15.4 The Chain Rule

In this section, we combine ideas based on the Chain Rule (Section 3.7) with what we know about partial derivatives (Section 15.3) to develop new methods for finding derivatives of functions of several variables. To illustrate the importance of these methods, consider the following situation.

Economists modeling manufacturing systems often works with *production functions* that relate the productivity (output) of the system to all the variables on which it depends (input). A simplified production function might take the form $P = F(L, K, R)$, where L, K, and R represent the availability of labor, capital, and natural resources, respectively. However, the variables L, K, and R may be intermediate variables that depend on other variables. For example, it might be that L is a function of the unemployment rate u, K is a function of the prime interest rate i, and R is a function of time t (seasonal availability of resources). Even in this simplified model we see that productivity, which is the dependent variable, is ultimately related to many other variables (**Figure 15.35**). Of critical interest to an economist is how changes in one variable determine changes in other variables. For instance, if the unemployment rate increases by 0.1% and the interest rate decreases by 0.2%, what is the effect on productivity? In this section we develop the tools needed to answer such questions.

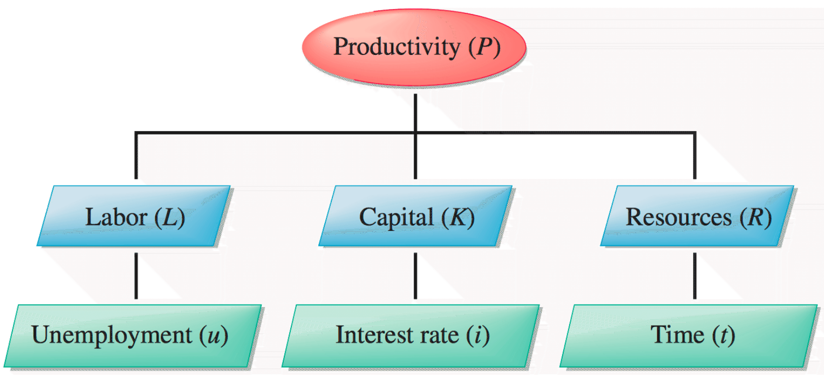


Figure 15.35

The Chain Rule with One Independent Variable »

Recall the basic Chain Rule: If y is a function of u and u is a function of t, then $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$. We first extend the Chain Rule to composite functions of the form $z = f(x, y)$, where x and y are functions of t. What is $\frac{dz}{dt}$?

We illustrate the relationships among the variables t, x, y, and z using a *tree diagram* (**Figure 15.36**). To find $\frac{dz}{dt}$, first notice that z depends on x, which in turn depends on t. The change in z with respect to x is $\frac{\partial z}{\partial x}$, while the change in x with respect to t is the ordinary derivative $\frac{dx}{dt}$. These derivatives appear on the correspond ing branches of the tree diagram. Using the Chain Rule idea, the product of these derivatives gives the change in z with respect to t through x.

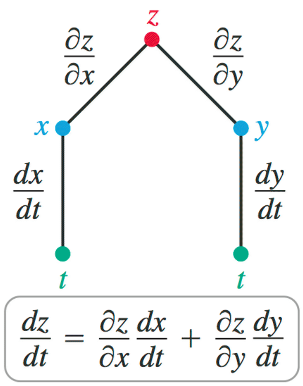


Figure 15.36

Similarly, z also depends on y. The change in z with respect to y is $\frac{\partial z}{\partial y}$, while the change in y with respect to t is $\frac{dy}{dt}$. The product of these derivatives, which appear on the corresponding branches of the tree, gives the change in z with respect to t through y. Summing the contributions to $\frac{dz}{dt}$ along each branch of the tree leads to the following theorem, whose proof is found in Appendix A.

THEOREM 15.7 Chain Rule (One Independent Variable)

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Note »

A subtle observation about notation should be made. If $z = f(x, y)$, where x and y are functions of another variable t, it is common to write $z = f(t)$ to show that z ultimately depends on t. However, these two functions denoted f are actually different. To be careful, we should write (or at least remember) that in fact $z = F(t)$, where F is a function other than f. This distinction is often overlooked for the sake of convenience.

Quick Check 1 Explain why Theorem 15.7 reduces to the Chain Rule for a function of one variable in the case that $z = f(x)$ and $x = g(t)$. ◆

Answer »

If $z = f(x(t))$, then $\frac{\partial z}{\partial y} = 0$, and the original Chain Rule results.

Before presenting examples, several comments are in order.

- With $z = f(x(t), y(t))$, the dependent variable is z and the sole independent variable is t. The variables x and y are **intermediate variables**.
- The choice of notation for partial and ordinary derivatives in the Chain Rule is important. We write ordinary derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ because x and y depend only on t. We write partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ because z

is a function of both x and y. Finally, we write $\frac{dz}{dt}$ as an ordinary derivative because z ultimately depends only on t.

- Theorem 15.7 generalizes directly to functions of more than two intermediate variables (**Figure 15.37**). For example, if $w = f(x, y, z)$, where x, y, and z are functions of the single independent variable t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

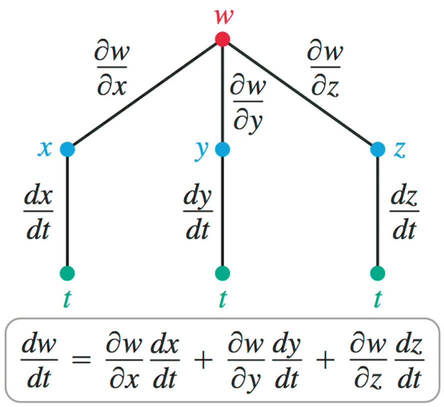


Figure 15.37

EXAMPLE 1 Chain Rule with one independent variable

Let $z = x^2 - 3y^2 + 20$, where $x = 2\cos t$ and $y = 2\sin t$.

a. Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{4}$.

b. Interpret the result geometrically.

Note »

> If f, x, and y are simple, as in Example 1, it is possible to substitute $x(t)$ and $y(t)$ into f, producing a function of t only, and then differentiate with respect to t. But this approach quickly becomes impractical with more complicated functions and the Chain Rule offers a great advantage.

SOLUTION »

a. Computing the intermediate derivatives and applying the Chain Rule (Theorem 15.7), we find that

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \underbrace{(2x)}_{\frac{\partial z}{\partial x}}\underbrace{(-2\sin t)}_{\frac{dx}{dt}} + \underbrace{(-6y)}_{\frac{\partial z}{\partial y}}\underbrace{(2\cos t)}_{\frac{dy}{dt}} && \text{Evaluate derivatives.} \\
&= -4x\sin t - 12y\cos t && \text{Simplify.} \\
&= -8\cos t\sin t - 24\sin t\cos t && \text{Substitute } x = 2\cos t,\ y = 2\sin t. \\
&= -16\sin 2t. && \text{Simplify; } \sin 2t = 2\sin t\cos t.
\end{aligned}$$

Substituting $t = \frac{\pi}{4}$ gives $\left.\frac{dz}{dt}\right|_{t=\pi/4} = -16$.

b. The parametric equations $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$, describe a circle C of radius 2 in the xy-plane. Imagine walking on the surface $z = x^2 - 3y^2 + 20$ directly above the circle C consistent with the positive (counterclockwise) orientation of C. Your path rises and falls as you walk (**Figure 15.38**); the rate of change of your elevation z with respect to t is given by $\frac{dz}{dt}$. For example, when $t = \frac{\pi}{4}$, the corresponding point on the surface is $\left(\sqrt{2}, \sqrt{2}, 16\right)$. At that point, z decreases at a rate of -16 (by part (a)) as you walk on the surface above C.

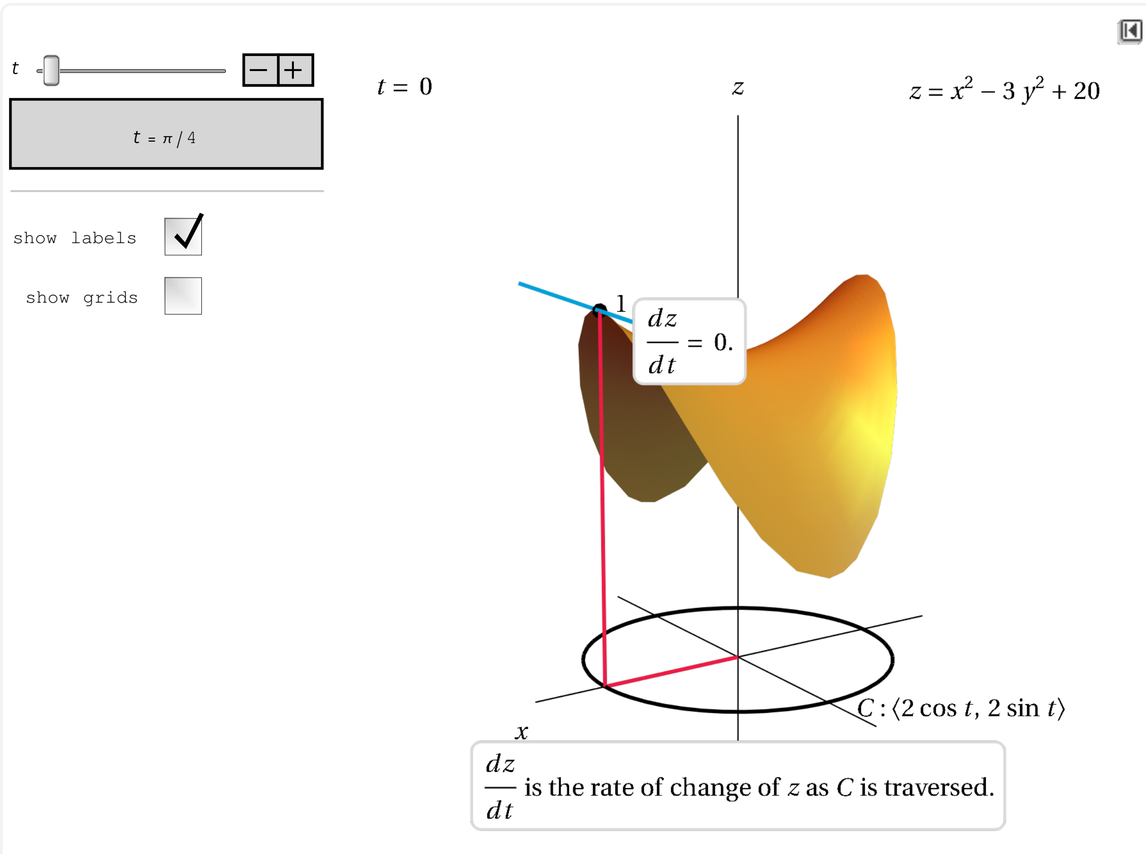
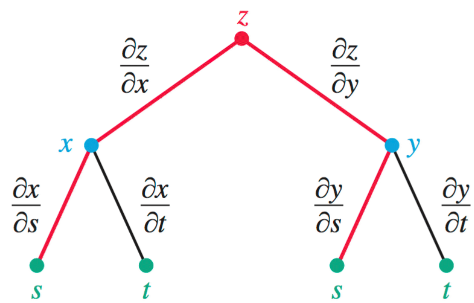


Figure 15.38

Related Exercises 10, 12 ◆

The Chain Rule with Several Independent Variables »

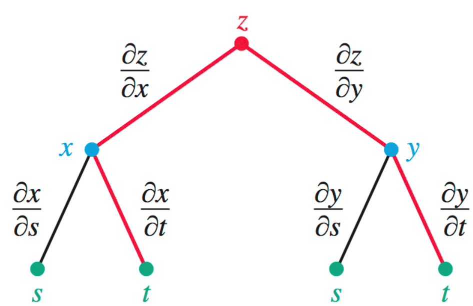
The ideas behind the Chain Rule of Theorem 15.7 can be modified to cover a variety of situations in which functions of several variables are composed with one another. For example, suppose z depends on two intermediate variables x and y, each of which depends on the independent variables s and t. Once again, a tree diagram (**Figure 15.39**) helps us organize the relationships among variables. The dependent variable z now ultimately depends on the two independent variables s and t, so it makes sense to ask about the rates of change of z with respect to either s or t, which are $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, respectively.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Figure 15.39

To compute $\frac{\partial z}{\partial s}$, we note that there are two paths in the tree (in red in Figure 15.39) that connect z to s and contribute to $\frac{\partial z}{\partial s}$. Along one path, z changes with respect to x (with rate of change $\frac{\partial z}{\partial x}$) and x changes with respect to s (with rate of change $\frac{\partial x}{\partial s}$). Along the other path, z changes with respect to y (with rate of change $\frac{\partial z}{\partial y}$) and y changes with respect to s (with rate of change $\frac{\partial y}{\partial s}$). We use a Chain Rule calculation along each path and combine the results. A similar argument leads to $\frac{\partial z}{\partial t}$ (**Figure 15.40**).



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Figure 15.40

THEOREM 15.8 **Chain Rule (Two Independent Variables)**

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Quick Check 2 Suppose $w = f(x, y, z)$, where $x = g(s, t)$, $y = h(s, t)$, and $z = p(s, t)$. Extend Theorem 15.8 to write a formula for $\frac{\partial w}{\partial t}$. ◆

Answer »

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

EXAMPLE 2 Chain Rule with two independent variables

Let $z = \sin 2x \cos 3y$, where $x = s + t$ and $y = s - t$. Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

SOLUTION »

The tree diagram in Figure 15.39 gives the Chain Rule formula for $\frac{\partial z}{\partial s}$: We form products of the derivatives along the branches connecting z to s and add the results. The partial derivative is

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= \underbrace{2\cos 2x \cos 3y}_{\frac{\partial z}{\partial x}} \cdot \underbrace{1}_{\frac{\partial x}{\partial s}} + \underbrace{(-3\sin 2x \sin 3y)}_{\frac{\partial z}{\partial y}} \cdot \underbrace{1}_{\frac{\partial y}{\partial s}}\\ &= 2\cos(2\underbrace{(s+t)}_{x})\cos(3\underbrace{(s-t)}_{y}) - 3\sin(2\underbrace{(s+t)}_{x})\sin(3\underbrace{(s-t)}_{y}).\end{aligned}$$

Following the branches of Figure 15.40 connecting z to t, we have

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ &= \underbrace{2\cos 2x \cos 3y}_{\frac{\partial z}{\partial x}} \cdot \underbrace{1}_{\frac{\partial x}{\partial t}} + \underbrace{(-3\sin 2x \sin 3y)}_{\frac{\partial z}{\partial y}} \cdot \underbrace{-1}_{\frac{\partial y}{\partial t}}\\ &= 2\cos(2\underbrace{(s+t)}_{x})\cos(3\underbrace{(s-t)}_{y}) + 3\sin(2\underbrace{(s+t)}_{x})\sin(3\underbrace{(s-t)}_{y}).\end{aligned}$$

Related Exercise 22 ◆

EXAMPLE 3 More variables

Let w be a function of x, y, and z, each of which is a function of s and t.

a. Draw a labeled tree diagram showing the relationships among the variables.

b. Write the Chain Rule formula for $\frac{\partial w}{\partial s}$.

SOLUTION »

a. Because w is a function of x, y, and z, the upper branches of the tree (**Figure 15.41**) are labeled with the partial derivatives w_x, w_y, and w_z. Each of x, y, and z is a function of two variables, so the lower branches of the tree also require partial derivative labels.

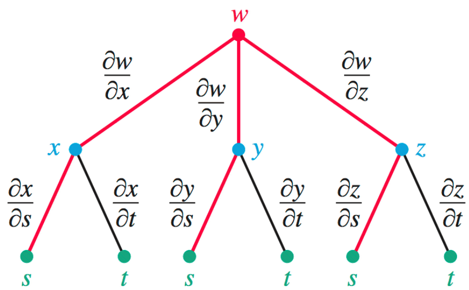


Figure 15.41

b. Extending Theorem 15.8, we take the three paths through the tree that connect w to s (red branches in Figure 15.41). Multiplying the derivatives that appear on each path and adding gives the result

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Related Exercises 25–26 ◆

Quick Check 3 If Q is a function of w, x, y, and z, each of which is a function of r, s, and t, how many dependent variables, intermediate variables, and independent variables are there? ◆

Answer »

One dependent variable, four intermediate variables, and three independent variables.

It is probably clear by now that we can create a Chain Rule for any set of relationships among variables. The key is to draw an accurate tree diagram and label the branches of the tree with the appropriate derivatives.

EXAMPLE 4 A different kind of tree

Let w be a function of z, where z is a function of x and y, and each of x and y is a function of t. Draw a labeled tree diagram and write the Chain Rule formula for $\frac{dw}{dt}$.

SOLUTION »

The dependent variable w is related to the independent variable t through two paths in the tree: $w \to z \to x \to t$ and $w \to z \to y \to t$ (**Figure 15.42**).

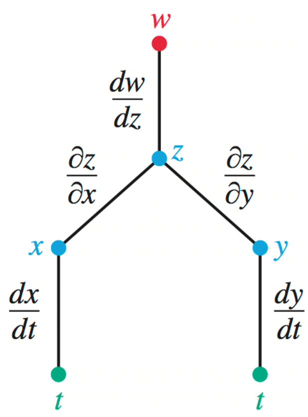


Figure 15.42

At the top of the tree, w is a function of the single variable z, so the rate of change is the ordinary derivative $\frac{dw}{dz}$. The tree below z looks like Figure 15.36. Multiplying the derivatives on each of the two branches connecting w to t, and adding the results, we have

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{dw}{dz}\frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right).$$

Related Exercise 31 ◆

Implicit Differentiation »

Using the Chain Rule for partial derivatives, the technique of implicit differentiation can be put in a larger perspective. Recall that if x and y are related through an implicit relationship, such as $\sin xy + \pi y^2 = x$, then $\frac{dy}{dx}$ is computed using implicit differentiation (Section 3.8). Another way to compute $\frac{dy}{dx}$ is to define the function $F(x, y) = \sin xy + \pi y^2 - x$. Notice that the original equation $\sin xy + \pi y^2 = x$ is $F(x, y) = 0$.

To find $\frac{dy}{dx}$, we treat x as the independent variable and differentiate both sides of $F(x, y(x)) = 0$ with respect to x. The derivative of the right side is 0. On the left side, we use the Chain Rule of Theorem 15.7:

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Noting that $\frac{dx}{dx} = 1$ and solving for $\frac{dy}{dx}$, we obtain the following theorem.

THEOREM 15.9 **Implicit Differentiation**

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Note »

The question of whether a relationship of the form $F(x, y) = 0$ or $F(x, y, z) = 0$ determines one or more functions is addressed by a theorem of advanced calculus called the Implicit Function Theorem.

EXAMPLE 5 Implicit differentiation

Find $\frac{dy}{dx}$ when $F(x, y) = \sin xy + \pi y^2 - x = 0$.

SOLUTION »

Computing the partial derivatives of F with respect to x and y, we find that

$$F_x = y\cos xy - 1 \quad \text{and} \quad F_y = x\cos xy + 2\pi y.$$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y\cos xy - 1}{x\cos xy + 2\pi y}.$$

As with many implicit differentiation calculations, the result is left in terms of both x and y. The same result is obtained using the methods of Section 3.8.

Note »

The method of Theorem 15.9 generalizes to computing $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with functions of the form $F(x, y, z) = 0$ (Exercise 56).

Related Exercise 37 ◆

Quick Check 4 Use the method of Example 5 to find $\frac{dy}{dx}$ when $F(x, y) = x^2 + xy - y^3 - 7 = 0$. Compare your solution to Example 3 in Section 3.8. Which method is easier? ◆

Answer »

$\frac{dy}{dx} = \frac{2x + y}{3y^2 - x}$; in this case, using $\frac{dy}{dx} = -\frac{F_x}{F_y}$ is more efficient.

EXAMPLE 6 Fluid flow

A basin of circulating water is represented by the square region $\{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$, where x is positive in the eastward direction and y is positive in the northward direction. The velocity components of the water are

$$\text{the east-west velocity } u(x, y) = 2\sin \pi x \cos \pi y \text{ and}$$
$$\text{the north-south velocity } v(x, y) = -2\cos \pi x \sin \pi y;$$

these velocity components produce the flow pattern shown in **Figure 15.43**. The *streamlines* shown in the figure are the paths followed by small parcels of water. The speed of the water at a point (x, y) is given by the function $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial s}{\partial y}$, the rates of change of the water speed in the x- and y-directions, respectively.

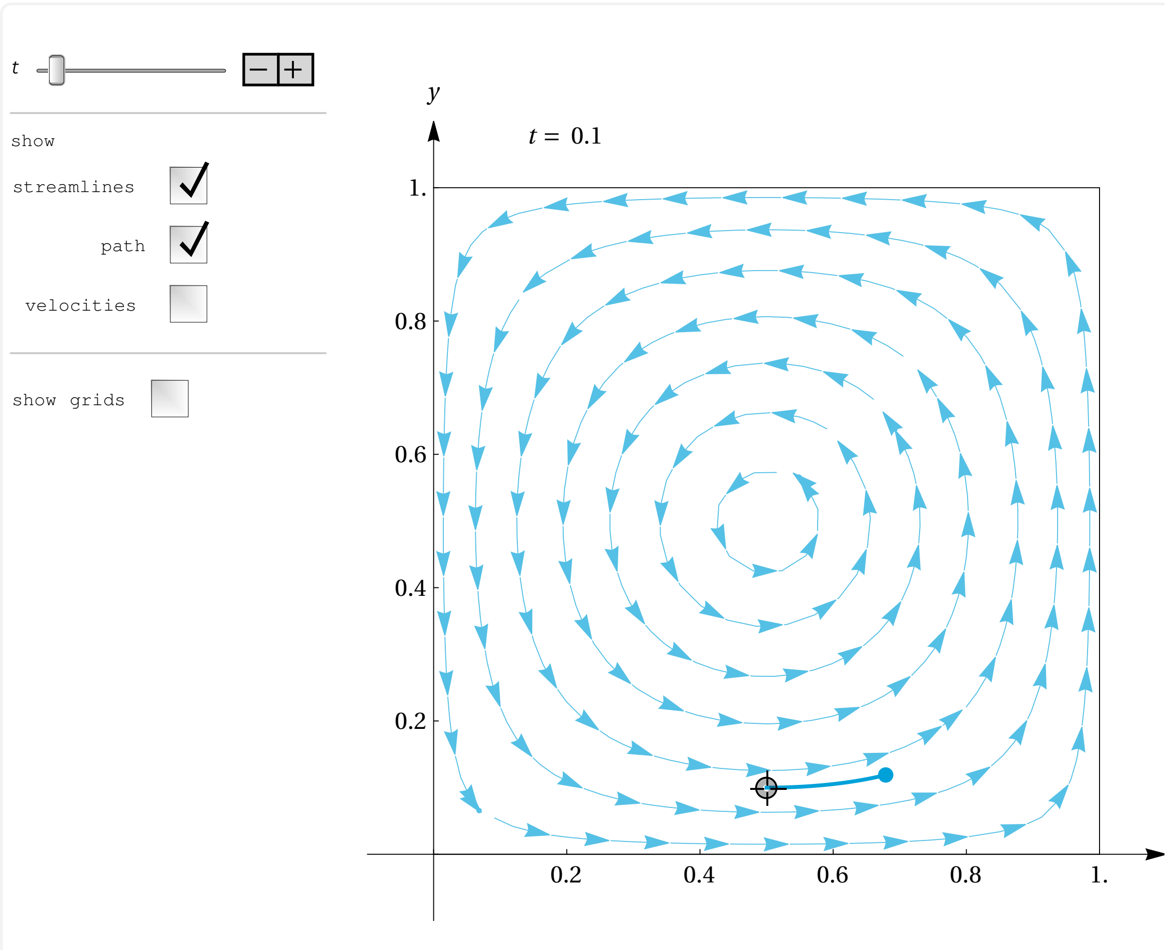


Figure 15.43

SOLUTION »

The dependent variable s depends on the independent variables x and y through the intermediate variables u and v (**Figure 15.44**).

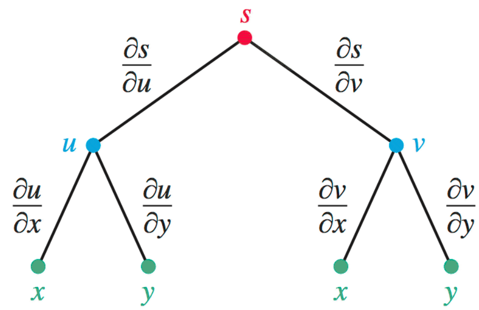


Figure 15.44

Theorem 15.8 applies here in the form

$$\frac{\partial s}{\partial x}=\frac{\partial s}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial x}\quad\text{and}\quad\frac{\partial s}{\partial y}=\frac{\partial s}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial y}.$$

The derivatives $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are easier to find if we square the speed function to obtain $s^2=u^2+v^2$ and then use implicit differentiation. To compute $\frac{\partial s}{\partial u}$, we differentiate both sides of $s^2=u^2+v^2$ with respect to u:

$$2\,s\frac{\partial s}{\partial u}=2\,u,\quad\text{which implies that }\frac{\partial s}{\partial u}=\frac{u}{s}.$$

Similarly, differentiating $s^2=u^2+v^2$ with respect to v gives

$$2\,s\frac{\partial s}{\partial v}=2\,v,\quad\text{which implies that }\frac{\partial s}{\partial v}=\frac{v}{s}.$$

Now the Chain Rule leads to $\frac{\partial s}{\partial x}$:

$$\begin{aligned}\frac{\partial s}{\partial x}&=\frac{\partial s}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial x}\\&=\underbrace{\frac{u}{s}}_{\frac{\partial s}{\partial u}}\underbrace{(2\,\pi\cos\pi x\cos\pi y)}_{\frac{\partial u}{\partial x}}+\underbrace{\frac{v}{2}}_{\frac{\partial s}{\partial v}}\underbrace{(2\,\pi\sin\pi x\sin\pi y)}_{\frac{\partial v}{\partial x}}\\&=\frac{2\,\pi}{s}(u\cos\pi x\cos\pi y+v\sin\pi x\sin\pi y).\end{aligned}$$

A similar calculation shows that

$$\frac{\partial s}{\partial y}=-\frac{2\,\pi}{s}(u\sin\pi x\sin\pi y+v\cos\pi x\cos\pi y).$$

As a final step, you could replace s, u and v, by their definitions in terms of x and y.

Related Exercises 41–42 ◆

EXAMPLE 7 Second derivatives

Let $z = f(x, y) = \frac{x}{y}$, where $x = s + t^2$ and $y = s^2 - t$. Compute $\frac{\partial^2 z}{\partial s^2} = z_{ss}$, $\frac{\partial^2 z}{\partial t\, \partial s} = z_{st}$, and $\frac{\partial^2 z}{\partial t^2} = z_{tt}$, and express the results in terms of s and t. We use subscripts for partial derivatives in this example to simplify the notation.

SOLUTION »

First, we need some ground rules. In this example, it is possible to express f in terms of s and t by substituting, after which the result could be differentiated directly to find the required partial derivatives. Unfortunately, this maneuver is not always possible in practice (see Exercises 72 and 73). Therefore, to make this example as useful as possible, we develop general formulas for the second partial derivatives and make substitutions only in the last step.

Figures 15.39 and 15.40 show the relationships among the variables, and Example 2 demonstrates the calculation of the first partial derivatives. Throughout these calculations, it is important to remember the meaning of differentiation with respect to s and t:

$$(\ )_s = (\ )_x\, x_s + (\ )_y\, y_s \text{ and } (\ )_t = (\ )_x\, x_t + (\ )_y\, y_t.$$

Let's compute the first partial derivatives:

$$z_s = z_x\, x_s + z_y\, y_s \text{ and } z_t = z_x\, x_t + z_y\, y_t.$$

Differentiating z_s with respect to s, we have

$$\begin{aligned}
z_{ss} &= \left(z_x\, x_s + z_y\, y_s\right)_s \\
&= (z_x)_s\, x_s + z_x\, x_{ss} + \left(z_y\right)_s y_s + z_y\, y_{ss} && \text{Product Rule (twice)} \\
&= \underbrace{(z_{xx}\, x_s + z_{xy}\, y_s)}_{(z_x)_s} x_s + z_x\, x_{ss} + \underbrace{(z_{yx}\, x_s + z_{yy}\, y_s)}_{(z_y)_s} y_s + z_y\, y_{ss} && \text{Differentiate } z_x \text{ and } z_y \text{ with respect to } s. \\
&= z_{xx}\, {x_s}^2 + 2\, z_{xy}\, x_s\, y_s + z_{yy}\, {y_s}^2 + z_x\, x_{ss} + z_y\, y_{ss}. && \text{Simplify with } z_{xy} = z_{yx}.
\end{aligned}$$

At this point, we substitute

$$z_x = \frac{1}{y},\ z_y = -\frac{x}{y^2},\ z_{xx} = 0,\ z_{xy} = -\frac{1}{y^2},\ z_{yy} = \frac{2\, x}{y^3},\ x_s = 1,\ x_{ss} = 0,\ y_s = 2\, s, \text{ and } y_{ss} = 2,$$

and simplify to find that

$$z_{ss} = \frac{2\left(s^3 + 3\, s\, t + 3\, s^2\, t^2 + t^3\right)}{\left(s^2 - t\right)^3}.$$

Differentiating z_s with respect to t, a similar procedure produces z_{st}:

$$\begin{aligned}
z_{st} &= \left(z_x\, x_s + z_y\, y_s\right)_t \\
&= (z_x)_t\, x_s + z_x\, x_{st} + \left(z_y\right)_t y_s + z_y\, y_{st} && \text{Product Rule (twice)} \\
&= \underbrace{(z_{xx}\, x_t + z_{xy}\, y_t)}_{(z_x)_t} x_s + z_x\, x_{st} + \underbrace{(z_{yx}\, x_t + z_{yy}\, y_t)}_{(z_y)_t} y_s + z_y\, y_{st} && \text{Differentiate } z_x \text{ and } z_y \text{ with respect to } t. \\
&= z_{xx}\, x_s\, x_t + z_{xy}\, x_s\, y_t + z_{xy}\, x_t\, y_s + z_{yy}\, y_s\, y_t + z_x\, x_{st} + z_y\, y_{st}. && \text{Simplify with } z_{xy} = z_{yx}.
\end{aligned}$$

Substituting in terms of s and t with $x_{st} = 0$ and $y_{st} = 0$, we have

$$z_{st} = -\frac{3\,s^2 + t + 4\,s^3\,t}{\left(s^2 - t\right)^3}.$$

An analogous calculation gives

$$z_{tt} = \frac{2\,s\left(1 + s^3\right)}{\left(s^2 - t\right)^3}.$$

Related Exercise 45 ◆

Exercises »

Getting Started »

Practice Exercises »

9–18. Chain Rule with one independent variable *Use Theorem 15.7 to find the following derivatives.*

9. $\dfrac{dz}{dt}$, where $z = x \sin y$, $x = t^2$, and $y = 4\,t^3$

10. $\dfrac{dz}{dt}$, where $z = x^2\,y - x\,y^3$, $x = t^2$, and $y = t^{-2}$

11. $\dfrac{dw}{dt}$, where $w = \cos 2\,x \sin 3\,y$, $x = \dfrac{t}{2}$, and $y = t^4$

12. $\dfrac{dz}{dt}$, where $z = \sqrt{r^2 + s^2}$, $r = \cos 2\,t$, and $s = \sin 2\,t$

13. $\dfrac{dz}{dt}$, where $z = (x + 2\,y)^{10}$, $x = \sin^2 t$, and $y = (3\,t + 4)^5$

14. $\dfrac{dz}{dt}$, where $z = \dfrac{x^{20}}{y^{10}}$, $x = \tan^{-1} t$, and $y = \ln\left(t^2 + 1\right)$

15. $\dfrac{dw}{dt}$, where $w = x\,y \sin z$, $x = t^2$, $y = 4\,t^3$, and $z = t + 1$

16. $\dfrac{dQ}{dt}$, where $Q = \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, and $z = \cos t$

17. $\dfrac{dV}{dt}$, where $V = x\,y\,z$, $x = e^t$, $y = 2\,t + 3$, and $z = \sin t$

18. $\dfrac{dU}{dt}$, where $U = \dfrac{x\,y^2}{z^8}$, $x = e^t$, $y = \sin 3\,t$, and $z = 4\,t + 1$

19–26. Chain Rule with several independent variables *Find the following derivatives.*

19. z_s and z_t, where $z = x^2 \sin y$, $x = s - t$, and $y = t^2$

20. z_s and z_t, where $z = \sin(2x + y)$, $x = s^2 - t^2$, and $y = s^2 + t^2$

21. z_s and z_t, where $z = xy - x^2 y$, $x = s + t$, and $y = s - t$

22. z_s and z_t, where $z = \sin x \cos 2y$, $x = s + t$, and $y = s - t$

23. z_s and z_t, where $z = e^{x+y}$, $x = st$, and $y = s + t$

24. z_s and z_t, where $z = \sin xy$, $x = s^2 t$, and $y = (s + t)^{10}$

25. w_s and w_t, where $w = \dfrac{x - z}{y + z}$, $x = s + t$, $y = st$, and $z = s - t$

26. w_r, w_s, and w_t, where $w = \sqrt{x^2 + y^2 + z^2}$, $x = st$, $y = rs$, and $z = rt$

27. **Changing cylinder** The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$.

a. Assume r and h are functions of t. Find $V'(t)$.

b. Suppose $r = e^t$ and $h = e^{-2t}$, for $t \geq 0$. Use part (a) to find $V'(t)$.

c. Does the volume of the cylinder in part (b) increase or decrease as t increases?

28. **Changing pyramid** The volume of a pyramid with a square base x units on a side and a height of h is $V = \dfrac{1}{3} x^2 h$.

a. Assume x and h are functions of t. Find $V'(t)$.

b. Suppose $x = \dfrac{t}{t+1}$ and $h = \dfrac{1}{t+1}$, for $t \geq 0$. Use part (a) to find $V'(t)$.

c. Does the volume of the pyramid in part (b) increase or decrease as t increases?

29–30. Derivative practice two ways *Find the indicated derivative in two ways:*

a. *Replace x and y to write z as a function of t and differentiate.*

b. *Use the Chain Rule.*

29. $z'(t)$, where $z = \dfrac{1}{x} + \dfrac{1}{y}$, $x = t^2 + 2t$, and $y = t^3 - 2$

30 $z'(t)$, where $z = \ln(x + y)$, $x = te^t$, and $y = e^t$

31–34. Making trees *Use a tree diagram to write the required Chain Rule formula.*

31. w is a function of z, where z is a function of p, q, and r, each of which is a function of t. Find $\dfrac{dw}{dt}$.

32. $w = f(x, y, z)$, where $x = g(t)$, $y = h(s, t)$, $z = p(r, s, t)$. Find $\dfrac{\partial w}{\partial t}$.

33. $u = f(v)$, where $v = g(w, x, y)$, $w = h(z)$, $x = p(t, z)$, $y = q(t, z)$. Find $\dfrac{\partial u}{\partial z}$.

34. $u = f(v, w, x)$, where $v = g(r, s, t)$, $w = h(r, s, t)$, $x = p(r, s, t)$, $r = F(z)$. Find $\dfrac{\partial u}{\partial z}$.

35–40. Implicit differentiation *Use Theorem 15.9 to evaluate* $\dfrac{dy}{dx}$*. Assume each equation implicitly defines y as a differentiable function of x.*

35. $x^2 - 2y^2 - 1 = 0$

36. $x^3 + 3xy^2 - y^5 = 0$

37. $2 \sin xy = 1$

38. $y e^{xy} - 2 = 0$

39. $\sqrt{x^2 + 2xy + y^4} = 3$

40. $y \ln\left(x^2 + y^2 + 4\right) = 3$

41–42. Fluid flow *The x- and y-components of a fluid moving in two dimensions are given by the following functions u and v. The speed of the fluid at* (x, y) *is* $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$*. Use the Chain Rule to find* $\dfrac{\partial s}{\partial x}$ *and* $\dfrac{\partial s}{\partial y}$*.*

41. $u(x, y) = 2y$ and $v(x, y) = -2x$; $x \geq 0$ and $y \geq 0$

42. $u(x, y) = x(1 - x)(1 - 2y)$ and $v(x, y) = y(y - 1)(1 - 2x)$; $0 \leq x \leq 1$ and $0 \leq y \leq 1$

43–48. Second derivatives *For the following sets of variables, find all the relevant second derivatives. In all cases, first find general expressions for the second derivatives and then substitute variables at the last step.*

43. $f(x, y) = x^2 y$, where $x = s + t$ and $y = s - t$

44. $f(x, y) = x^2 y - xy^2$, where $x = st$ and $y = \dfrac{s}{t}$

45. $f(x, y) = \dfrac{y}{x}$, where $x = s^2 + t^2$ and $y = s^2 - t^2$

46. $f(x, y) = e^{x-y}$, where $x = s^2$ and $y = 3t^2$

47. $f(x, y, z) = xy + xz - yz$, where $x = s^2 - 2s$, $y = \dfrac{2}{s^2}$, and $z = 3s^2 - 2$

48. $f(x, y) = xy$, where $x = s + 2t - u$ and $y = s + 2t + u$

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume all partial derivatives exist.

a. If $z = (x + y) \sin xy$, where x and y are functions of s, then $\dfrac{\partial z}{\partial s} = \dfrac{dz}{dx}\dfrac{dx}{ds}$.

b. Given that $w = f(x(s, t), y(s, t), z(s, t))$, the rate of change of w with respect to t is $\frac{dw}{dt}$.

50–54. Derivative practice *Find the indicated derivative for the following functions.*

50. $\frac{\partial z}{\partial p}$, where $z = \frac{x}{y}$, $x = p + q$, and $y = p - q$

51. $\frac{dw}{dt}$, where $w = xyz$, $x = 2t^4$, $y = 3t^{-1}$, and $z = 4t^{-3}$

52. $\frac{\partial w}{\partial x}$, where $w = \cos z - \cos x \cos y + \sin x \sin y$, and $z = x + y$

53. $\frac{\partial z}{\partial x}$, where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

54. $\frac{\partial z}{\partial x}$, where $xy - z = 1$

55. Change on a line Suppose $w = f(x, y, z)$ and ℓ is the line $\mathbf{r}(t) = \langle at, bt, ct \rangle$, for $-\infty < t < \infty$.

a. Find $w'(t)$ on ℓ (in terms of a, b, c, w_x, w_y, and w_z).

b. Apply part (a) to find $w'(t)$ when $f(x, y, z) = xyz$.

c. Apply part (a) to find $w'(t)$ when $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

d. For a general twice differentiable function $w = f(x, y, z)$, find $w''(t)$.

56. Implicit differentiation rule with three variables Assume $F(x, y, z(x, y)) = 0$ implicitly defines z as a differentiable function of x and y. Extend Theorem 15.9 to show that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

57–59. Implicit differentiation with three variables *Use the result of Exercise 56 to evaluate* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *for the following relations.*

57. $xy + xz + yz = 3$

58. $x^2 + 2y^2 - 3z^2 = 1$

59. $xyz + x + y - z = 0$

60. More than one way Let $e^{xyz} = 2$. Find z_x and z_y in three ways (and check for agreement).

a. Use the result of Exercise 56.

b. Take logarithms of both sides and differentiate $xyz = \ln 2$.

c. Solve for z and differentiate $z = \frac{\ln 2}{xy}$.

61–64. Walking on a surface *Consider the following surfaces specified in the form $z = f(x, y)$ and the oriented curve C in the xy-plane.*

- ***a.*** *In each case, find $z'(t)$.*
- ***b.*** *Imagine that you are walking on the surface directly above the curve C in the direction of positive orientation. Find the values of t for which you are walking uphill (that is, z is increasing).*

61. $z = x^2 + 4y^2 + 1$, $C: x = \cos t,\ y = \sin t;\ 0 \le t \le 2\pi$

62. $z = 4x^2 - y^2 + 1$, $C: x = \cos t,\ y = \sin t;\ 0 \le t \le 2\pi$

63. $z = \sqrt{1 - x^2 - y^2}$, $C: x = e^{-t},\ y = e^{-t};\ t \ge \frac{1}{2}\ln 2$

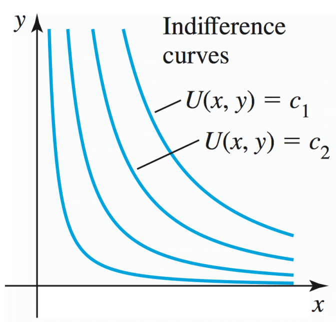
64. $z = 2x^2 + y^2 + 1$, $C: x = 1 + \cos t,\ y = \sin t;\ 0 \le t \le 2\pi$

65. Conservation of energy A projectile with mass m is launched into the air on a parabolic trajectory. For $t \ge 0$, its horizontal and vertical coordinates are $x(t) = u_0 t$ and $y(t) = -\frac{1}{2} g t^2 + v_0 t$, respectively, where u_0 is the initial horizontal velocity, v_0 is the initial vertical velocity, and g is the acceleration due to gravity. Recalling that $u(t) = x'(t)$ and $v(t) = y'(t)$ are the components of the velocity, the energy of the projectile (kinetic plus potential) is

$$E(t) = \frac{1}{2} m \left(u^2 + v^2\right) + m g y$$

Use the Chain Rule to compute $E'(t)$ and show that $E'(t) = 0$, for all $t \ge 0$. Interpret the result.

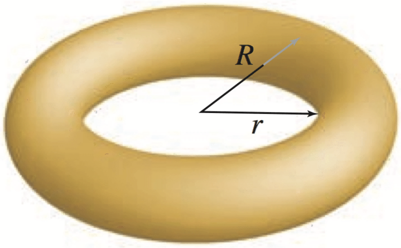
66. Utility functions in economics Economists use *utility functions* to describe consumers' relative preference for two or more commodities (for example, vanilla vs. chocolate ice cream or leisure time vs. material goods). The Cobb-Douglas family of utility functions has the form $U(x, y) = x^a y^{1-a}$, where x and y are the amounts of two commodities and $0 < a < 1$ is a parameter. Level curves on which the utility function is constant are called *indifference curves*; the preference is the same for all combinations of x and y along an indifference curve (see figure).



- **a.** The marginal utilities of the commodities x and y are defined to be $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, respectively. Compute the marginal utilities for the utility function $U(x, y) = x^a y^{1-a}$.
- **b.** The marginal rate of substitution (MRS) is the slope of the indifference curve at the point (x, y). Use the Chain Rule to show that for $U(x, y) = x^a y^{1-a}$, the MRS is $-\frac{a}{1-a}\frac{y}{x}$.

c. Find the MRS for the utility function $U(x, y) = x^{0.4}\, y^{0.6}$ at $(x, y) = (8, 12)$.

67. Constant volume tori The volume of a solid torus is given by $V = \frac{\pi^2}{4}(R + r)(R - r)^2$, where r and R are the inner and outer radii and $R > r$ (see figure).



a. If R and r increase at the same rate, does the volume of the torus increase, decrease, or remain constant?

b. If R and r decrease at the same rate, does the volume of the torus increase, decrease, or remain constant?

68. Body surface area One of several empirical formulas that relates the surface area S of a human body to the height h and weight w of the body is the Mosteller formula $S(h, w) = \frac{1}{60}\sqrt{h\,w}$, where h is measured in cm, w is measured in kg, and S is measured in square meters. Suppose h and w are functions of t.

a. Find $S'(t)$.

b. Show that the condition that the surface area remains constant as h and w change is $w\,h'(t) + h\,w'(t) = 0$.

c. Show that part (b) implies that for constant surface area, h and w must be inversely related; that is, $h = \frac{C}{w}$, where C is a constant.

69. The Ideal Gas Law The pressure, temperature, and volume of an ideal gas are related by $P\,V = k\,T$, where $k > 0$ is a constant. Any two of the variables may be considered independent, which determines the dependent variable.

a. Use implicit differentiation to compute the partial derivatives $\frac{\partial P}{\partial V}$, $\frac{\partial T}{\partial P}$, and $\frac{\partial V}{\partial T}$.

b. Show that $\frac{\partial P}{\partial V}\frac{\partial T}{\partial P}\frac{\partial V}{\partial T} = -1$. (See Exercise 75 for a generalization.)

70. Variable density The density of a thin circular plate of radius 2 is given by $\rho(x, y) = 4 + x\,y$. The edge of the plate is described by the parametric equations $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$.

a. Find the rate of change of the density with respect to t on the edge of the plate.

b. At what point(s) on the edge of the plate is the density a maximum?

T 71. Spiral through a domain Suppose you follow the helical path $C: x = \cos t$, $y = \sin t$, $z = t$, for $t \ge 0$, through the domain of the function $w = f(x, y, z) = \frac{x\,y\,z}{z^2 + 1}$.

a. Find $w'(t)$ along C.

b. Estimate the point (x, y, z) on C at which w has its maximum value.

Explorations and Challenges »

72. Change of coordinates Recall that Cartesian and polar coordinates are related through the transformation equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

a. Evaluate the partial derivatives x_r, y_r, x_θ, and y_θ.

b. Evaluate the partial derivatives r_x, r_y, θ_x, and θ_y.

c. For a function $z = f(x, y)$, find z_r and z_θ, where x and y are expressed in terms of r and θ.

d. For a function $z = g(r, \theta)$, find z_x and z_y, where r and θ are expressed in terms of x and y.

e. Show that $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

73. Change of coordinates continued An important derivative operation in many applications is called the Laplacian; in Cartesian coordinates, for $z = f(x, y)$, the Laplacian is $z_{xx} + z_{yy}$. Determine the Laplacian in polar coordinates using the following steps.

a. Begin with $z = g(r, \theta)$ and write z_x and z_y in terms of polar coordinates (see Exercise 72).

b. Use the Chain Rule to find $z_{xx} = \dfrac{\partial}{\partial x}(z_x)$. There should be two major terms, which, when expanded and simplified, result in five terms.

c. Use the Chain Rule to find $z_{yy} = \dfrac{\partial}{\partial y}\left(z_y\right)$. There should be two major terms, which, when expanded and simplified, result in five terms.

d. Combine parts (b) and (c) to show that

$$z_{xx} + z_{yy} = z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}.$$

74. Geometry of implicit differentiation Suppose x and y are related by the equation $F(x, y) = 0$. Interpret the solution of this equation as the set of points (x, y) that lie on the intersection of the surface $z = F(x, y)$ with the xy-plane $(z = 0)$.

a. Make a sketch of a surface and its intersection with the xy-plane. Give a geometric interpretation of the result that $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

b. Explain geometrically what happens at points where $F_y = 0$.

75. General three-variable relationship In the implicit relationship $F(x, y, z) = 0$, any two of the variables may be considered independent, which then determines the dependent variable. To avoid confusion, we may use a subscript to indicate which variable is held fixed in a derivative calculation; for example $\left(\dfrac{\partial z}{\partial x}\right)_y$ means that y is held fixed in taking the partial derivative of z with respect to x. (In this context, the subscript does *not* mean a derivative.)

a. Differentiate $F(x, y, z) = 0$ with respect to x holding y fixed to show that $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.

b. As in part (a), find $\left(\frac{\partial y}{\partial z}\right)_x$ and $\left(\frac{\partial x}{\partial y}\right)_z$.

c. Show that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$.

d. Find the relationship analogous to part (c) for the case $F(w, x, y, z) = 0$.

76. Second derivative Let $f(x, y) = 0$ define y as a twice differentiable function of x.

a. Show that $y''(x) = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}$.

b. Verify part (a) using the function $f(x, y) = xy - 1$.

77. Subtleties of the Chain Rule Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all (x, y, z) in $\mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

a. We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = f(x, y, z(x, y))$. Differentiate with respect to x holding y fixed to show that $\left(\frac{\partial w}{\partial x}\right)_y = 18$, where the subscript y indicates that y is held fixed.

b. Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = f(x, y(x, z), z)$ and differentiate with respect to x holding z fixed. Show that $\left(\frac{\partial w}{\partial x}\right)_z = 8$, where the subscript z indicates that z is held fixed.

c. Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

d. Repeat the arguments of parts (a) and (b) to find $\left(\frac{\partial w}{\partial y}\right)_x$, $\left(\frac{\partial w}{\partial y}\right)_z$, $\left(\frac{\partial w}{\partial z}\right)_x$, and $\left(\frac{\partial w}{\partial z}\right)_y$.