15.5 Directional Derivatives and the Gradient

Partial derivatives tell us a lot about the rate of change of a function on its domain. However, they do not *directly* answer some important questions. For example, suppose you are standing at a point $(a, b, f(a, b))$ on the surface $z = f(x, y)$. The partial derivatives f_x and f_y tell you the rate of change (or slope) of the surface at that point in the directions parallel to the x-axis and y-axis, respectively. But you could walk in an infinite number of directions from that point and find a different rate of change in every direction. With this observation in mind, we pose several questions.

- Suppose you are standing on a surface and you walk in a direction *other* than a coordinate direction—say, northwest or south-southeast. What is the rate of change of the function in such a direction?
- Suppose you are standing on a surface and you release a ball at your feet and let it roll. In which direction will it roll?
- If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the steepest path?

These questions will be answered in this section as we introduce the *directional derivative*, followed by one of the central concepts of calculus—the *gradient*.

Directional Derivatives »

Let $(a, b, f(a, b))$ be a point on the surface $z = f(x, y)$ and let $\mathbf{u}$ be a unit vector in the xy-plane (**Figure 15.45**). Our aim is to find the rate of change of f in the direction $\mathbf{u}$ at $P_0(a, b)$. In general, this rate of change is neither $f_x(a, b)$ nor $f_y(a, b)$ (unless $\mathbf{u} = \langle 1, 0\rangle$ or $\mathbf{u} = \langle 0, 1\rangle$), but it turns out to be a combination of $f_x(a, b)$ and $f_y(a, b)$.

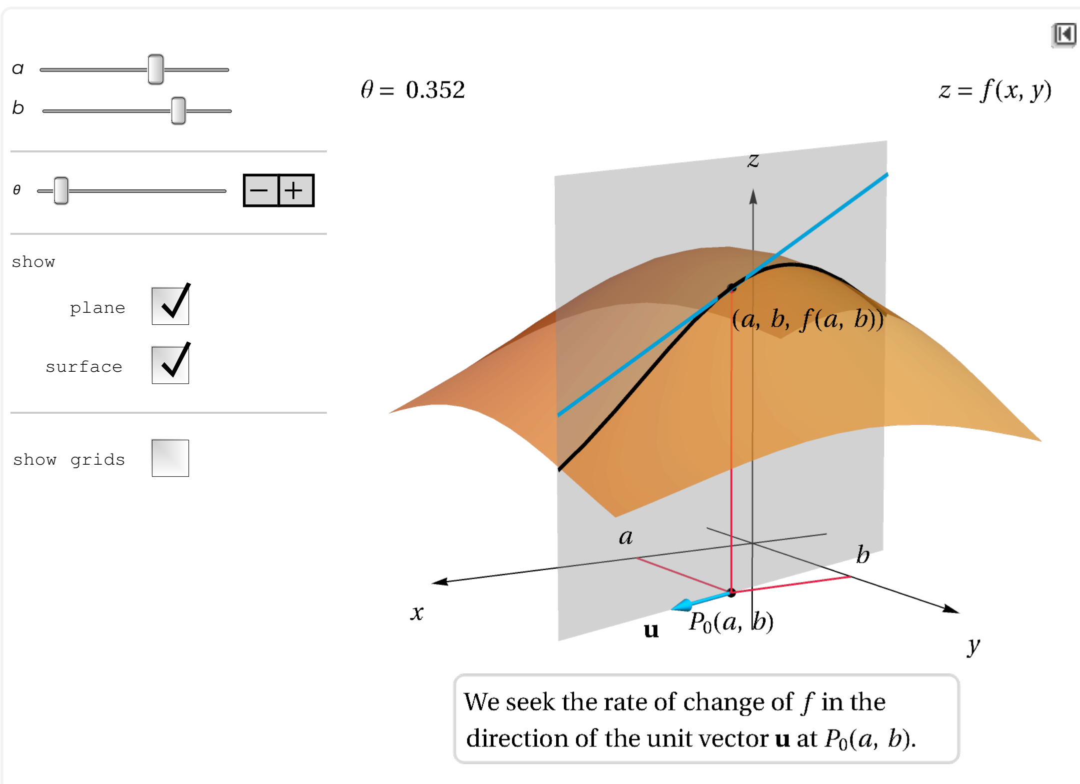


Figure 15.45

Figure 15.46a shows the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$; its x- and y-components are u_1 and u_2, respectively.

The derivative we seek must be computed along the line ℓ in the xy-plane through P_0 in the direction of **u**. A neighboring point P, which is h units from P_0 along ℓ, has coordinates $P(a + h\,u_1,\ b + h\,u_2)$ (**Figure 15.46b**).

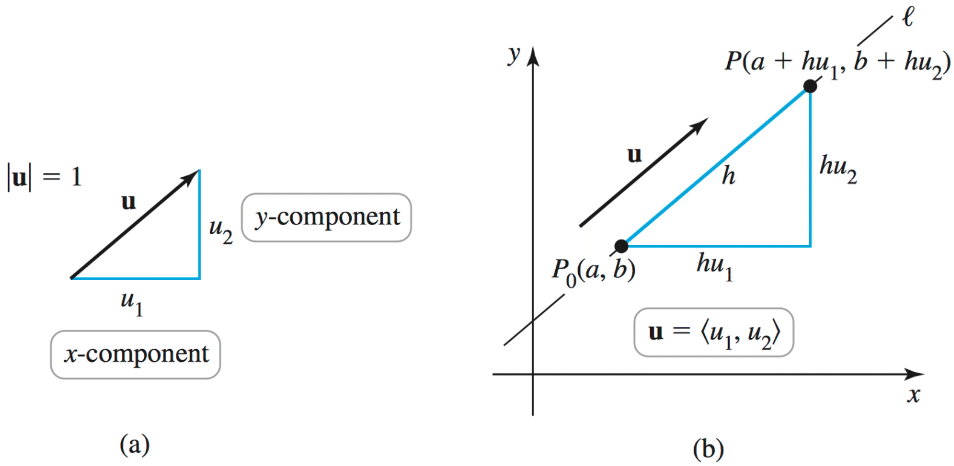


Figure 15.46

Now imagine the plane Q perpendicular to the xy-plane, containing ℓ. This plane cuts the surface $z = f(x,\ y)$ in a curve C. Consider two points on C corresponding to P_0 and P; they have z-coordinates $f(a,\ b)$ and $f(a + h\,u_1,\ b + h\,u_2)$ (**Figure 15.47**). The slope of the secant line between these points is

$$\frac{f(a + h\,u_1,\ b + h\,u_2) - f(a,\ b)}{h}.$$

The derivative of f in the direction of **u** is obtained by letting $h \to 0$; when the limit exists, it is called the *directional derivative of f at* $(a,\ b)$ *in the direction of* **u**. It gives the slope of the line tangent to the curve C in the plane Q.

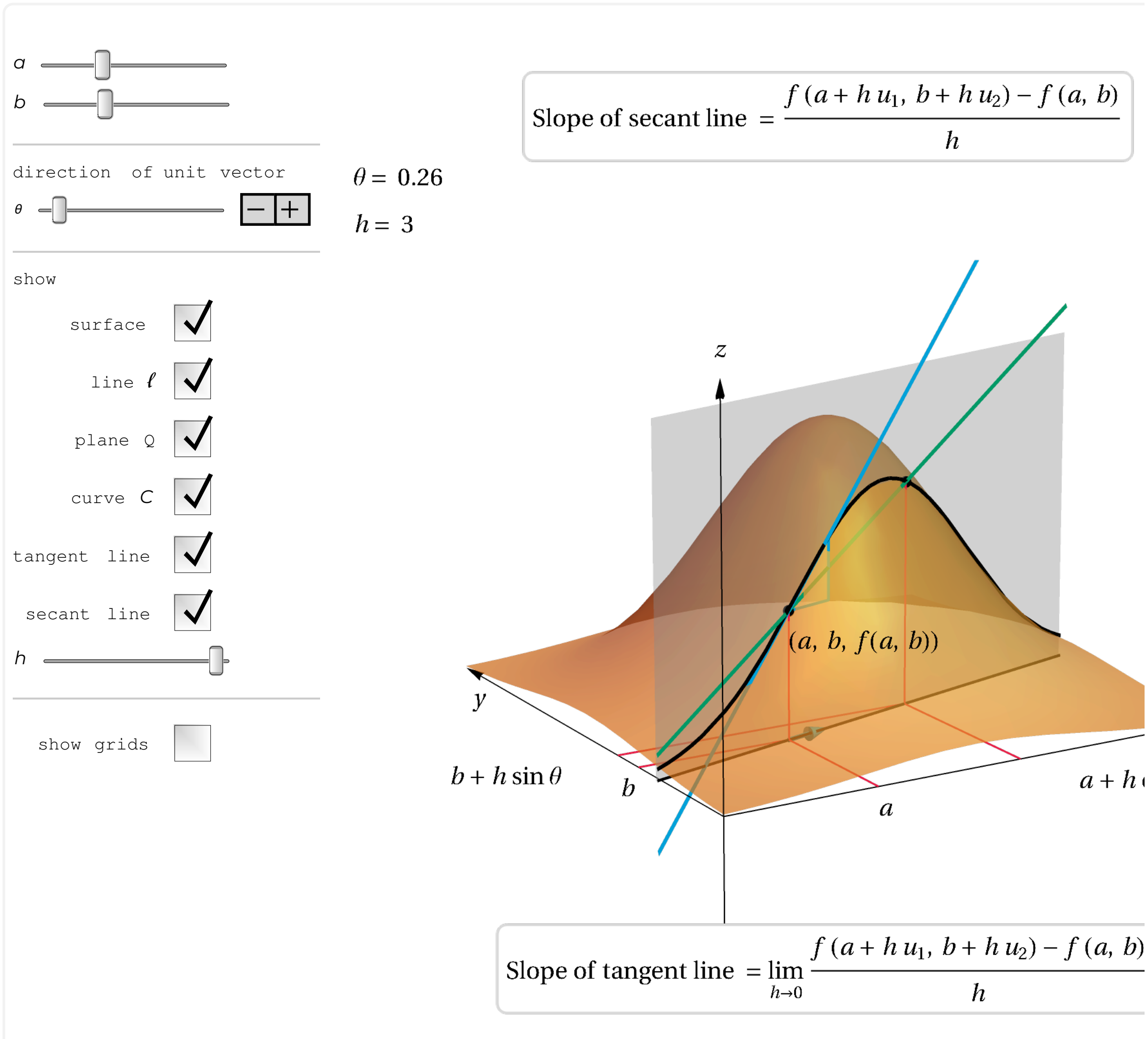


Figure 15.47

DEFINITION **Directional Derivative**

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\, u_1,\, b + h\, u_2) - f(a, b)}{h},$$

provided the limit exists.

Note »

As motivation, it is instructive to see how the directional derivative includes the ordinary derivative in one variable. Setting $u_2 = 0$ in the definition of the directional derivative and ignoring the second variable gives the rate of change of f in the x-direction. The directional derivative becomes

$$\lim_{h \to 0} \frac{f(a + h\, u_1) - f(a)}{h}.$$

Multiplying the numerator and denominator of this quotient by u_1, we have

$$u_1 \underbrace{\lim_{h \to 0} \frac{f(a + h\,u_1) - f(a)}{h}}_{f'(a)} = u_1\, f'(a).$$

Only because **u** is a unit vector and $u_1 = 1$ does the directional derivative reduce to the ordinary derivative $f'(a)$ in the x-direction. A similar argument may be used in the y-direction. Choosing **u** to be a unit vector gives the simplest formulas for the directional derivative.

Quick Check 1 Explain why, when $\mathbf{u} = \langle 1, 0 \rangle$ in the definition of the directional derivative, the result is $f_x(a, b)$ and when $\mathbf{u} = \langle 0, 1 \rangle$, the result is $f_y(a, b)$. ◆

Answer »

If $\mathbf{u} = \langle u_1, u_2 \rangle = \langle 1, 0 \rangle$, then

$$D_{\mathbf{u}}\, f(a, b) = \lim_{h \to 0} \frac{f(a + h\,u_1,\, b + h\,u_2) - f(a, b)}{h}$$
$$= \lim_{h \to 0} \frac{f(a + h,\, b) - f(a, b)}{h} = f_x(a, b).$$

Similarly, when $\mathbf{u} = \langle 0, 1 \rangle$, the partial derivative $f_y(a, b)$ results.

As with ordinary derivatives, we would prefer to evaluate directional derivatives without taking limits. Fortunately, there is an easy way to express the directional derivative in terms of partial derivatives.

The key is to define a function that is equal to f along the line ℓ through (a, b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$. The points on ℓ satisfy the parametric equations

$$x = a + s\,u_1 \quad \text{and} \quad y = b + s\,u_2,$$

where $-\infty < s < \infty$. Because **u** is a unit vector, the parameter s corresponds to arc length. As s increases, the points (x, y) move along ℓ in the direction of **u** with $s = 0$ corresponding to (a, b). Now we define the function

$$g(s) = f\Big(\underbrace{a + s\,u_1}_{x},\, \underbrace{b + s\,u_2}_{y}\Big),$$

which gives the values of f along ℓ. The derivative of f along ℓ is $g'(s)$, and when evaluated at $s = 0$, it is the directional derivative of f at (a, b); that is, $g'(0) = D_{\mathbf{u}}\, f(a, b)$.

Note »

To see that s is an arc length parameter, note that the line ℓ may be written in the form

$$\mathbf{r}(s) = \langle a + s\,u_1,\, b + s\,u_2 \rangle.$$

Therefore, $\mathbf{r}'(s) = \langle u_1, u_2 \rangle$ and $|\mathbf{r}'(s)| = 1$. It follows by the discussion in Section 14.4 that s is an arc length parameter. Note that $g'(s)$ does not correctly measure the slope of f along ℓ unless **u** is a unit vector.

Noting that $\dfrac{dx}{ds} = u_1$ and $\dfrac{dy}{ds} = u_2$, we apply the Chain Rule to find that

$$\begin{aligned} D_{\mathbf{u}} f(a, b) = g'(0) &= \left.\left(\frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{u_2}\right)\right|_{s=0} && \text{Chain Rule} \\ &= f_x(a, b)\, u_1 + f_y(a, b)\, u_2 && s = 0 \text{ corresponds to } (a, b). \\ &= \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle. && \text{Identify dot product.} \end{aligned}$$

We see that the directional derivative is a weighted average of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$, with the components of $\mathbf{u}$ serving as the weights. In other words, knowing the slope of the surface in the x- and y-directions allows us to find the slope in any direction. Notice that the directional derivative can be written as a dot product, which provides a practical formula for computing directional derivatives.

Quick Check 2 In the parametric description $x = a + s\,u_1$ and $y = b + s\,u_2$, where $\mathbf{u} = \langle u_1, u_2\rangle$ is a unit vector, show that any positive change Δs in s produces a line segment of length Δs. ◆

Answer »

THEOREM 15.10 Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle.$$

EXAMPLE 1 Computing directional derivatives

Consider the paraboloid $z = f(x, y) = \frac{1}{4}\left(x^2 + 2y^2\right) + 2$. Let P_0 be the point $(3, 2)$ and consider the unit vectors

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \text{and} \quad \mathbf{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle.$$

a. Find the directional derivative of f at P_0 in the directions of $\mathbf{u}$ and $\mathbf{v}$.

b. Graph the surface and interpret the directional derivatives.

SOLUTION »

a. We see that $f_x = \frac{x}{2}$ and $f_y = y$; evaluated at $(3, 2)$, we have $f_x(3, 2) = \frac{3}{2}$ and $f_y(3, 2) = 2$. The directional derivatives in the directions $\mathbf{u}$ and $\mathbf{v}$ are

$$\begin{aligned} D_{\mathbf{u}} f(3, 2) &= \langle f_x(3, 2), f_y(3, 2)\rangle \cdot \langle u_1, u_2\rangle \\ &= \frac{3}{2} \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = \frac{7}{2\sqrt{2}} \approx 2.47 \text{ and} \\ D_{\mathbf{v}} f(3, 2) &= \langle f_x(3, 2), f_y(3, 2)\rangle \cdot \langle v_1, v_2\rangle \\ &= \frac{3}{2} \cdot \frac{1}{2} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4} - \sqrt{3} \approx -0.98. \end{aligned}$$

b. In the direction of $\mathbf{u}$, the directional derivative is approximately 2.47. Because it is positive, the function is increasing at $(3, 2)$ in this direction. Equivalently, if Q is the vertical plane containing $\mathbf{u}$ and C is the curve along which the surface intersects Q, then the slope of the line tangent to C is approximately 2.47 (**Figure 15.48**). In the direction of $\mathbf{v}$, the directional derivative is approximately -0.98. Because it is negative, the function is decreasing in this direction. In this case, the vertical plane Q contains $\mathbf{v}$ and again C is the curve along which the surface intersects Q; the slope of the line tangent to C is approximately -0.98.

Note »

> It is understood that the line tangent to C in the direction of $\mathbf{u}$ lies in the vertical plane containing $\mathbf{u}$.

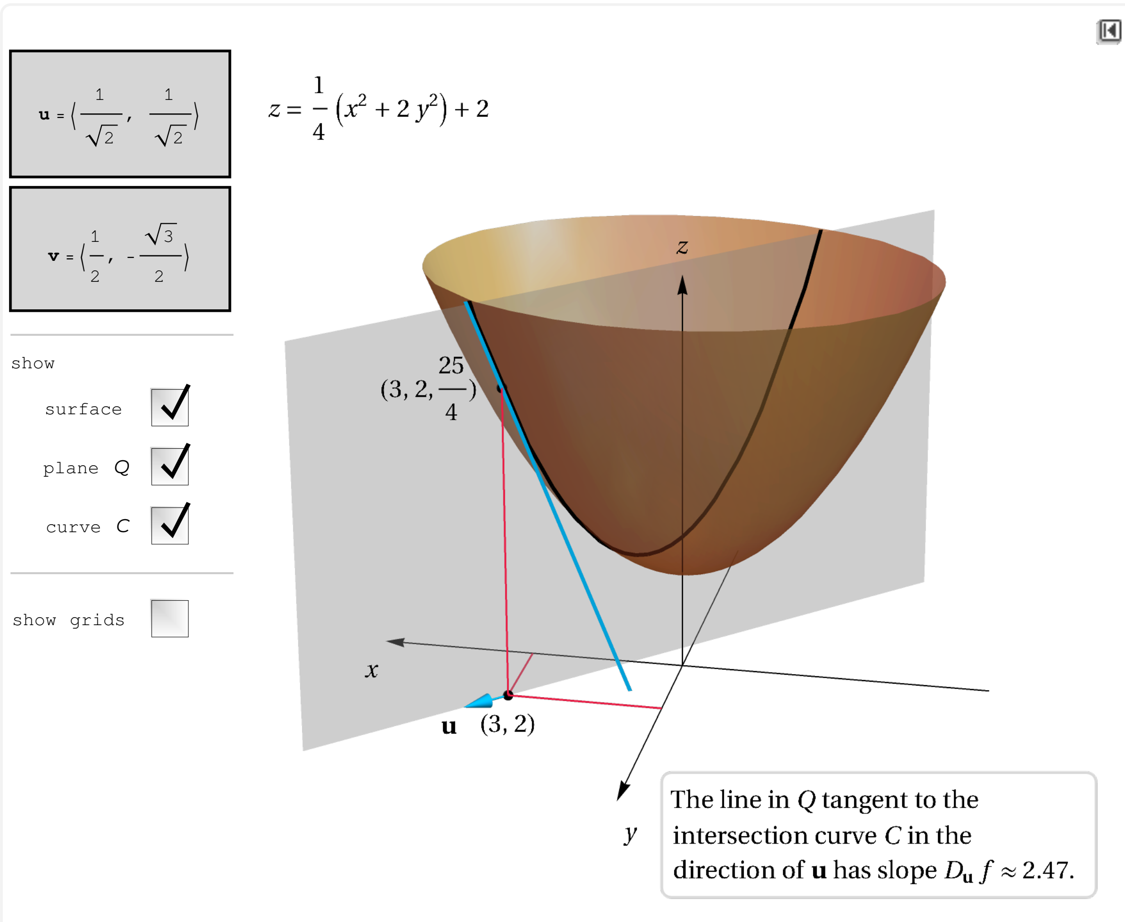


Figure 15.48

Related Exercises 11–12 ◆

Quick Check 3 In Example 1, evaluate $D_{-\mathbf{u}}\, f(3, 2)$ and $D_{-\mathbf{v}}\, f(3, 2)$. ◆

Answer »

The Gradient Vector »

We have seen that the directional derivative can be written as a dot product: $D_{\mathbf{u}}\, f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle$. The vector $\langle f_x(a, b), f_y(a, b)\rangle$ that appears in the dot product is important in its own right and is called the *gradient* of f.

DEFINITION **Gradient (Two Dimensions)**

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y)\right\rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

Note »

With the definition of the gradient, the directional derivative of f at (a, b) in the direction of the unit vector $\mathbf{u}$ can be written

$$D_{\mathbf{u}}\, f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

The gradient satisfies sum, product, and quotient rules analogous to those for ordinary derivatives (Exercise 85).

EXAMPLE 2 Computing gradients

Find $\nabla f(3, 2)$ for $f(x, y) = x^2 + 2\,x\,y - y^3$.

SOLUTION »

Computing $f_x = 2\,x + 2\,y$ and $f_y = 2\,x - 3\,y^2$, we have

$$\nabla f(x, y) = \left\langle 2\,(x+y), 2\,x - 3\,y^2\right\rangle = 2\,(x+y)\,\mathbf{i} + \left(2\,x - 3\,y^2\right)\mathbf{j}.$$

Substituting $x = 3$ and $y = 2$ gives

$$\nabla f(3, 2) = \langle 10, -6\rangle = 10\,\mathbf{i} - 6\,\mathbf{j}.$$

Related Exercises 13–15 ◆

EXAMPLE 3 Computing directional derivatives with gradients

Let $f(x, y) = 3 - \dfrac{x^2}{10} + \dfrac{x\,y^2}{10}$.

a. Compute $\nabla f(3, -1)$.

b. Compute $D_{\mathbf{u}}\, f(3, -1)$, where $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right\rangle$.

c. Compute the directional derivative of f at $(3, -1)$ in the direction of the vector $\langle 3, 4\rangle$.

SOLUTION »

a. Note that $f_x = -\dfrac{x}{5} + \dfrac{y^2}{10}$ and $f_y = \dfrac{x\,y}{5}$. Therefore,

$$\nabla f(3, -1) = \left\langle -\frac{x}{5} + \frac{y^2}{10}, \frac{x\,y}{5}\right\rangle\Bigg|_{(3,-1)} = \left\langle -\frac{1}{2}, -\frac{3}{5}\right\rangle.$$

b. Before computing the directional derivative, it is important to verify that $\mathbf{u}$ is a unit vector (in this case, it is). The required directional derivative is

$$D_{\mathbf{u}} f(3, -1) = \nabla f(3, -1) \cdot \mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \frac{1}{10\sqrt{2}}.$$

Figure 15.49 shows the line tangent to the trace in the plane corresponding to **u** whose slope is $D_{\mathbf{u}} f(3, -1)$.

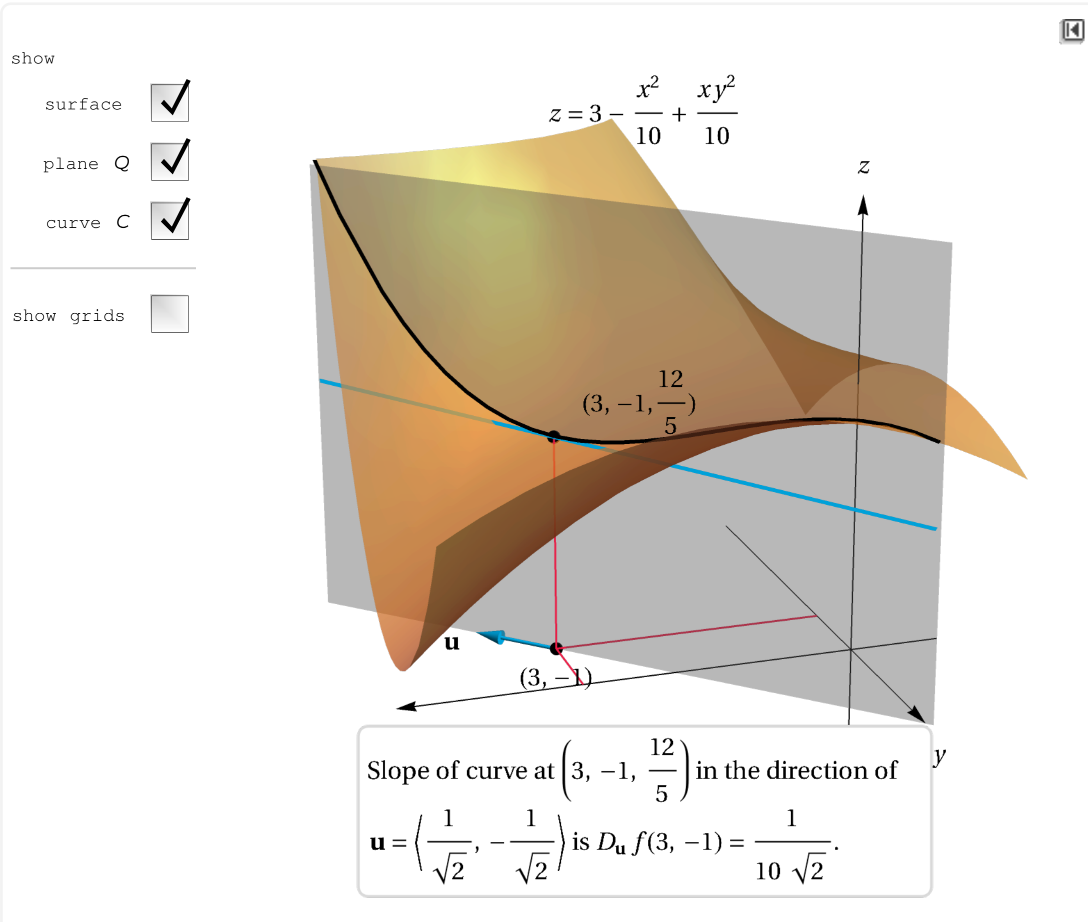


Figure 15.49

c. In this case, the direction is given in terms of a nonunit vector. The vector $\langle 3, 4 \rangle$ has length 5, so the unit vector in the direction of $\langle 3, 4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. The directional derivative at $(3, -1)$ in the direction of **u** is

$$D_{\mathbf{u}} f(3, -1) = \nabla f(3, -1) \cdot \mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{39}{50},$$

which gives the slope of the surface in the direction of $\langle 3, 4 \rangle$ at $(3, -1)$.

Related Exercises 22, 27 ◆

Interpretations of the Gradient »

The gradient is important not only in calculating directional derivatives; it plays many other roles in multivariable calculus. Our present goal is to develop some intuition about the meaning of the gradient.

We have seen that the directional derivative of f at (a, b) in the direction of the unit vector **u** is

$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$. Using properties of the dot product, we have

Note »

$$\begin{aligned} D_{\mathbf{u}} f(a, b) &= \nabla f(a, b) \cdot \mathbf{u} \\ &= |\nabla f(a, b)|\,|\mathbf{u}| \cos\theta \\ &= |\nabla f(a, b)| \cos\theta, \qquad |\mathbf{u}| = 1 \end{aligned}$$

where θ is the angle between $\nabla f(a, b)$ and $\mathbf{u}$. If follows that $D_{\mathbf{u}} f(a, b)$ has its maximum value when $\cos\theta = 1$, which corresponds to $\theta = 0$. Therefore, $D_{\mathbf{u}} f(a, b)$ has its maximum value and f has its greatest rate of *increase* when $\nabla f(a, b)$ and $\mathbf{u}$ point in the same direction. Notice that when $\cos\theta = 1$, the actual rate of increase is $D_{\mathbf{u}} f(a, b) = |\nabla f(a, b)|$ (**Figure 15.50**).

Note »

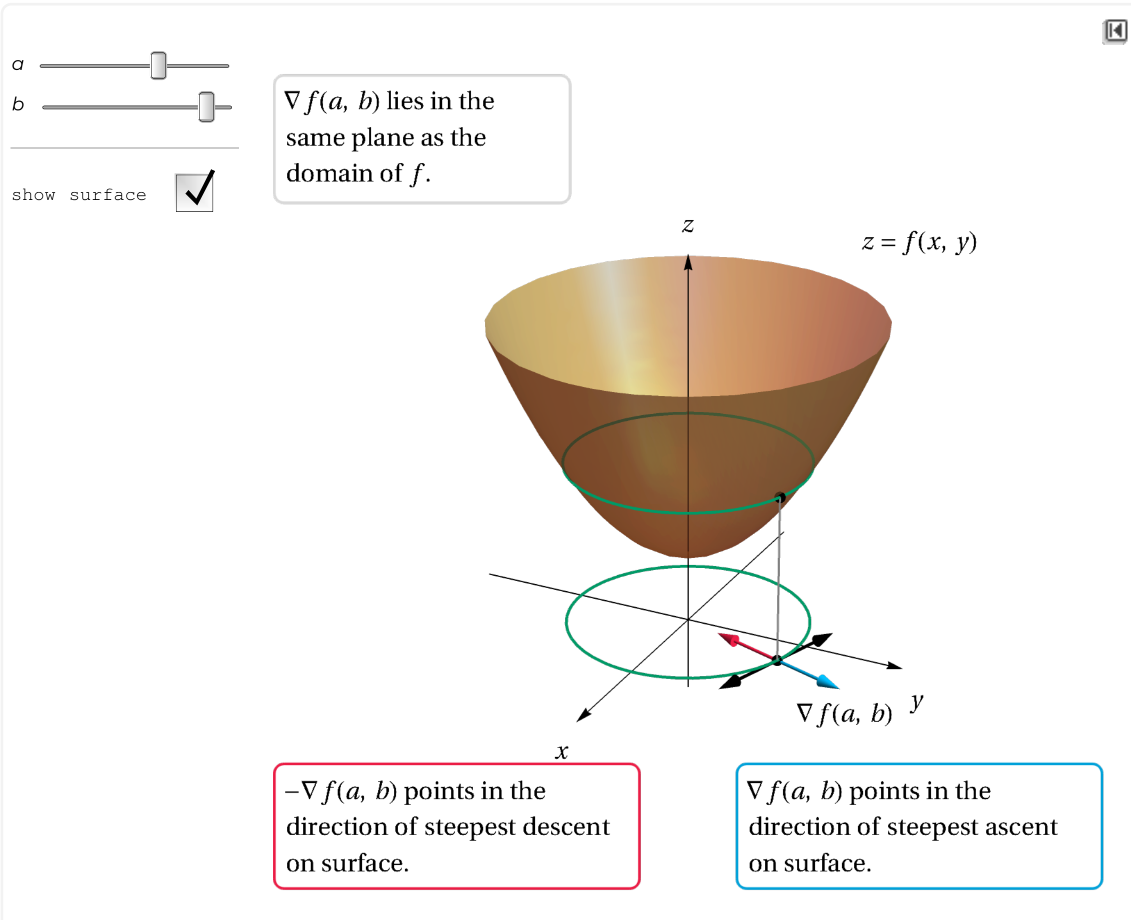


Figure 15.50

Similarly, when $\theta = \pi$, we have $\cos\theta = -1$, and f has its greatest rate of *decrease* when $\nabla f(a, b)$ and $\mathbf{u}$ point in opposite directions. The actual rate of decrease is $D_{\mathbf{u}} f(a, b) = -|\nabla f(a, b)|$. These observations are summarized as follows: The gradient $\nabla f(a, b)$ points in the *direction of steepest ascent* at (a, b), while $-\nabla f(a, b)$ points in the *direction of steepest descent*.

Notice that $D_{\mathbf{u}} f(a, b) = 0$ when the angle between $\nabla f(a, b)$ and $\mathbf{u}$ is $\frac{\pi}{2}$, which means $\nabla f(a, b)$ and $\mathbf{u}$ are orthogonal (Figure 15.50). These observations justify the following theorem.

THEOREM 15.11 **Directions of Change**

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.

2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

EXAMPLE 4 **Steepest ascent and descent**

Consider the bowl-shaped paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$.

a. If you are located on the paraboloid at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you move in order to *ascend* on the surface at the maximum rate? What is the rate of change?

b. If you are located at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you walk in order to *descend* on the surface at the maximum rate? What is the rate of change?

c. At the point $(3, 1, 16)$, in what direction(s) is there no change in the function values?

SOLUTION »

EXAMPLE 5 **Interpreting directional derivatives**

Consider the function $f(x, y) = 3x^2 - 2y^2$.

a. Compute $\nabla f(x, y)$ and $\nabla f(2, 3)$.

b. Let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. At $(2, 3)$, for what values of θ (measured relative to the positive x-axis), with $0 \le \theta < 2\pi$, does the directional derivative have its maximum and minimum values and what are those values?

SOLUTION »

a. The gradient is $\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 6x, -4y \rangle$, and at $(2, 3)$, we have $\nabla f(2, 3) = \langle 12, -12 \rangle$.

b. The gradient $\nabla f(2, 3) = \langle 12, -12 \rangle$ makes an angle of $\frac{7\pi}{4}$ with the positive x-axis. So, the maximum rate of change of f occurs in this direction, and that rate of change is $|\nabla f(2, 3)| = |\langle 12, -12 \rangle| = 12\sqrt{2} \approx 17$. The direction of maximum decrease is opposite to the direction of the gradient, which corresponds to $\theta = \frac{3\pi}{4}$. The maximum rate of decrease is the negative of the maximum rate of increase, or $-12\sqrt{2} \approx -17$. The function has zero change in the directions orthogonal to the gradient, which correspond to $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

Figure 15.52 summarizes these conclusions. Notice that the gradient at $(2, 3)$ appears to be orthogonal to the level curve of f passing through (2, 3). We next see that this is always the case.

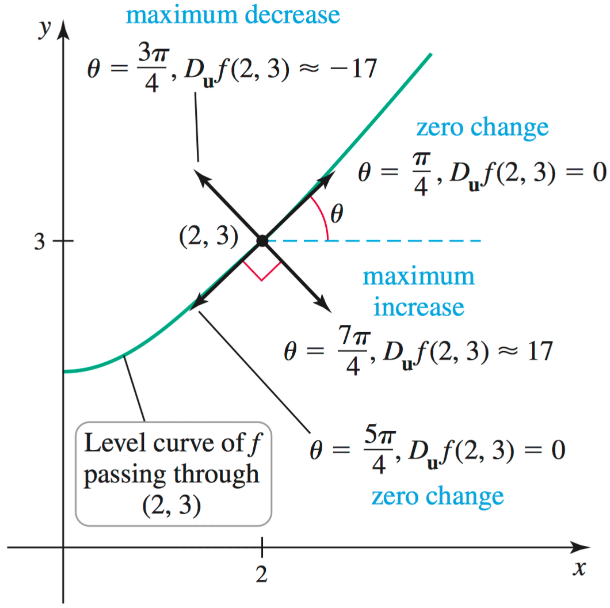


Figure 15.52

Related Exercises 37–38 ◆

The Gradient and Level Curves »

Theorem 15.11 states that in any direction orthogonal to the gradient $\nabla f(a, b)$, the function f does not change at (a, b). Recall from Section 15.1 that the curve $f(x, y) = z_0$, where z_0 is a constant, is a *level curve*, on which function values are constant. Combining these two observations, we conclude that the gradient $\nabla f(a, b)$ is orthogonal to the line tangent to the level curve through (a, b).

THEOREM 15.12 **The Gradient and Level Curves**

Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

Proof: Consider the function $z = f(x, y)$ and its level curve $f(x, y) = z_0$, where the constant z_0 is chosen so that the curve passes through the point (a, b). Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a parametrization for the level curve near (a, b) (where it is smooth) and let $\mathbf{r}(t_0)$ correspond to the point (a, b). We now differentiate $f(x, y) = z_0$ with respect to t. The derivative of the right side is 0. Applying the Chain Rule to the left side results in

$$\begin{aligned} \frac{d}{dt}(f(x, y)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{\nabla f(x,y)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\mathbf{r}'(t)} \\ &= \nabla f(x, y) \cdot \mathbf{r}'(t). \end{aligned}$$

Substituting $t = t_0$, we have $\nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0$, which implies that $\mathbf{r}'(t_0)$ (the tangent vector at (a, b)) is orthogonal to $\nabla f(a, b)$. **Figure 15.53** illustrates the geometry of the theorem. ◆

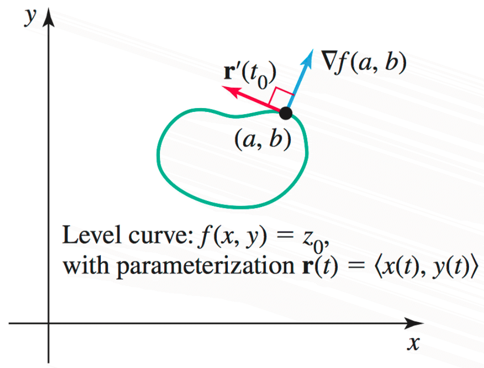


Figure 15.53

An immediate consequence of Theorem 15.12 is an alternative equation of the tangent line. The curve described by $f(x, y) = z_0$ can be viewed as a level curve for a surface. By Theorem 15.12, the line tangent to the curve at (a, b) is orthogonal to $\nabla f(a, b)$. Therefore, if (x, y) is a point on the tangent line, then $\nabla f(a, b)\cdot\langle x - a, y - b\rangle = 0$, which, when simplified, gives an equation of the line tangent to the curve $f(x, y) = z_0$:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0.$$

Quick Check 4 Draw a circle in the xy-plane centered at the origin and regard it is as a level curve of the surface $z = x^2 + y^2$. At the point (a, a) of the level curve, the slope of the tangent line is -1. Show that the gradient at (a, a) is orthogonal to the tangent line. ◆

Answer »

> The gradient is $\langle 2x, 2y\rangle$, which, evaluated at (a, a), is $\langle 2a, 2a\rangle$. Taking the dot product of the gradient and the vector $\langle -1, 1\rangle$ (a vector parallel to a line of slope -1), we see that $\langle 2a, 2a\rangle\cdot\langle -1, 1\rangle = 0$.

EXAMPLE 6 Gradients and level curves

Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

a. Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

b. Find an equation of the line tangent to the level curve at $(1, 1)$.

SOLUTION »

EXAMPLE 7 Path of steepest descent

The paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$ is shown in **Figure 15.55**. A ball is released at the point $(3, 4, 61)$ on the surface and it follows the path of steepest descent C to the vertex of the paraboloid.

a. Find an equation of the projection of C in the xy-plane.

b. Find an equation of C on the paraboloid.

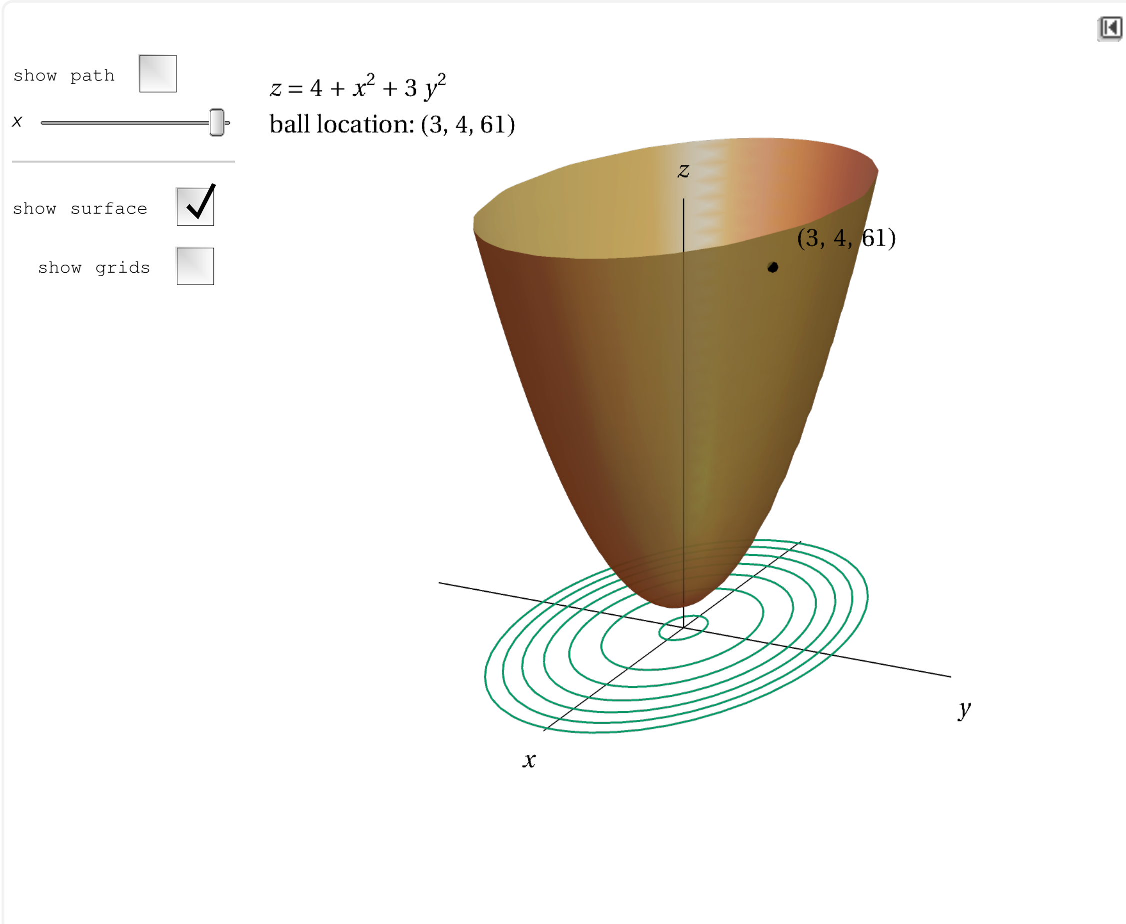


Figure 15.55

SOLUTION »

a. The projection of C in the xy-plane points in the direction of $-\nabla f(x, y) = \langle -2x, -6y \rangle$, which means that at the point (x, y) the line tangent to the path has slope $y'(x) = \frac{-6y}{-2x} = \frac{3y}{x}$. Therefore, the path in the xy-plane satisfies $y'(x) = \frac{3y}{x}$ and passes through the initial point $(3, 4)$. You can verify that the solution to this differential equation is $y = \frac{4x^3}{27}$. Therefore, the projection of the path of steepest descent in the xy-plane is the curve $y = \frac{4x^3}{27}$. The descent ends at $(0, 0)$, which corresponds to the vertex of the paraboloid (Figure 15.55). At all points of the descent, the curve in the xy-plane is orthogonal to the level curves of the surface.

b. To find a parametric description of C, it is easiest to define the parameter $t = x$. Using part (a), we find that

$$y = \frac{4x^3}{27} = \frac{4t^3}{27} \text{ and } z = 4 + x^2 + 3y^2 = 4 + t^2 + \frac{16}{243}t^6.$$

Because $0 \le x \le 3$, the parameter t varies over the interval $0 \le t \le 3$. A parametric description of C is

$$C : \mathbf{r}(t) = \left\langle t, \frac{4t^3}{27}, 4 + t^2 + \frac{16}{243} t^6 \right\rangle, \text{ for } 0 \le t \le 3.$$

With this parameterization, C is traced from $\mathbf{r}(0) = \langle 0, 0, 4 \rangle$ to $\mathbf{r}(3) = \langle 3, 4, 61 \rangle$—in the direction opposite to that of the ball's descent.

Related Exercise 57 ◆

Quick Check 5 Verify that $y = \frac{4x^3}{27}$ satisfies the equation $y'(x) = \frac{3y}{x}$, with $y(3) = 4$. ◄

The Gradient in Three Dimensions »

The directional derivative, the gradient, and the idea of a level curve extend immediately to functions of three variables of the form $w = f(x, y, z)$. The main differences are that the gradient is a vector in $\mathbb{R}^3$ and level curves become *level surfaces* (Section 15.1). Here is how the gradient looks when we step up one dimension.

The easiest way to visualize the surface $w = f(x, y, z)$ is to picture its level surfaces—the surfaces in $\mathbb{R}^3$ on which f has a constant value. The level surfaces are given by the equation $f(x, y, z) = C$, where C is a constant (**Figure 15.56**). The level surfaces *can* be graphed, and they may be viewed as layers of the full four-dimensional surface (like layers of an onion). With this image in mind, we now extend the concept of a gradient.

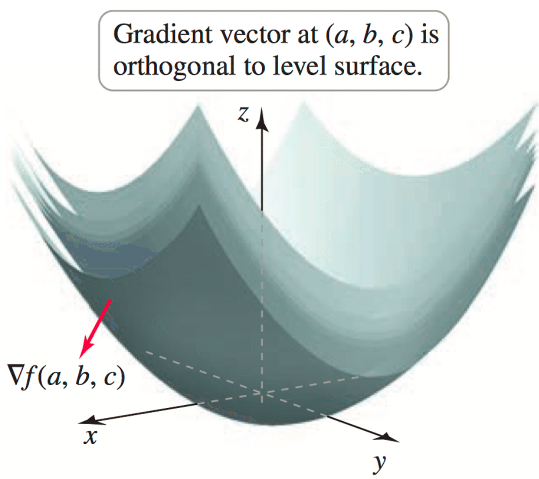


Figure 15.56

Given the function $w = f(x, y, z)$, we begin just as we did in the two-variable case and define the directional derivative and the gradient.

DEFINITION **Directional Derivative and Gradient in Three Dimensions**

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of u** is

$$D_{\mathbf{u}} f(a, b, c) = \lim_{h \to 0} \frac{f(a + h u_1, b + h u_2, c + h u_3) - f(a, b, c)}{h},$$

provided this limit exists.

The **gradient** of f at the point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x, y, z) &= \left\langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \right\rangle \\ &= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}. \end{aligned}$$

An argument similar to that given in two dimensions leads from the definition of the directional derivative to a computational formula. Given a unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, the directional derivative of f in the direction of $\mathbf{u}$ at the point (a, b, c) is

$$D_{\mathbf{u}} f(a, b, c) = f_x(a, b, c)\, u_1 + f_y(a, b, c)\, u_2 + f_z(a, b, c)\, u_3.$$

As before, we recognize this expression as a dot product of the vector $\mathbf{u}$ and the vector $\nabla f(a, b, c) = \left\langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \right\rangle$, which is the gradient evaluated at (a, b, c). These observations lead to Theorem 15.13, which mirrors Theorems 15.10 and 15.11.

THEOREM 15.13 **Directional Derivative and Interpreting the Gradient**

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \left\langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \right\rangle \cdot \langle u_1, u_2, u_3 \rangle. \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.

2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

Note »

When we introduce the tangent plane in Section 15.6, we can also claim that $\nabla f(a, b, c)$ is orthogonal to the level surface that passes through (a, b, c).

Quick Check 6 Compute $\nabla f(-1, 2, 1)$ when $f(x, y, z) = \dfrac{xy}{z}$. ◆

Answer »

$\langle 2, -1, 2 \rangle$

EXAMPLE 8 Gradients in three dimensions

Consider the function $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$ and its level surface $f(x, y, z) = 3$.

a. Find and interpret the gradient at the points $P(2, 0, 0)$, $Q\left(0, \sqrt{2}, 0\right)$, $R(0, 0, 1)$, and $S\left(1, 1, \frac{1}{2}\right)$ on the level surface.

b. What are the actual rates of change of f in the directions of the gradients in part (a)?

SOLUTION »

a. The gradient is

$$\nabla f = \left\langle f_x, f_y, f_z \right\rangle = \langle 2x, 4y, 8z \rangle.$$

Evaluating the gradient at the four points we find that

$$\nabla f(2, 0, 0) = \langle 4, 0, 0 \rangle, \qquad \nabla f\left(0, \sqrt{2}, 0\right) = \left\langle 0, 4\sqrt{2}, 0 \right\rangle,$$
$$\nabla f(0, 0, 1) = \langle 0, 0, 8 \rangle, \text{ and } \quad \nabla f\left(1, 1, \frac{1}{2}\right) = \langle 2, 4, 4 \rangle.$$

The level surface $f(x, y, z) = 3$ is an ellipsoid (**Figure 15.57**), which is one layer of a four-dimensional surface. The four points P, Q, R, and S are shown on the level surface with the respective gradient vectors. In each case, the gradient points in the direction that f has its maximum rate of increase. Of particular importance is the fact—to be made clear in the next section—that at each point, the gradient is orthogonal to the level surface.

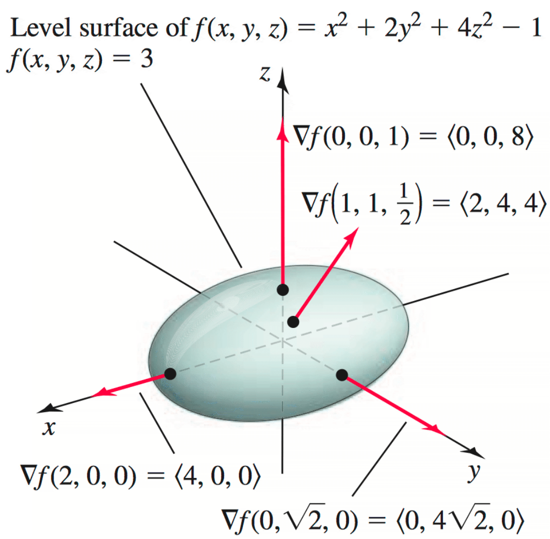


Figure 15.57

b. The actual rate of increase of f at (a, b, c) in the direction of the gradient is $|\nabla f(a, b, c)|$. At P, the rate of increase of f in the direction of the gradient is $|\langle 4, 0, 0 \rangle| = 4$; at Q, the rate of increase is $\left|\left\langle 0, 4\sqrt{2}, 0 \right\rangle\right| = 4\sqrt{2}$; at R the rate of increase is $|\langle 0, 0, 8 \rangle| = 8$; and at S, the rate of increase is $|\langle 2, 4, 4 \rangle| = 6$.

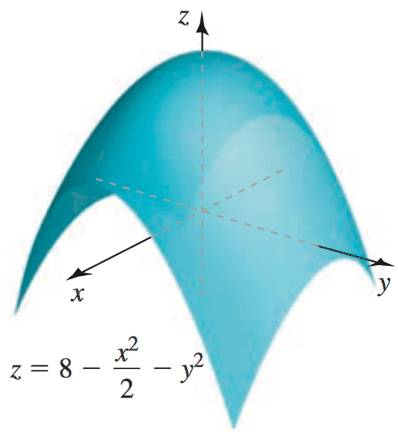
Related Exercises 59–60 ◆

Exercises »

Getting Started »

Practice Exercises »

11. Directional derivatives Consider the function $f(x, y) = 8 - \dfrac{x^2}{2} - y^2$, whose graph is a paraboloid (see figure).

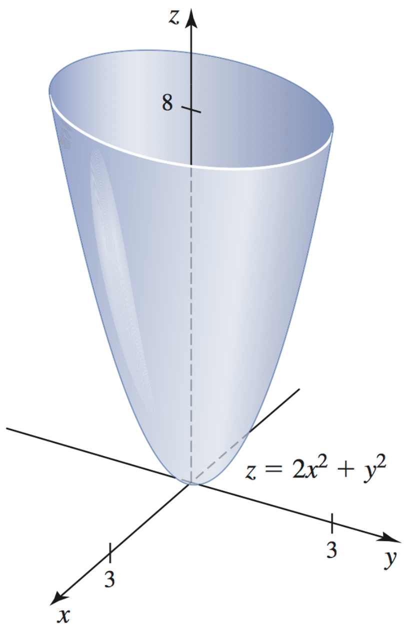


	$(a, b) = (2, 0)$	$(a, b) = (0, 2)$	$(a, b) = (1, 1)$
$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$			

a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions given by the unit vectors **u**, **v**, and **w**.

b. Interpret each of the directional derivatives computed in part (a) at the point (2, 0).

12. Directional derivatives Consider the function $f(x, y) = 2x^2 + y^2$, whose graph is a paraboloid (see figure).



	$(a, b) = (1, 0)$	$(a, b) = (1, 1)$	$(a, b) = (1, 2)$
$\mathbf{u} = \langle 1, 0\rangle$			
$\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{w} = \langle 0, 1\rangle$			

a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions given by the unit vectors **u**, **v**, and **w**.

b. Interpret each of the directional derivatives computed in part (a) at the point (1, 0).

13–20. Computing gradients *Compute the gradient of the following functions and evaluate it at the given point P.*

13. $f(x, y) = 2 + 3x^2 - 5y^2$; $P(2, -1)$

14. $f(x, y) = 4x^2 - 2xy + y^2$; $P(-1, -5)$

15. $g(x, y) = x^2 - 4x^2 y - 8xy^2$; $P(-1, 2)$

16. $p(x, y) = \sqrt{12 - 4x^2 - y^2}$; $P(-1, -1)$

17. $f(x, y) = x e^{2xy}$; $P(1, 0)$

18. $f(x, y) = \sin(3x + 2y)$; $P\left(\pi, \frac{3\pi}{2}\right)$

19. $F(x, y) = e^{-x^2 - 2y^2}$; $P(-1, 2)$

20. $h(x, y) = \ln\left(1 + x^2 + 2y^2\right)$; $P(2, -3)$

21–30. Computing directional derivatives with the gradient *Compute the directional derivative of the following functions at the given point P in the direction of the given vector. Be sure to use a unit vector for the direction vector.*

21. $f(x, y) = x^2 - y^2$; $P(-1, -3)$; $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$

22. $f(x, y) = 3x^2 + y^3$; $P(3, 2)$; $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

23. $f(x, y) = 10 - 3x^2 + \frac{y^4}{4}$; $P(2, -3)$; $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

24. $g(x, y) = \sin \pi(2x - y)$; $P(-1, -1)$; $\left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$

25. $f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

26. $f(x, y) = 13e^{xy}$; $P(1, 0)$; $\langle 5, 12 \rangle$

27. $f(x, y) = 3x^2 + 2y + 5$; $P(1, 2)$; $\langle -3, 4 \rangle$

28. $h(x, y) = e^{-x-y}$; $P(\ln 2, \ln 3)$; $\langle 1, 1 \rangle$

29. $g(x, y) = \ln\left(4 + x^2 + y^2\right)$; $g(-1, 2)$; $\langle 2, 1 \rangle$

30. $f(x, y) = \frac{x}{x - y}$; $P(4, 1)$; $\langle -1, 2 \rangle$

31–36. Direction of steepest ascent and descent *Consider the following functions and points P.*

- ***a.*** *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*
- ***b.*** *Find a vector that points in a direction of no change in the function at P.*

31. $f(x, y) = x^2 - 4y^2 - 9$; $P(1, -2)$

32. $f(x, y) = x^2 + 4xy - y^2$; $P(2, 1)$

33. $f(x, y) = x^4 - x^2 y + y^2 + 6$; $P(-1, 1)$

34. $p(x, y) = \sqrt{20 + x^2 + 2xy - y^2}$; $P(1, 2)$

35. $F(x, y) = e^{-x^2/2 - y^2/2}$; $P(-1, 1)$

36. $f(x, y) = 2\sin(2x - 3y)$; $P(0, \pi)$

37–42. Interpreting directional derivatives *A function f and a point P are given. Let θ correspond to the direction of the directional derivative.*

- ***a.*** *Find the gradient and evaluate it at P.*

b. *Find the angles θ (with respect to the positive x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*

c. *Write the directional derivative at P as a function of θ; call this function g.*

d. *Find the value of θ that maximizes $g(\theta)$ and find the maximum value.*

e. *Verify that the value of θ that maximizes g corresponds to the direction of the gradient. Verify that the maximum value of g equals the magnitude of the gradient.*

37. $f(x, y) = 10 - 2x^2 - 3y^2$; $P(3, 2)$

38. $f(x, y) = 8 + x^2 + 3y^2$; $P(-3, -1)$

39. $f(x, y) = \sqrt{2 + x^2 + y^2}$; $P(\sqrt{3}, 1)$

40. $f(x, y) = \sqrt{12 - x^2 - y^2}$; $P\left(-1, -\frac{1}{\sqrt{3}}\right)$

41. $f(x, y) = e^{-x^2 - 2y^2}$; $P(-1, 0)$

T **42.** $f(x, y) = \ln(1 + 2x^2 + 3y^2)$; $P\left(\frac{3}{4}, -\sqrt{3}\right)$

43–46. Directions of change *Consider the following functions f and points P. Sketch the xy-plane showing P and the level curve through P. Indicate (as in Figure 15.52) the directions of maximum increase, maximum decrease, and no change for f.*

43. $f(x, y) = 8 + 4x^2 + 2y^2$; $P(2, -4)$

44. $f(x, y) = -4 + 6x^2 + 3y^2$; $P(-1, -2)$

T **45.** $f(x, y) = x^2 + xy + y^2 + 7$; $P(-3, 3)$

T **46.** $f(x, y) = \tan(2x + 2y)$; $P\left(\frac{\pi}{16}, \frac{\pi}{16}\right)$

47–50. Level curves *Consider the paraboloid* $f(x, y) = 16 - \frac{x^2}{4} - \frac{y^2}{16}$ *and the point P on the given level curve of f. Compute the slope of the line tangent to the level curve at P and verify that the tangent line is orthogonal to the gradient at that point.*

47. $f(x, y) = 0$; $P(0, 16)$

48. $f(x, y) = 0$; $P(8, 0)$

49. $f(x, y) = 12$; $P(4, 0)$

50. $f(x, y) = 12$; $P(2\sqrt{3}, 4)$

51–54. Level curves *Consider the upper half of the ellipsoid* $f(x, y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}$ *and the point P on the given level curve of f. Compute the slope of the line tangent to the level curve at P and verify that the tangent line is orthogonal to the gradient at that point.*

51. $f(x, y) = \frac{\sqrt{3}}{2}$; $P\left(\frac{1}{2}, \sqrt{3}\right)$

52. $f(x, y) = \frac{1}{\sqrt{2}}$; $P\left(0, \sqrt{8}\right)$

53. $f(x, y) = \frac{1}{\sqrt{2}}$; $P\left(\sqrt{2}, 0\right)$

54. $f(x, y) = \frac{1}{\sqrt{2}}$; $P(1, 2)$

55–58. Path of steepest descent *Consider each of the following surfaces and the point P on the surface.*

- ***a.*** *Find the gradient of f.*
- ***b.*** *Let C ' be the path of steepest descent on the surface beginning at P and let C be the projection of C ' on the xy-plane. Find an equation of C in the xy-plane.*
- ***c.*** *Find parametric equations for the path C ' on the surface.*

55. $f(x, y) = 4 + x$ (a plane); $P(4, 4, 8)$

56. $f(x, y) = y + x$ (a plane); $P(2, 2, 4)$

57. $f(x, y) = 4 - x^2 - 2y^2$ (a paraboloid); $P(1, 1, 1)$

58. $f(x, y) = y + x^{-1}$; $P(1, 2, 3)$

59–66. Gradients in three dimensions *Consider the following functions f, points P, and unit vectors* **u**.

- ***a.*** *Compute the gradient of f and evaluate it at P.*
- ***b.*** *Find the unit vector in the direction of maximum increase of f at P.*
- ***c.*** *Find the rate of change of the function in the direction of maximum increase at P.*
- ***d.*** *Find the directional derivative at P in the direction of the given vector.*

59. $f(x, y, z) = x^2 + 2y^2 + 4z^2 + 10$; $P(1, 0, 4)$; $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$

60. $f(x, y, z) = 4 - x^2 + 3y^2 + \frac{z^2}{2}$; $P(0, 2, -1)$; $\left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

61. $f(x, y, z) = 1 + 4xyz$; $P(1, -1, -1)$; $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

62. $f(x, y, z) = xy + yz + xz + 4$; $P(2, -2, 1)$; $\left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

63. $f(x, y, z) = 1 + \sin(x + 2y - z)$; $P\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)$; $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

64. $f(x, y, z) = e^{xyz-1}$; $P(0, 1, -1)$; $\left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

65. $f(x, y, z) = \ln\left(1 + x^2 + y^2 + z^2\right)$; $P(1, 1, -1)$; $\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

66. $f(x, y, z) = \frac{x - z}{y - z}$; $P(3, 2, -1)$; $\left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$

67. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

- **a.** If $f(x, y) = x^2 + y^2 - 10$, then $\nabla f(x, y) = 2x + 2y$.
- **b.** Because the gradient gives the direction of maximum increase of a function, the gradient is always positive.
- **c.** The gradient of $f(x, y, z) = 1 + xyz$ has four components.
- **d.** If $f(x, y, z) = 4$, then $\nabla f = \mathbf{0}$.

68. **Gradient of a composite function** Consider the function $F(x, y, z) = e^{xyz}$.

- **a.** Write F as a composite function $f \circ g$, where f is a function of one variable and g is a function of three variables.
- **b.** Relate ∇F to ∇g.

69–72. Directions of zero change *Find the directions in the xy-plane in which the following functions have zero change at the given point. Express the directions in terms of unit vectors.*

69. $f(x, y) = 12 - 4x^2 - y^2$; $P(1, 2, 4)$

70. $f(x, y) = x^2 - 4y^2 - 8$; $P(4, 1, 4)$

71. $f(x, y) = \sqrt{3 + 2x^2 + y^2}$; $P(1, -2, 3)$

72. $f(x, y) = e^{1-xy}$; $P(1, 0, e)$

73. **Steepest ascent on a plane** Suppose a long sloping hillside is described by the plane $z = ax + by + c$, where a, b, and c are constants. Find the path in the xy-plane, beginning at (x_0, y_0), that corresponds to the path of steepest ascent on the hillside.

74. **Gradient of a distance function** Let (a, b) be a given point in $\mathbb{R}^2$ and let $d = f(x, y)$ be the distance between (a, b) and the variable point (x, y).

- **a.** Show that the graph of f is a cone.
- **b.** Show that the gradient of f at any point other than (a, b) is a unit vector.
- **c.** Interpret the direction and magnitude of ∇f.

75–78. Looking ahead—tangent planes *Consider the following surfaces* $f(x, y, z) = 0$, *which may be regarded as a level surface of the function* $w = f(x, y, z)$. *A point* $P(a, b, c)$ *on the surface is also given.*

- ***a.*** *Find the (three-dimensional) gradient of* f *and evaluate it at* P.
- ***b.*** *The set of all vectors orthogonal to the gradient with their tails at* P *form a plane. Find an equation of that plane (soon to be called the tangent plane).*

75. $f(x, y, z) = x^2 + y^2 + z^2 - 3 = 0$; $P(1, 1, 1)$

76. $f(x, y, z) = 8 - xyz = 0$; $P(2, 2, 2)$

77. $f(x, y, z) = e^{x+y-z} - 1 = 0$; $P(1, 1, 2)$

78. $f(x, y, z) = xy + xz - yz - 1 = 0$; $P(1, 1, 1)$

T **79.** **A traveling wave** A snapshot (frozen in time) of a set of water waves is described by the function $z = 1 + \sin(x - y)$, where z gives the height of the waves and (x, y) are coordinates in the horizontal plane $z = 0$.

- **a.** Use a graphing utility to graph $z = 1 + \sin(x - y)$.
- **b.** The crests and the troughs of the waves are aligned in the direction in which the height function has zero change. Find the direction in which the crests and troughs are aligned.
- **c.** If you were surfing on one of these waves and wanted the steepest descent from the crest to the trough, in which direction would you point your surfboard (given in terms of a unit vector in the xy-plane)?
- **d.** Check that your answers to parts (b) and (c) are consistent with the graph of part (a).

80. **Traveling waves in general** Generalize Exercise 79 by considering a set of waves described by the function $z = A + \sin(ax - by)$, where a, b, and A are real numbers.

- **a.** Find the direction in which the crests and troughs of the waves are aligned. Express your answer as a unit vector in terms of a and b.
- **b.** Find the surfer's direction—that is, the direction of steepest descent from a crest to a trough. Express your answer as a unit vector in terms of a and b.

Explorations and Challenges »

81–83. Potential functions *Potential functions arise frequently in physics and engineering. A potential function has the property that a field of interest (for example, an electric field, a gravitational field, or a velocity field) is the gradient of the potential (or sometimes the negative of the gradient of the potential). (Potential functions are considered in depth in Chapter 17.)*

81. **Electric potential due to a point charge** The electric field due to a point charge of strength Q at the origin has a potential function $\phi = \frac{kQ}{r}$, where $r^2 = x^2 + y^2 + z^2$ is the square of the distance between a variable point $P(x, y, z)$ and the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E} = -\nabla\phi$, where $\nabla\phi$ is the gradient in three dimensions.

- **a.** Show that the three-dimensional electric field due to a point charge is given by

$$\mathbf{E}(x, y, z) = kQ\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle.$$

b. Show that the electric field at a point has a magnitude $|\mathbf{E}| = \frac{kQ}{r^2}$. Explain why this relationship is called an inverse square law.

82. Gravitational potential The gravitational potential associated with two objects of mass M and m is $\phi = -\frac{GMm}{r}$, where G is the gravitational constant. If one of the objects is at the origin and the other object is at $P(x, y, z)$, then $r^2 = x^2 + y^2 + z^2$ is the square of the distance between the objects. The gravitational field at P is given by $\mathbf{F} = -\nabla\phi$, where $\nabla\phi$ is the gradient in three dimensions. Show that the force has a magnitude $|\mathbf{F}| = \frac{GMm}{r^2}$. Explain why this relationship is called an inverse square law.

83. Velocity potential In two dimensions, the motion of an ideal fluid (an incompressible and irrotational fluid) is governed by a velocity potential ϕ. The velocity components of the fluid, u in the x-direction and v in the y-direction, are given by $\langle u, v\rangle = \nabla\phi$. Find the velocity components associated with the velocity potential $\phi(x, y) = \sin \pi x \sin 2\pi y$.

84. Gradients for planes Prove that for the plane described by $f(x, y) = Ax + By$, where A and B are nonzero constants, the gradient is constant (independent of (x, y)). Interpret this result.

85. Rules for gradients Use the definition of the gradient (in two or three dimensions), assume f and g are differentiable functions on $\mathbb{R}^2$ or $\mathbb{R}^3$, and let c be a constant. Prove the following gradient rules.

a. Constants Rule: $\nabla(cf) = c\nabla f$

b. Sum Rule: $\nabla(f + g) = \nabla f + \nabla g$

c. Product Rule: $\nabla(fg) = (\nabla f)g + f\nabla g$

d. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

e. Chain Rule: $\nabla(f \circ g) = f'(g)\nabla g$, where f is a function of one variable

86–91. Using gradient rules *Use the gradient rules of Exercise 85 to find the gradient of the following functions.*

86. $f(x, y) = xy\cos(xy)$

87. $f(x, y) = \frac{x + y}{x^2 + y^2}$

88. $f(x, y) = \ln\left(1 + x^2 + y^2\right)$

89. $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$

90. $f(x, y, z) = (x + y + z)e^{xyz}$

91. $f(x, y, z) = \frac{x + yz}{y + xz}$