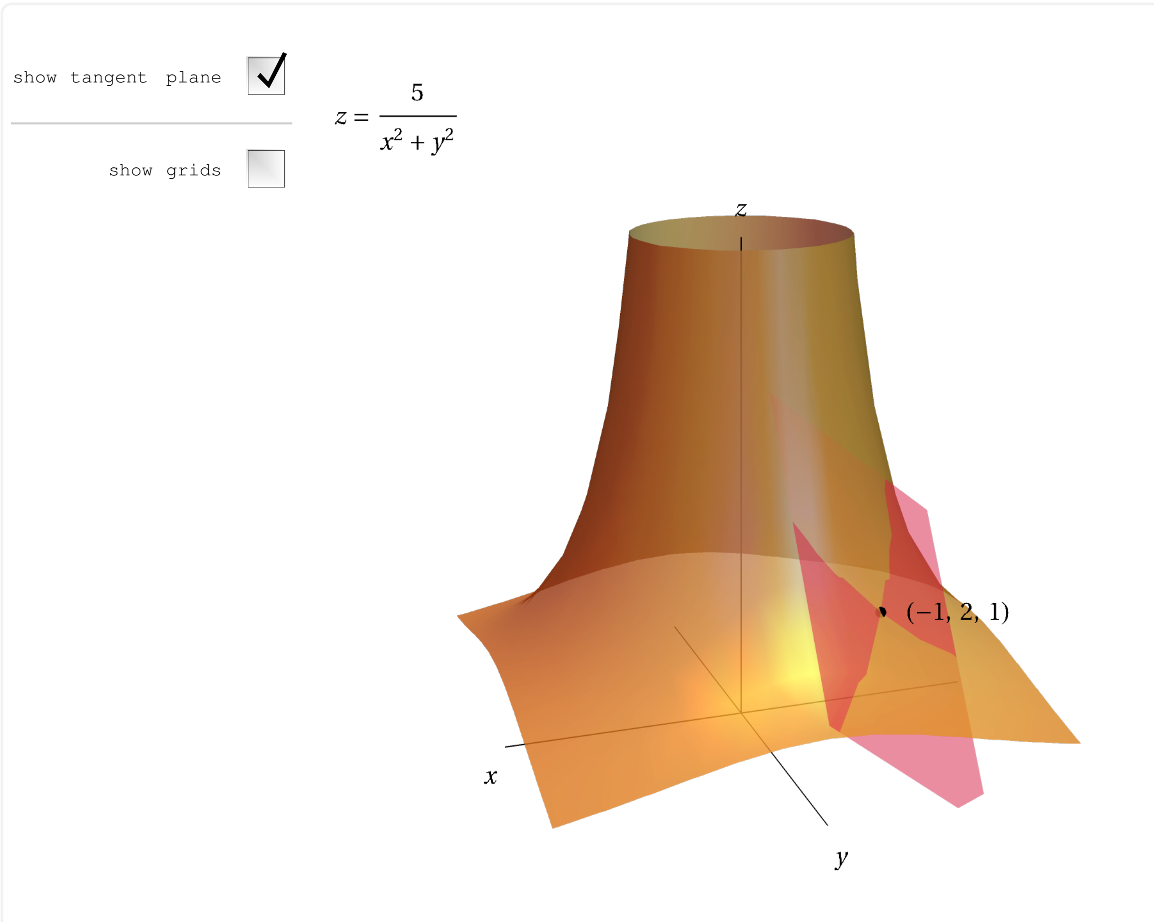


Figure 15.64

b. The value of the function at the point $(-1.05, 2.1)$ is approximated by the value of the linear approximation at that point, which is

$$L(-1.05, 2.1) = 0.4\,(-1.05) - 0.8\,(2.1) + 3 = 0.90.$$

In this case, we can easily evaluate $f(-1.05, 2.1) \approx 0.907$ and compare the linear approximation with the exact value; the approximation has a relative error of about 0.8 %.

Note »

$$\text{Relative error} = \frac{|\text{approximation} - \text{exact value}|}{|\text{exact value}|}$$

Related Exercise 36 ◆

Quick Check 2 Look at the graph of the surface in Example 3 (Figure 15.64) and explain why $f_x(-1, 2) > 0$ and $f_y(-1, 2) < 0$. ◆

Answer »

> If you walk in the positive x-direction from $(-1, 2, 1)$, then you walk uphill. If you walk in the positive y-direction from $(-1, 2, 1)$, then you walk downhill.

Differentials and Change »

Recall that for a function of the form $y = f(x)$, if the independent variable changes from x to $x + dx$, the corresponding change Δy in the dependent variable is approximated by the differential $dy = f'(x)\,dx$, which is the change in the linear approximation. Therefore, $\Delta y \approx dy$, with the approximation improving as dx approaches 0.

For functions of the form $z = f(x, y)$, we start with the linear approximation to the surface

$$f(x, y) \approx L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

The exact change in the function between the points (a, b) and (x, y) is

$$\Delta z = f(x, y) - f(a, b).$$

Replacing $f(x, y)$ by its linear approximation, the change Δz is approximated by

$$\Delta z \approx \underbrace{L(x, y) - f(a, b)}_{dz} = f_x(a, b)\underbrace{(x - a)}_{dx} + f_y(a, b)\underbrace{(y - b)}_{dy}.$$

The change in the x-coordinate is $dx = x - a$ and the change in the y-coordinate is $dy = y - b$ (**Figure 15.65**).

Note »

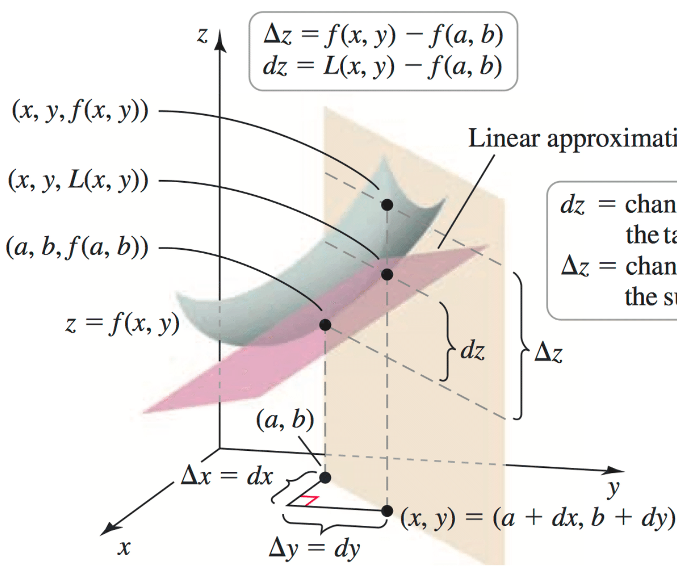


Figure 15.65

As before, we let the differential dz denote the change in the linear approximation. Therefore, the approximate change in the z-coordinate is

$$\Delta z \approx dz = \underbrace{f_x(a, b)\,dx}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } x}} + \underbrace{f_y(a, b)\,dy}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } y}}.$$

This expression says that if we move the independent variables from (a, b) to $(x, y) = (a + dx, b + dy)$, the corresponding change in the dependent variable Δz has two contributions—one due to the change in x and one due to the change in y. If dx and dy are small in magnitude, then so is Δz. The approximation $\Delta z \approx dz$ improves as dx and dy approach 0. The relationships among the differentials are illustrated in Figure 15.65.

Quick Check 3 Explain why, if $dx = 0$ or $dy = 0$ in the change formula for Δz, the result is the change formula for one variable. ◆

Answer »

> If $\Delta x = 0$, then the change formula becomes $\Delta z \approx f_y(a, b)\,\Delta y$, which is the change formula for the single variable y. If $\Delta y = 0$, then the change formula becomes $\Delta z \approx f_x(a, b)\,\Delta x$, which is the change formula for the single variable x.

More generally, we replace the fixed point (a, b) in the previous discussion with the variable point (x, y) to arrive at the following definition.

DEFINITION **The differential dz**

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + dx, y + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

EXAMPLE 4 Approximating function change

Let $z = f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the independent variables change from $(-1, 2)$ to $(-0.93, 1.94)$.

SOLUTION »

If the independent variables change from $(-1, 2)$ to $(-0.93, 1.94)$, then $dx = 0.07$ (an increase) and $dy = -0.06$ (a decrease). Using the values of the partial derivatives evaluated in Example 3, the corresponding change in z is approximately

$$\begin{aligned} dz &= f_x(-1, 2)\,dx + f_y(-1, 2)\,dy \\ &= 0.4\,(0.07) + (-0.8)\,(-0.06) \\ &= 0.076. \end{aligned}$$

Again, we can check the accuracy of the approximation. The actual change is $f(-0.93, 1.94) - f(-1, 2) \approx 0.080$, so the approximation has a 5% error.

Related Exercise 40 ◆

EXAMPLE 5 Body mass index

The body mass index (BMI) for an adult human is given by the function $B(w, h) = \dfrac{w}{h^2}$, where w is weight measured in kilograms and h is height measured in meters.

a. Use differentials to approximate the change in the BMI when weight increases from 55 to 56.5 kg and height increases from 1.65 to 1.66 m.

b. Which produces a greater *percentage* change in the BMI, a 1% change in the weight (at a constant height) or a 1% change in the height (at a constant weight)?

SOLUTION »

a. The approximate change in the BMI is $dB = B_w\, dw + B_h\, dh$, where the derivatives are evaluated at $w = 55$ and $h = 1.65$, and the changes in the independent variables are $dw = 1.5$ and $dh = 0.01$. Evaluating the partial derivatives, we find that

$$B_w(w,\ h) = \frac{1}{h^2}, \qquad B_w(55,\ 1.65) \approx 0.37,$$
$$B_h(w,\ h) = -\frac{2\,w}{h^3}, \quad \text{and} \quad B_h(55,\ 1.65) \approx -24.49.$$

Therefore, the approximate change in the BMI is

$$\begin{aligned} dB &= B_w(55,\ 1.65)\, dw + B_h(55,\ 1.65)\, dh \\ &\approx (0.37)(1.5) + (-24.49)(0.01) \\ &\approx 0.56 - 0.25 \\ &= 0.31. \end{aligned}$$

As expected, an increase in weight *increases* the BMI, while an increase in height *decreases* the BMI. In this case, the two contributions combine for a net increase in the BMI.

b. The changes dw, dh, and dB that appear in the differential change formula in part (a) are *absolute changes*. The corresponding *relative*, or *percentage*, changes are $\frac{dw}{w}$, $\frac{dh}{h}$, and $\frac{dB}{B}$. To introduce relative changes into the change formula, we divide both sides of $dB = B_w\, dw + B_h\, dh$ by $B = \frac{w}{h^2} = w\,h^{-2}$. The result is

$$\begin{aligned} \frac{dB}{B} &= B_w\,\frac{dw}{w\,h^{-2}} + B_h\,\frac{dh}{w\,h^{-2}} \\ &= \frac{1}{h^2}\,\frac{dw}{w\,h^{-2}} - \frac{2\,w}{h^3}\,\frac{dh}{w\,h^{-2}} && \text{Substitute for } B_w \text{ and } B_h. \\ &= \underbrace{\frac{dw}{w}}_{\substack{\text{relative}\\ \text{change in } w}} - \underbrace{2\,\frac{dh}{h}}_{\substack{\text{relative}\\ \text{change in } h}}. && \text{Simplify.} \end{aligned}$$

This expression relates the relative changes in w, h, and B. With h constant $(dh = 0)$, a 1% change in w $(\frac{dw}{w} = 0.01)$ produces approximately a 1% change of the same sign in B. With w constant $(dw = 0)$, a 1% change in h $(\frac{dh}{h} = 0.01)$ produces approximately a 2% change in B of the opposite sign. We see that the BMI formula is more sensitive to small changes in h than in w.

Note »

See Exercises 68–69 for general results about relative or percentage changes in functions.

Related Exercise 44 ◆

Quick Check 4 In Example 5, interpret the facts that $B_w > 0$ and $B_h < 0$ for $w, h > 0$. ◆

Answer »

The differential for functions of two variables extends naturally to more variables. For example, if f is differentiable at (x, y, z) with $w = f(x, y, z)$, then

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz.$$

The differential dw (or df) gives the approximate change in f at the point (x, y, z) due to changes of dx, dy, and dz in the independent variables.

EXAMPLE 6 Manufacturing errors

A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10$ cm, a height of $h = 50$ cm, and a thickness of $t = 0.1$ cm (**Figure 15.66**). The manufacturing process produces tubes with a maximum error of ± 0.05 cm in the radius and height, and a maximum error of ± 0.0005 cm in the thickness. The volume of the cylindrical tube is $V(r, h, t) = \pi h t (2r - t)$. Use differentials to estimate the maximum error in the volume of a tube.

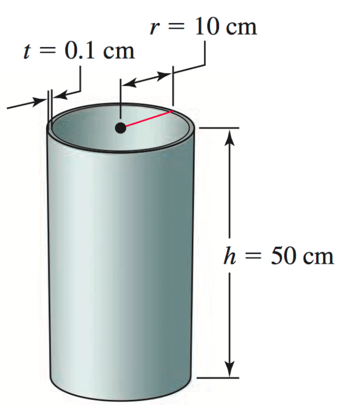


Figure 15.66

SOLUTION »

The approximate change in the volume of a tube due to changes dr, dh, and dt in the radius, height, and thickness, respectively, is

$$dV = V_r\,dr + V_h\,dh + V_t\,dt.$$

The partial derivatives evaluated at $r = 10$, $h = 50$, and $t = 0.1$ are

$$\begin{aligned} V_r(r, h, t) &= 2\pi h t, & V_r(10, 50, 0.1) &= 10\pi, \\ V_h(r, h, t) &= \pi t(2r - t), & V_h(10, 50, 0.1) &= 1.99\pi, \\ V_t(r, h, t) &= 2\pi h(r - t), & V_t(10, 50, 0.1) &= 990\pi. \end{aligned}$$

We let $dr = dh = 0.05$ and $dt = 0.0005$ be the maximum errors in the radius, height, and thickness, respectively. The maximum error in the volume is approximately

$$\begin{aligned} dV &= V_r(10, 50, 0.1)\,dr + V_h(10, 50, 0.1)\,dh + V_t(10, 50, 0.1)\,dt \\ &= 10\,\pi\,(0.05) + 1.99\,\pi\,(0.05) + 990\,\pi\,(0.0005) \\ &\approx 1.57 + 0.31 + 1.56 \\ &= 3.44. \end{aligned}$$

The maximum error in the volume is approximately 3.44 cm^3. Notice that the "magnification factor" for the thickness ($990\,\pi$) is roughly 100 and 500 times greater than the magnification factors for the radius and height, respectively. This means that for the same errors in r, h, and t, the volume is far more sensitive to errors in the thickness. The partial derivatives allow us to do a sensitivity analysis to determine which independent (input) variables are most critical in producing change in the dependent (output) variable.

Related Exercise 52 ◆

Exercises »

Getting Started »

Practice Exercises »

13–28. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given points (two planes and two equations).*

13. $x^2 + y + z = 3$; $(1, 1, 1)$ and $(2, 0, -1)$

14. $x^2 + y^3 + z^4 = 2$; $(1, 0, 1)$ and $(-1, 0, 1)$

15. $x\,y + x\,z + y\,z - 12 = 0$; $(2, 2, 2)$ and $(2, 0, 6)$

16. $x^2 + y^2 - z^2 = 0$; $(3, 4, 5)$ and $(-4, -3, 5)$

17. $z = 4 - 2\,x^2 - y^2$; $(2, 2, -8)$ and $(-1, -1, 1)$

18. $z = 2 + 2\,x^2 + \dfrac{y^2}{2}$; $\left(-\dfrac{1}{2}, 1, 3\right)$ and $(3, -2, 22)$

19. $z = e^{x\,y}$; $(1, 0, 1)$ and $(0, 1, 1)$

20. $z = \sin x\,y + 2$; $(1, 0, 2)$ and $(0, 5, 2)$

21. $xy \sin z = 1$; $\left(1, 2, \dfrac{\pi}{6}\right)$ and $\left(-2, -1, \dfrac{5\,\pi}{6}\right)$

22. $yz\,e^{xz} - 8 = 0$; $(0, 2, 4)$ and $(0, -8, -1)$

23. $z^2 - \dfrac{x^2}{16} - \dfrac{y^2}{9} - 1 = 0$; $\left(4, 3, -\sqrt{3}\right)$ and $\left(-8, 9, \sqrt{14}\right)$

24. $2\,x + y^2 - z^2 = 0$; $(0, 1, 1)$ and $(4, 1, -3)$

25. $z = x^2\,e^{x-y}$; $(2, 2, 4)$ and $(-1, -1, 1)$

26. $z = \ln(1 + x\,y)$; $(1, 2, \ln 3)$ and $(-2, -1, \ln 3)$

27. $z = \dfrac{x - y}{x^2 + y^2}$; $\left(1, 2, -\dfrac{1}{5}\right)$ and $\left(2, -1, \dfrac{3}{5}\right)$

28. $z = 2\cos(x - y) + 2$; $\left(\dfrac{\pi}{6}, -\dfrac{\pi}{6}, 3\right)$ and $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, 4\right)$

29–32. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given point.*

29. $z = \tan^{-1}(xy)$; $\left(1, 1, \dfrac{\pi}{4}\right)$

30. $z = \tan^{-1}(x + y)$; $(0, 0, 0)$

31. $\sin xyz = \dfrac{1}{2}$; $\left(\pi, 1, \dfrac{1}{6}\right)$

32. $\dfrac{x + z}{y - z} = 2$; $(4, 2, 0)$

33–38. Linear approximation

- ***a.*** *Find the linear approximation to the function f at the given point.*
- ***b.*** *Use part (a) to estimate the given function value.*

33. $f(x, y) = xy + x - y$; $(2, 3)$; estimate $f(2.1, 2.99)$.

34. $f(x, y) = 12 - 4x^2 - 8y^2$; $(-1, 4)$; estimate $f(-1.05, 3.95)$.

35. $f(x, y) = -x^2 + 2y^2$; $(3, -1)$; estimate $f(3.1, -1.04)$.

36. $f(x, y) = \sqrt{x^2 + y^2}$; $(3, -4)$; estimate $f(3.06, -3.92)$.

37. $f(x, y, z) = \ln(1 + x + y + 2z)$; $(0, 0, 0)$; estimate $f(0.1, -0.2, 0.2)$.

38. $f(x, y, z) = \dfrac{x + y}{x - z}$; $(3, 2, 4)$; estimate $f(2.95, 2.05, 4.02)$.

39–42. Approximate function change *Use differentials to approximate the change in z for the given changes in the independent variables.*

39. $z = 2x - 3y - 2xy$ when (x, y) changes from $(1, 4)$ to $(1.1, 3.9)$

40. $z = -x^2 + 3y^2 + 2$ when (x, y) changes from $(-1, 2)$ to $(-1.05, 1.9)$

41. $z = e^{x+y}$ when (x, y) changes from $(0, 0)$ to $(0.1, -0.05)$

42. $z = \ln(1 + x + y)$ when (x, y) changes from $(0, 0)$ to $(-0.1, 0.03)$

43. Changes in torus surface area The surface area of a torus with an inner radius r and an outer radius $R > r$ is $S = 4\pi^2\left(R^2 - r^2\right)$.

- **a.** If r increases and R decreases, does S increase or decrease, or is it impossible to say?
- **b.** If r increases and R increases, does S increase or decrease, or is it impossible to say?

c. Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 3.05$ and R changes from $R = 5.50$ to $R = 5.65$.

d. Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 2.95$ and R changes from $R = 7.00$ to $R = 7.04$.

e. Find the relationship between the changes in r and R that leaves the surface area (approximately) unchanged.

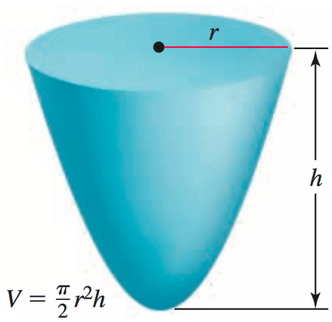
44. Changes in cone volume The volume of a right circular cone with radius r and height h is $V = \pi r^2 h/3$.

a. Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$.

b. Approximate the change in the volume of the cone when the radius changes from $r = 5.40$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$.

45. Area of an ellipse The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi a b$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

46. Volume of a paraboloid The volume of a segment of a circular paraboloid (see figure) with radius r and height h is $V = \pi r^2 h/2$. Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%.



47–50. Differentials with more than two variables *Write the differential dw in terms of the differentials of the independent variables.*

47. $w = f(x, y, z) = x y^2 + x^2 z + y z^2$

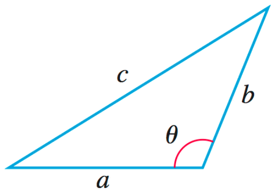
48. $w = f(x, y, z) = \sin(x + y - z)$

49. $w = f(u, x, y, z) = \dfrac{u + x}{y + z}$

50. $w = f(p, q, r, s) = \dfrac{pq}{rs}$

T **51. Law of Cosines** The side lengths of any triangle are related by the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$



a. Estimate the change in the side length c when a changes from $a = 2$ to $a = 2.03$, b changes from $b = 4.00$ to $b = 3.96$, and θ changes from $\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{3} + \frac{\pi}{90}$.

b. If a changes from $a = 2$ to $a = 2.03$ and b changes from $b = 4.00$ to $b = 3.96$, is the resulting change in c greater in magnitude when $\theta = \frac{\pi}{20}$ (small angle) or when $\theta = \frac{9\pi}{20}$ (close to a right angle)?

52. Travel cost The cost of a trip that is L miles long, driving a car that gets m miles per gallon, with gas costs of $\$p/\text{gal}$ is $C = \frac{Lp}{m}$ dollars. Suppose you plan a trip of $L = 1500$ mi in a car that gets $m = 32$ mi/gal, with gas costs of $p = \$3.80/\text{gal}$.

a. Explain how the cost function is derived.

b. Compute the partial derivatives C_L, C_m, and C_p. Explain the meaning of the signs of the derivatives in the context of this problem.

c. Estimate the change in the total cost of the trip if L changes from $L = 1500$ to $L = 1520$, m changes from $m = 32$ to $m = 31$, and p changes from $p = \$3.80$ to $p = \$3.85$.

d. Is the total cost of the trip (with $L = 1500$ mi, $m = 32$ mi/gal, and $p = \$3.80$) more sensitive to a 1% change in L, in m, or in p (assuming the other two variables are fixed)? Explain.

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The planes tangent to the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ all have the form $ax + bz + c = 0$.

b. Suppose $w = \frac{xy}{z}$, for $x > 0$, $y > 0$, and $z > 0$. A decrease in z with x and y fixed results in an increase in w.

c. The gradient $\nabla F(a, b, c)$ lies in the plane tangent to the surface $F(x, y, z) = 0$ at (a, b, c).

54–57. Horizontal tangent planes *Find the points at which the following surfaces have horizontal tangent planes.*

54. $x^2 + 2y^2 + z^2 - 2x - 2z - 2 = 0$

55. $x^2 + y^2 - z^2 - 2x + 2y + 3 = 0$

56. $z = \sin(x - y)$ in the region $-2\pi \le x \le 2\pi$, $-2\pi \le y \le 2\pi$

57. $z = \cos 2x \sin y$ in the region $-\pi \le x \le \pi$, $-\pi \le y \le \pi$

58. Heron's formula The area of a triangle with sides of length a, b, and c is given by a formula from antiquity called Heron's formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$ is the *semiperimeter* of the triangle.

a. Find the partial derivatives A_a, A_b, and A_c.

b. A triangle has sides of length $a = 2$, $b = 4$, $c = 5$. Estimate the change in the area when a increases by 0.03, b decreases by 0.08, and c increases by 0.6.

c. For an equilateral triangle with $a = b = c$, estimate the percent change in the area when all sides increase in length by $p\%$.

59. Surface area of a cone A cone with height h and radius r has a lateral surface area (the curved surface only, excluding the base) of $S = \pi r \sqrt{r^2 + h^2}$.

a. Estimate the change in the surface area when r increases from $r = 2.50$ to $r = 2.55$ and h decreases from $h = 0.60$ to $h = 0.58$.

b. When $r = 100$ and $h = 200$, is the surface area more sensitive to a small change in r or a small change in h? Explain.

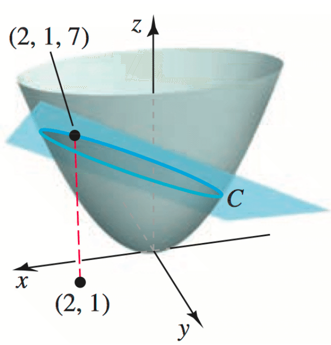
60. Line tangent to an intersection curve Consider the paraboloid $z = x^2 + 3y^2$ and the plane $z = x + y + 4$, which intersects the paraboloid in a curve C at $(2, 1, 7)$ (see figure). Find the equation of the line tangent to C at the point $(2, 1, 7)$. Proceed as follows.

a. Find a vector normal to the plane at $(2, 1, 7)$.

b. Find a vector normal to the plane tangent to the paraboloid at $(2, 1, 7)$.

c. Argue that the line tangent to C at $(2, 1, 7)$ is orthogonal to both normal vectors found in parts (a) and (b). Use this fact to find a direction vector for the tangent line.

d. Knowing a point on the tangent line and the direction of the tangent line, write an equation of the tangent line in parametric form.



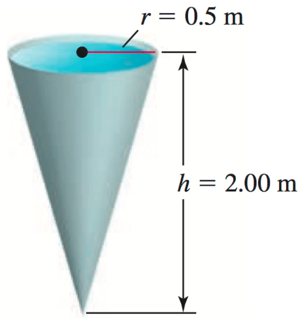
61. Batting averages Batting averages in baseball are defined by $A = x/y$, where $x \geq 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \leq A \leq 1$.

a. Use differentials to estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

b. If a batter currently has a batting average of $A = 0.350$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

c. Does the answer to part (b) depend on the current batting average? Explain.

62. Water-level changes A conical tank with radius 0.50 m and height 2.00 m is filled with water (see figure). Water is released from the tank, and the water level drops by 0.05 m (from 2.00 m to 1.95 m). Approximate the change in the volume of water in the tank. (*Hint:* When the water level drops, both the radius and height of the cone of water change.)

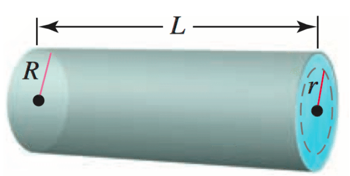


63. Flow in a cylinder Poiseuille's Law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4Lv}\left(R^2 - r^2\right)$, where P is the difference in the pressure between the ends of the cylinder and v is the viscosity of the fluid (see figure). Assuming P and v are constant, the velocity V along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where k is a constant that we will take to be $k = 1$.

a. Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3$ cm to $R = 3.05$ cm and the length increases from $L = 50$ cm to $L = 50.5$ cm.

b. Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length L increases by 2%.

c. Complete the following sentence: If the radius of the cylinder increases by p%, then the length of the cylinder must increase by approximately ___% in order for the velocity to remain constant.



Explorations and Challenges »

64. Floating-point operations In general, real numbers (with infinite decimal expansions) cannot be represented exactly in a computer by floating-point numbers (with finite decimal expansions). Suppose floating-point numbers on a particular computer carry an error of at most 10^{-16}. Estimate the maximum error that is committed in evaluating the following functions. Express the error in absolute and relative (percent) terms.

a. $f(x, y) = xy$

b. $f(x, y) = \frac{x}{y}$

c. $F(x, y, z) = xyz$

d. $F(x, y, z) = \dfrac{x/y}{z}$

65. Probability of at least one encounter Suppose in a large group of people, a fraction $0 \le r \le 1$ of the people have flu. The probability that in n random encounters you will meet at least one person with flu is $P = f(n, r) = 1 - (1 - r)^n$. Although n is a positive integer, regard it as a positive real number.

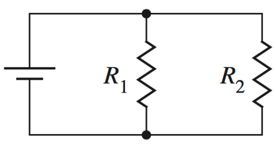
a. Compute f_r and f_n.

b. How sensitive is the probability P to the flu rate r? Suppose you meet $n = 20$ people. Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c. Approximately how much does the probability P increase if the flu rate increases from $r = 0.9$ to $r = 0.91$ with $n = 20$?

d. Interpret the results of parts (b) and (c).

66. Two electrical resistors When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R, measured in ohms (Ω), is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.



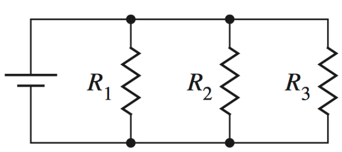
a. Estimate the change in R if R_1 increases from $2\ \Omega$ to $2.05\ \Omega$ and R_2 decreases from $3\ \Omega$ to $2.95\ \Omega$.

b. Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c. Is it true that if R_1 and R_2 increase, then R increases? Explain.

d. Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

67. Three electrical resistors Extending Exercise 66, when three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$, and $R_3 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R, measured in ohms (Ω), is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. Estimate the change in R if R_1 increases from $2\ \Omega$ to $2.05\ \Omega$, R_2 decreases from $3\ \Omega$ to $2.95\ \Omega$, and R_3 increases from $1.5\ \Omega$ to $1.55\ \Omega$.



68. Power functions and percent change Suppose that $z = f(x, y) = x^a y^b$, where a and b are real numbers. Let $\dfrac{dx}{x}$, $\dfrac{dy}{y}$, and $\dfrac{dz}{z}$ be the approximate relative (percent) changes in x, y, and z, respectively. Show that $\dfrac{dz}{z} = \dfrac{a(dx)}{x} + \dfrac{b(dy)}{y}$; that is, the relative changes are additive when weighted by the exponents a and b.

69. Logarithmic differentials Let f be a differentiable function of one or more variables that is positive on its domain.

a. Show that $d(\ln f) = \dfrac{df}{f}$.

b. Use part (a) to explain the statement that the absolute change in $\ln f$ is approximately equal to the relative change in f.

c. Let $f(x, y) = x\,y$, note that $\ln f = \ln x + \ln y$, and show that relative changes add; that is $\dfrac{df}{f} = \dfrac{dx}{x} + \dfrac{dy}{y}$.

d. Let $f(x, y) = \dfrac{x}{y}$, note that $\ln f = \ln x - \ln y$, and show that relative changes subtract; that is $\dfrac{df}{f} = \dfrac{dx}{x} - \dfrac{dy}{y}$.

e. Show that in a product of n numbers, $f = x_1\,x_2 \cdots x_n$, the relative change in f is approximately equal to the sum of the relative changes in the variables.

70. Distance from a plane to an ellipsoid (Adapted from 1938 Putnam Exam) Consider the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and the plane P given by $A\,x + B\,y + C\,z + 1 = 0$. Let $h = \left(A^2 + B^2 + C^2\right)^{-1/2}$ and $m = \left(a^2\,A^2 + b^2\,B^2 + c^2\,C^2\right)^{1/2}$.

a. Find the equation of the plane tangent to the ellipsoid at the point (p, q, r).

b. Find the two points on the ellipsoid at which the tangent plane is parallel to P and find equations of the tangent planes.

c. Show that the distance between the origin and the plane P is h.

d. Show that the distance between the origin and the tangent planes is $h\,m$.

e. Find a condition that guarantees the plane P does not intersect the ellipsoid.