15.3 Partial Derivatives

The derivative of a function of one variable, $y = f(x)$, measures the rate of change of y with respect to x, and it gives slopes of tangent lines. The analogous idea for functions of several variables presents a new twist: Derivatives may be defined with respect to any of the independent variables. For example, we can compute the derivative of $f(x, y)$ with respect to x or y. The resulting derivatives are called *partial derivatives*; they still represent rates of change and they are associated with slopes of tangents. Therefore, much of what you have learned about derivatives applies to functions of several variables. However, much is also different.

Derivatives with Two Variables »

Consider a function f defined on a domain D in the xy-plane. Suppose f represents the elevation of the land (above sea level) over D. Imagine that you are on the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ and you are asked to determine the slope of the surface where you are standing. Your answer should be, it *depends*!

Figure 15.30a shows a function that resembles the landscape in **Figure 15.30b**. Suppose you are standing at the point $P(0, 0, f(0, 0))$, which lies on the pass or the saddle. The surface behaves differently, depending on the direction in which you walk. If you walk east (positive x-direction), the elevation increases and your path takes you upward on the surface. If you walk north (positive y-direction), the elevation decreases and your path takes you downward on the surface. In fact, in every direction you walk from the point P, the function values change at different rates. So how should the slope or the rate of change at a given point be defined?

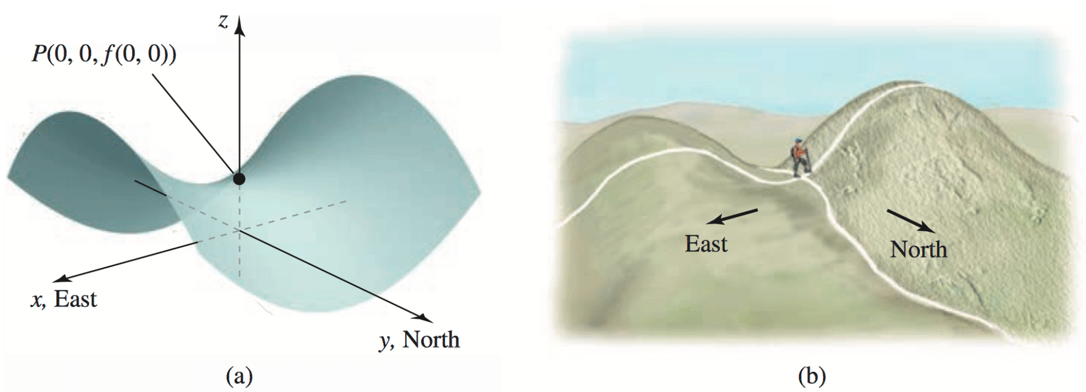


Figure 15.30

The answer to this question involves *partial derivatives*, which arise when we hold all but one independent variable fixed and then compute an ordinary derivative with respect to the remaining variable. Suppose we move along the surface $z = f(x, y)$, starting at the point $(a, b, f(a, b))$ in such a way that $y = b$ is fixed and only x varies. The resulting path is a curve (a trace) on the surface that varies in the x-direction (**Figure 15.31**). This curve is the intersection of the surface with the vertical plane $y = b$; it is described by $z = f(x, b)$, which is a function of the single variable x. We know how to compute the slope of this curve: It is the ordinary derivative of $f(x, b)$ with respect to x. This derivative is called the *partial derivative of f with respect to x*, denoted $\dfrac{\partial f}{\partial x}$ or f_x. When evaluated at (a, b) its value is defined by the limit

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

provided this limit exists. Notice that the y-coordinate is fixed at $y = b$ in this limit. If we replace (a, b) by the variable point (x, y), then f_x becomes a function of x and y.

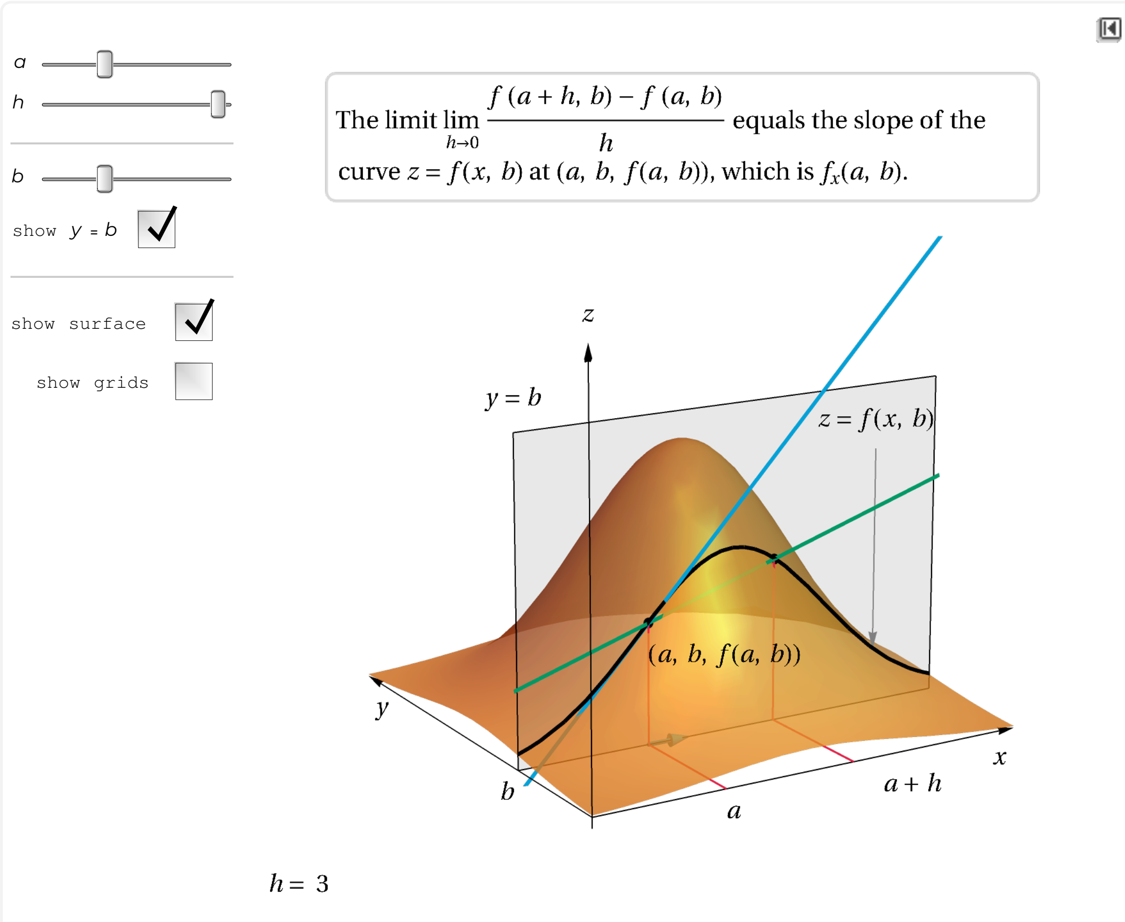


Figure 15.31

In a similar way, we can move along the surface $z = f(x, y)$ from the point $(a, b, f(a, b))$ in such a way that $x = a$ is fixed and only y varies. Now, the result is a trace described by $z = f(a, y)$, which is the intersection of the surface and the plane $x = a$ (**Figure 15.32**). The slope of this curve at (a, b) is given by the ordinary derivative of $f(a, y)$ with respect to y. This derivative is called the *partial derivative of f with respect to y*, denoted $\frac{\partial f}{\partial y}$ or f_y. When evaluated at (a, b), it is defined by the limit

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

provided this limit exists. If we replace (a, b) by the variable point (x, y), then f_y becomes a function of x and y.

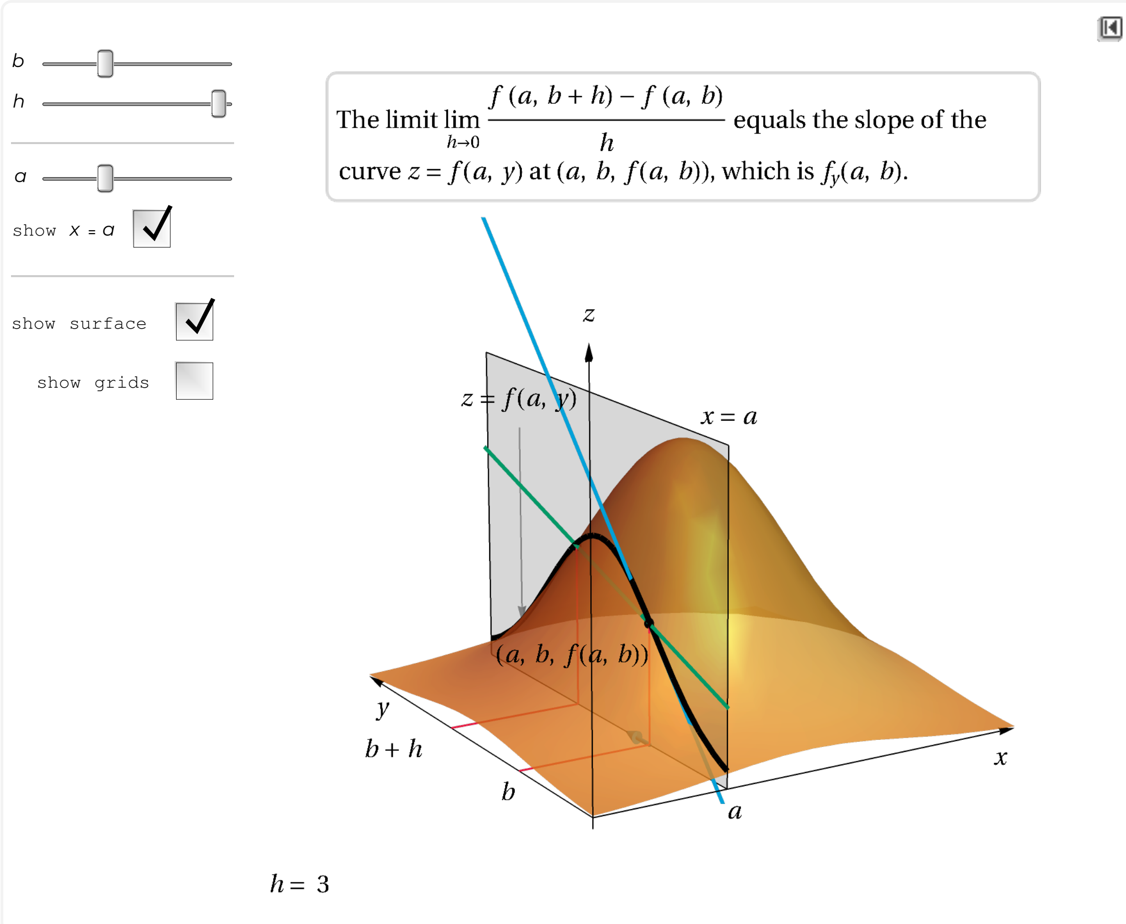


Figure 15.32

DEFINITION **Partial Derivatives**

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

provided these limits exist.

Notation

The partial derivatives evaluated at a point (a, b) are denoted in any of the following ways:

$$\frac{\partial f}{\partial x}(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

Notice that the d in the ordinary derivative $\dfrac{df}{dx}$ has been replaced by ∂ in the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. The notation $\dfrac{\partial}{\partial x}$ is an instruction or operator: It says, "take the partial derivative with respect to x of the function that follows."

Note »

Recall that f' is a function, while $f'(a)$ is the value of the derivative at $x = a$. In the same way, f_x and f_y are functions of x and y, while $f_x(a, b)$ and $f_y(a, b)$ are their values at (a, b).

Calculating Partial Derivatives

We begin by calculating partial derivatives using the limit definition. The procedure in Example 1 should look familiar. It echoes the method used in Chapter 3 when we first introduced ordinary derivatives.

EXAMPLE 1 Partial derivatives from the definition

Suppose $f(x, y) = x^2 y$. Use the limit definition of partial derivatives to compute $f_x(x, y)$ and $f_y(x, y)$.

SOLUTION »

We compute the partial derivatives at an arbitrary point (x, y) in the domain. The partial derivative with respect to x is

$$\begin{aligned}
f_x(x, y) &= \lim_{x \to 0} \frac{f(x+h, y) - f(x, y)}{h} && \text{Definition of } f_x \text{ at } (x, y) \\
&= \lim_{x \to 0} \frac{(x+h)^2 y - x^2 y}{h} && \text{Substitute for } f(x+h, y) \text{ and } f(x, y). \\
&= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 - x^2\right) y}{h} && \text{Factor and expand.} \\
&= \lim_{h \to 0} (2x + h) y && \text{Simplify and cancel } h. \\
&= 2xy. && \text{Evaluate limit.}
\end{aligned}$$

In a similar way, the partial derivative with respect to y is

$$\begin{aligned}
f_y(x, y) &= \lim_{x \to 0} \frac{f(x, y+h) - f(x, y)}{h} && \text{Definition of } f_y \text{ at } (x, y) \\
&= \lim_{h \to 0} \frac{x^2(y+h) - x^2 y}{h} && \text{Substitute for } f(x, y+h) \text{ and } f(x, y). \\
&= \lim_{h \to 0} \frac{x^2(y+h-y)}{h} && \text{Factor.} \\
&= x^2. && \text{Simplify and evaluate limit.}
\end{aligned}$$

Related Exercise 11 ◆

A careful examination of Example 1 reveals a shortcut for evaluating partial derivatives. To compute the partial derivative of f with respect to x, we treat y as a constant and take an ordinary derivative with respect to x:

$$\frac{\partial}{\partial x}\left(x^2 y\right) = y \underbrace{\frac{\partial}{\partial x}\left(x^2\right)}_{2x} = 2xy. \quad \text{Treat } y \text{ as a constant.}$$

Similarly, we treat x (and therefore x^2) as a constant to evaluate the partial derivative of f with respect to y:

$$\frac{\partial}{\partial y}\left(x^2 y\right) = x^2 \underbrace{\frac{\partial}{\partial y}(y)}_{1} = x^2. \quad \text{Treat } x \text{ as a constant.}$$

The next two examples illustrate the process.

EXAMPLE 2 Partial derivatives

Let $f(x, y) = x^3 - y^2 + 4$.

a. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

b. Evaluate each derivative at $(2, -4)$.

SOLUTION »

a. We compute the partial derivative with respect to x assuming y is a constant; the Power Rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\underbrace{x^3}_{\text{variable}} - \underbrace{y^2 + 4}_{\substack{\text{constant with} \\ \text{respect to } x}}\right) = 3x^2 + 0 = 3x^2.$$

The partial derivative with respect to y is computed by treating x as a constant; using the Power Rule gives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\underbrace{x^3}_{\substack{\text{constant with} \\ \text{repsect to } y}} - \underbrace{y^2}_{\text{variable}} + \underbrace{4}_{\text{constant}}\right) = -2y.$$

b. It follows that $f_x(2, -4) = \left(3x^2\right)\big|_{(2,-4)} = 12$ and $f_y(2, -4) = (-2y)\big|_{(2,-4)} = 8$.

Related Exercise 16 ◆

Quick Check 1 Compute f_x and f_y for $f(x, y) = 2xy$. ◆

Answer »

> $f_x = 2y$; $f_y = 2x$

EXAMPLE 3 Partial derivatives

Compute the partial derivatives of the following functions.

a. $f(x, y) = \sin xy$

b. $g(x, y) = x^2 e^{xy}$

SOLUTION »

a. Treating y as a constant and differentiating with respect to x, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sin xy) = y \cos xy .$$

Note »

Holding x fixed and differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\sin xy) = x \cos xy.$$

b. To compute the partial derivative with respect to x, we call on the Product Rule. Holding y fixed, we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 e^{xy}\right) \\ &= \frac{\partial}{\partial x}\left(x^2\right) e^{xy} + x^2 \frac{\partial}{\partial x}(e^{xy}) && \text{Product Rule} \\ &= 2\,x\,e^{xy} + x^2\,y\,e^{xy} && \text{Evaluate partial derivatives.} \\ &= x\,e^{xy}(2 + x\,y). && \text{Simplify.}\end{aligned}$$

Treating x as a constant, the partial derivative with respect to y is

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(x^2 e^{xy}\right) = x^2 \underbrace{\frac{\partial}{\partial y}(e^{xy})}_{x\,e^{xy}} = x^3 e^{xy}.$$

Note »

Because x and y are *independent* variables,

$$\frac{\partial}{\partial x}(y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y}(x) = 0.$$

Related Exercises 17, 21 ◆

Higher-Order Partial Derivatives »

Just as we have higher-order derivatives of functions of one variable, we also have higher-order partial derivatives. For example, given a function f and its partial derivative f_x, we can take the derivative of f_x with respect to x or with respect to y, which accounts for two of the four possible *second-order partial derivatives*. Table 15.3 summarizes the notation for second partial derivatives.

Table 15.3

Notation 1	Notation 2	What we say ...
$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	*d squared f dx squared* or *f-x-x*
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	*d squared f dy squared* or *f-y-y*
$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x\,\partial y}$	$(f_y)_x = f_{yx}$	*f-y-x*
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y\,\partial x}$	$(f_x)_y = f_{xy}$	*f-x-y*

The order of differentiation can make a difference in the **mixed partial derivatives** f_{xy} and f_{yx}. So, it is important to use the correct notation to reflect the order in which derivatives are taken. For example, the notations $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and f_{yx} both mean $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$; that is, differentiate first with respect to y, then with respect to x.

Quick Check 2 Which of the following expressions are equivalent to each other: (a) f_{xy}, (b) f_{yx}, or (c) $\dfrac{\partial^2 f}{\partial y\,\partial x}$? Write $\dfrac{\partial^2 f}{\partial p\,\partial q}$ in subscript notation. ◆

Answer »

EXAMPLE 4 Second partial derivatives

Find the four second partial derivatives of $f(x, y) = 3x^4 y - 2xy + 5xy^3$.

SOLUTION »

First, we compute

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(3x^4 y - 2xy + 5xy^3\right) = 12x^3 y - 2y + 5y^3$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^4 y - 2xy + 5xy^3\right) = 3x^4 - 2x + 15xy^2.$$

For the second partial derivatives, we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(12 x^3 y - 2 y + 5 y^3\right) = 36 x^2 y,$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3 x^4 - 2 x + 15 x y^2\right) = 30 x y,$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3 x^4 - 2 x + 15 x y^2\right) = 12 x^3 - 2 + 15 y^2, \text{ and}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(12 x^3 y - 2 y + 5 y^3\right) = 12 x^3 - 2 + 15 y^2.$$

Related Exercises 39–40 ◆

Quick Check 3 Compute f_{xxx} and f_{xxy} for $f(x, y) = x^3 y$. ◆

Answer »

$f_{xxx} = 6 y$; $f_{xxy} = 6 x$

Equality of Mixed Partial Derivatives

Notice that the two mixed partial derivatives in Example 4 are equal; that is, $f_{xy} = f_{yx}$. It turns out that most of the functions we encounter in this book have this property. Sufficient conditions for equality of mixed partial derivatives are given in a theorem attributed to the French mathematician Alexis Clairaut (1713–1765). The proof is found in advanced texts.

THEOREM 15.4 **(Clairaut) Equality of Mixed Partial Derivatives**

Assume f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Assuming sufficient continuity, Theorem 15.4 can be extended to higher derivatives of f. For example, $f_{xyx} = f_{xxy} = f_{yxx}$.

Functions of Three Variables »

Everything we learned about partial derivatives of functions with two variables carries over to functions of three or more variables, as illustrated in Example 5.

EXAMPLE 5 **Partial derivatives with more than two variables**

Find f_x, f_y, and f_z when $f(x, y, z) = e^{-xy} \cos z$.

SOLUTION »

To find f_x, we treat y and z as constants and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\underbrace{e^{-xy}}_{\substack{y \text{ is} \\ \text{constant}}} \cdot \underbrace{\cos z}_{\text{constant}}\right) = -y e^{-xy} \cos z.$$

Holding x and z constant and differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\underbrace{e^{-xy}}_{\substack{x \text{ is} \\ \text{constant}}} \cdot \underbrace{\cos z}_{\text{constant}}\right) = -x\, e^{-xy} \cos z.$$

To find f_z, we hold x and y constant and differentiate with respect to z:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\underbrace{e^{-xy}}_{\text{constant}} \cos z\right) = -e^{-xy} \sin z$$

Related Exercises 55–56 ◆

Quick Check 4 Compute f_{xz} and f_{zz} for $f(x, y, z) = x\,y\,z - x^2\,z + y\,z^2$. ◆

Answer »

$f_{xz} = y - 2\,x$; $f_{zz} = 2\,y$

Applications of Partial Derivatives

When functions are used in realistic applications (for example, to describe velocity, pressure, investment fund balance, or population), they often involve more than one independent variable. For this reason, partial derivatives appear frequently in mathematical modeling.

EXAMPLE 6 Ideal Gas Law

The pressure P, volume V, and temperature T of an ideal gas are related by the equation $P\,V = k\,T$, where $k > 0$ is a constant depending on the amount of gas.

a. Determine the rate of change of the pressure with respect to the volume at constant temperature. Interpret the result.

b. Determine the rate of change of the pressure with respect to the temperature at constant volume. Interpret the result.

c. Explain these results using level curves.

SOLUTION »

Expressing the pressure as a function of volume and temperature, we have $P = k\,\frac{T}{V}$.

a. We find the partial derivative $\frac{\partial P}{\partial V}$ by holding T constant and differentiating P with respect to V:

$$\frac{\partial P}{\partial V} = \frac{\partial}{\partial V}\left(k\,\frac{T}{V}\right) = k\,T\,\frac{\partial}{\partial V}\left(V^{-1}\right) = -\frac{k\,T}{V^2}.$$

Note »

Recognizing that P, V, and T are always positive, we see that $\frac{\partial P}{\partial V} < 0$, which means that the pressure is a decreasing function of volume at a constant temperature.

Note »

In the Ideal Gas Law, temperature is a positive variable because it is measured in kelvins.

b. The partial derivative $\frac{\partial P}{\partial T}$ is found by holding V constant and differentiating P with respect to T:

$$\frac{\partial P}{\partial T} = \frac{\partial}{\partial T}\left(k\frac{T}{V}\right) = \frac{k}{V}.$$

In this case $\frac{\partial P}{\partial T} > 0$, which says that the pressure is an increasing function of temperature at constant volume.

c. The level curves (Section 15.1) of the pressure function are curves in the VT-plane that satisfy $k\frac{T}{V} = P_0$, where P_0 is a constant. Solving for T, the level curves are given by $T = \frac{1}{k} P_0 V$. Because $\frac{P_0}{k}$ is a positive constant, the level curves are lines in the first quadrant passing through the origin (**Figure 15.33**) with slope $\frac{P_0}{k}$. The fact that $\frac{\partial P}{\partial V} < 0$ (from part (a)), means that if we hold $T > 0$ fixed and move in the direction of increasing V on a *horizontal* line, we cross level curves corresponding to decreasing pressures. Similarly, $\frac{\partial P}{\partial T} > 0$ (from part (b)) means that if we hold $V > 0$ fixed and move in the direction of increasing T on a *vertical* line, we cross level curves corresponding to increasing pressures.

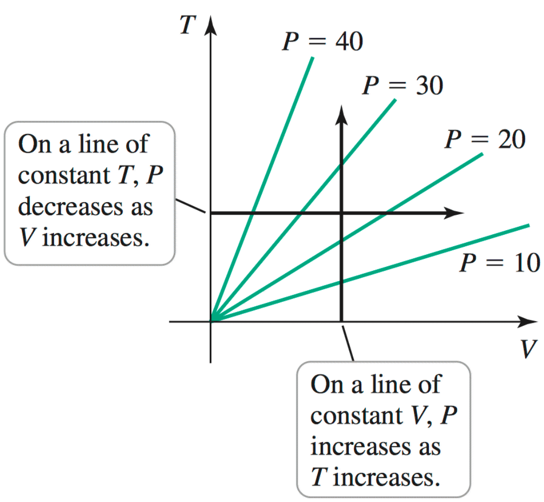


Figure 15.33

Related Exercise 69 ◆

Quick Check 5 Explain why, in Figure 15.33, the slopes of the level curves increase as the pressure increases. ◆

Answer »

> The equations of the level curves are $T = \frac{1}{k} P_0 V$. As the pressure P_0 increases, the slopes of theses lines increase.

Differentiability »

We close this section with a technical matter that bears on the remainder of the chapter. Although we know how to compute partial derivatives of a function of several variables, we have not said what it means for such a function to be *differentiable* at a point. It is tempting to conclude that if the partial derivatives f_x and f_y exist at a point, then f is differentiable there. However, it is not so simple.

Recall that a function f of one variable is differentiable at $x = a$ provided the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

exists. If f is differentiable at a, it means that the curve is smooth at the point $(a, f(a))$ (no jumps, corners, or cusps); furthermore, the curve has a unique tangent line at that point with slope $f'(a)$. Differentiability for a function of several variables should carry the same properties: The surface should be smooth at the point in question and something analogous to a unique tangent line should exist at the point.

Staying with the one-variable case, we define the quantity

$$\varepsilon = \underbrace{\frac{f(a + \Delta x) - f(a)}{\Delta x}}_{\text{slope of secant line}} - \underbrace{f'(a)}_{\substack{\text{slope of} \\ \text{tangent line}}},$$

where ε is viewed as a function of Δx. Notice that ε is the difference between the slopes of secant lines and the slope of the tangent line at the point $(a, f(a))$. If f is differentiable at a, then this difference approaches zero as $\Delta x \to 0$; therefore, $\lim_{\Delta x \to 0} \varepsilon = 0$. Multiplying both sides of the definition of ε by Δx gives

$$\varepsilon \, \Delta x = f(a + \Delta x) - f(a) - f'(a) \, \Delta x.$$

Rearranging, we have the change in the function $y = f(x)$:

$$\Delta y = f(a + \Delta x) - f(a) = f'(a) \, \Delta x + \underbrace{\varepsilon}_{\substack{\varepsilon \to 0 \\ \text{as } \Delta x \to 0}} \Delta x.$$

This expression says that in the one-variable case, if f is differentiable at a, then the change in f between a and a nearby point $a + \Delta x$ is represented by $f'(a) \, \Delta x$ plus a quantity $\varepsilon \, \Delta x$, where $\lim_{\Delta x \to 0} \varepsilon = 0$.

Note »

> Notice that $f'(a) \, \Delta x$ is the approximate change in the function given by a linear approximation.

The analogous requirement with several variables is the definition of differentiability for functions of two (or more) variables.

DEFINITION **Differentiability**

The function $z = f(x, y)$ is **differentiable at (a, b)** provided $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. A function is **differentiable** on an open region R if it is differentiable at every point of R.

Several observations are needed here. First, the definition extends to functions of more than two variables. Second, we show how differentiability is related to linear approximation and the existence of a *tangent plane* in Section 15.6. Finally, the conditions of the definition are generally difficult to verify. The following theorem may be useful in checking differentiability.

THEOREM 15.5 **Conditions for Differentiability**

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

As shown in Example 7, the existence of f_x and f_y at (a, b) is not enough to ensure differentiability of f at (a, b). However, by Theorem 15.5, if f_x and f_y are continuous at (a, b) (and defined in an open set containing (a, b)), then we can conclude f is differentiable there. Polynomials and rational functions are differentiable at all points of their domains, as are compositions of exponential, logarithmic, and trigonometric functions with other differentiable functions. The proof of this theorem is given in Appendix A.

We close with the analog of Theorem 3.1, which states that differentiability implies continuity.

THEOREM 15.6 **Differentiability Implies Continuity**

If a function f is differentiable at (a, b), then it is continuous at (a, b).

Proof: By the definition of differentiability,

$$\Delta z = f_x\,(a, b)\,\Delta x + f_y\,(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. Because f is assumed to be differentiable, we see that as Δx and Δy approach 0,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \Delta z = 0.$$

Also, because $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$, it follows that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b),$$

which implies continuity of f at (a, b). ◆

Note »

Recall that continuity requires that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

which is equivalent to

$$\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b).$$

EXAMPLE 7 A nondifferentiable function

Discuss the differentiability and continuity of the function

$$f(x, y) = \begin{cases} \dfrac{3xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

SOLUTION »

As a rational function, f is continuous and differentiable at all points $(x, y) \neq (0, 0)$. The interesting behavior occurs at the origin. Using calculations similar to those in Example 4 in Section 15.2, it can be shown that if the origin is approached along the line $y = mx$, then

$$\lim_{\substack{(x,y)\to(0,0) \\ (\text{along } y = mx)}} \frac{3xy}{x^2 + y^2} = \frac{3m}{m^2 + 1}.$$

Therefore, the value of the limit depends on the direction of approach, which implies that the limit does not exist, and f is not continuous at $(0, 0)$. By Theorem 15.6, it follows that f is not differentiable at $(0, 0)$. **Figure 15.34** shows the discontinuity of f at the origin.

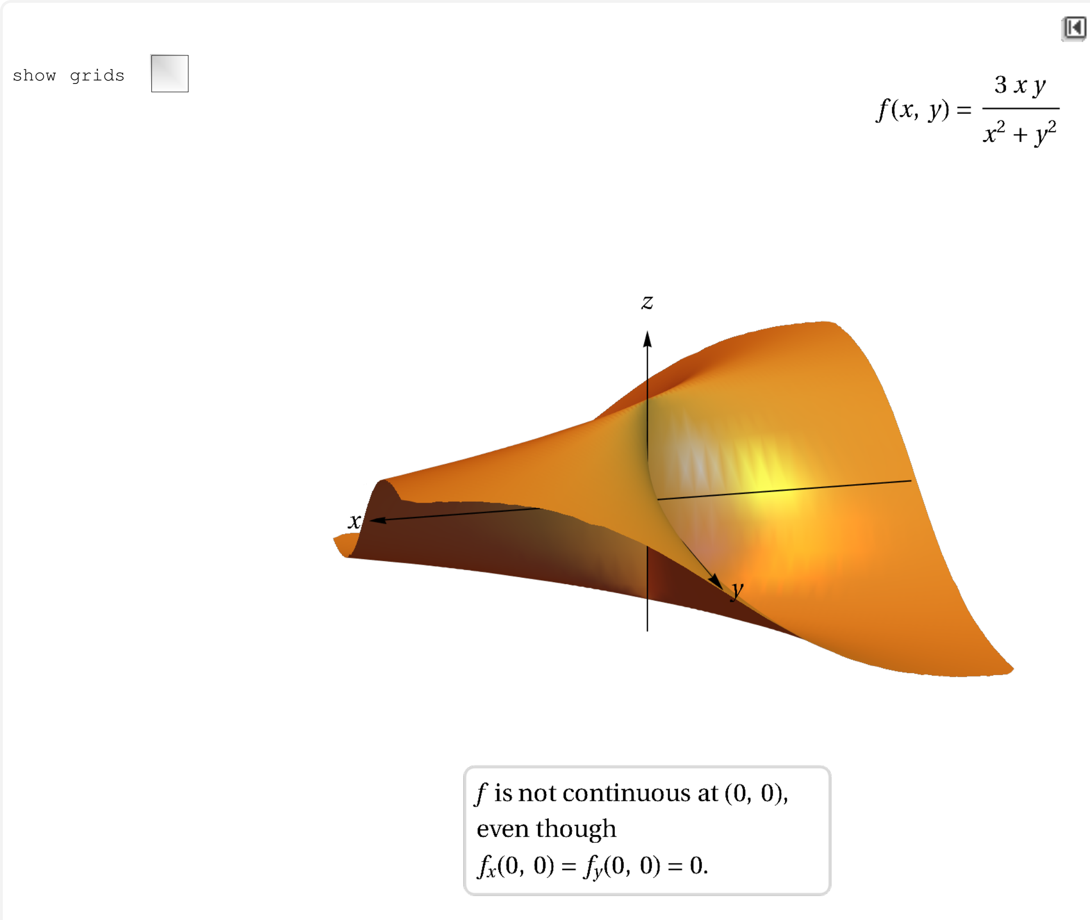


Figure 15.34

Let's look at the first partial derivatives of f at $(0, 0)$. A short calculation shows that

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Despite the fact that its first partial derivatives exist at $(0, 0)$, f is not differentiable at $(0, 0)$. As noted earlier, the existence of first partial derivatives at a point is not enough to ensure differentiability at that point.

Note »

> The relationships between the existence and continuity of partial derivatives and whether a function is differentiable are further explored in Exercises 96–97.

Related Exercises 77–78 ◆

Exercises »

Getting Started »

Practice Exercises »

11–14. Evaluating partial derivatives using limits *Use the limit definition of partial derivatives to evaluate $f_x(x, y)$ and $f_y(x, y)$ for the following functions.*

11. $f(x, y) = 5\,x\,y$

12. $f(x, y) = x + y^2 + 4$

13. $f(x, y) = \dfrac{x}{y}$

14. $f(x, y) = \sqrt{x\,y}$

15–37. Partial derivatives *Find the first partial derivatives of the following functions.*

15. $f(x, y) = x\,e^y$

16. $f(x, y) = 4\,x^3\,y^2 + 3\,x^2\,y^3 + 10$

17. $f(x, y) = e^{x^2\,y}$

18. $f(x, y) = \left(3\,x\,y + 4\,y^2 + 1\right)^5$

19. $f(w, z) = \dfrac{w}{w^2 + z^2}$

20. $f(s, t) = \dfrac{s - t}{s + t}$

21. $f(x, y) = x \cos x\,y$

22. $f(x, y) = \tan^{-1} \dfrac{x^2}{y^2}$

23. $s(y, z) = z^2 \tan yz$

24. $g(x, z) = x \ln\left(z^2 + x^2\right)$

25. $G(s, t) = \dfrac{\sqrt{s\,t}}{s + t}$

26. $F(p, q) = \sqrt{p^2 + pq + q^2}$

27. $f(x, y) = x^{2\,y}$

28. $g(x, y) = \cos^5\left(x^2\,y^3\right)$

29. $h(x, y) = x - \sqrt{x^2 - 4\,y}$

30. $h(u, v) = \sqrt{\dfrac{uv}{u - v}}$

31. $f(x, y) = \displaystyle\int_x^{y^3} e^{t^2}\, dt$

32. $g(x, y) = y \sin^{-1} \sqrt{xy}$

33. $f(x, y) = 1 - \tan^{-1}\left(x^2 + y^2\right)$

34. $f(x, y) = \ln\left(1 + e^{-xy}\right)$

35. $h(x, y) = (1 + 2y)^x$

36. $f(x, y) = 1 - \cos(2(x + y)) + \cos^2(x + y)$

37. $f(x, y) = \displaystyle\int_x^y h(s)\, ds$, where h is continuous for all real numbers

38–47. Second partial derivatives *Find the four second partial derivatives of the following functions.*

38. $f(x, y) = x^2 \sin y$

39. $h(x, y) = x^3 + xy^2 + 1$

40. $f(x, y) = 2x^5 y^2 + x^2 y$

41. $f(x, y) = y^3 \sin 4x$

42. $f(x, y) = \sin^2\left(x^3 y\right)$

43. $p(u, v) = \ln\left(u^2 + v^2 + 4\right)$

44. $Q(r, s) = \dfrac{e^{r^3 s}}{s}$

45. $F(r, s) = r e^s$

46. $H(x, y) = \sqrt{4 + x^2 + y^2}$

47. $f(x, y) = \tan^{-1}\left(x^3 y^2\right)$

48–53. Equality of mixed partial derivatives *Verify that $f_{xy} = f_{yx}$ for the following functions.*

48. $f(x, y) = 3x^2 y^{-1} - 2x^{-1} y^2$

49. $f(x, y) = e^{x+y}$

50. $f(x, y) = \sqrt{xy}$

51. $f(x, y) = \cos xy$

52. $f(x, y) = e^{\sin xy}$

53. $f(x, y) = \left(2x - y^3\right)^4$

54–62. Partial derivatives with more than two variables *Find the first partial derivatives of the following functions.*

54. $G(r, s, t) = \sqrt{rs + rt + st}$

55. $h(x, y, z) = \cos(x + y + z)$

56. $g(x, y, z) = 2x^2 y - 3xz^4 + 10y^2 z^2$

57. $F(u, v, w) = \dfrac{u}{v + w}$

58. $Q(x, y, z) = \tan xyz$

59. $G(r, s, t) = \sqrt{rs^3 t^5}$

60. $g(w, x, y, z) = \cos(w + x)\sin(y - z)$

61. $h(w, x, y, z) = \dfrac{wz}{xy}$

62. $F(w, x, y, z) = w\sqrt{x + 2y + 3z}$

63. Exploiting patterns Let $R(t) = \dfrac{at + b}{ct + d}$ and $g(x, y, z) = \dfrac{4x - 2y - 2z}{-6x + 3y - 3z}$.

- **a.** Verify that $R'(t) = \dfrac{ad - bc}{(ct + d)^2}$.
- **b.** Use the derivative $R'(t)$ to find the first partial derivatives of g.

64–67. Estimating partial derivatives from a table *The following table shows values of a function* $f(x, y)$ *for values of x from* 2 *to* 2.5 *and values of y from* 3 *to* 3.5. *Use this table to estimate the values of the following partial derivatives.*

$y \backslash x$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
3	4.243	4.347	4.450	4.550	4.648	4.743
3.1	4.384	4.492	4.598	4.701	4.802	4.902
3.2	4.525	4.637	4.746	4.853	4.957	5.060
3.3	4.667	4.782	4.895	5.005	5.112	5.218
3.4	4.808	4.930	5.043	5.156	5.267	5.376
3.5	4.950	5.072	5.191	5.308	5.422	5.534

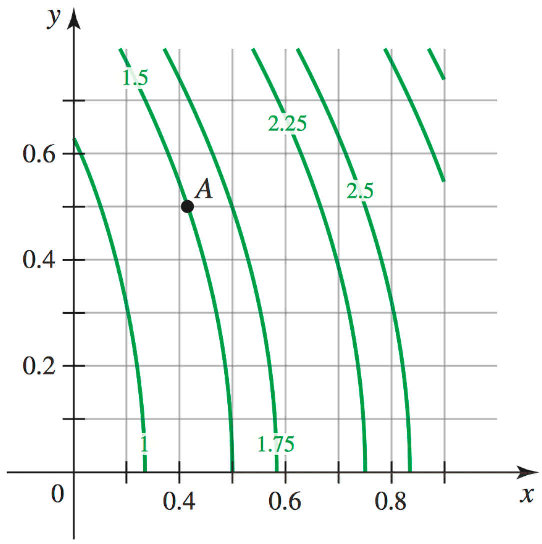
64. $f_x(2, 3)$

65. $f_y(2, 3)$

66. $f_x(2.2, 3.4)$

67. $f_y(2.4, 3.3)$

68. Estimating partial derivatives from a graph Use the level curves of f (see figure) to estimate the values of f_x and f_y at $A(0.42, 0.5)$.



69. Gas law calculations Consider the Ideal Gas Law $PV = kT$, where $k > 0$ is a constant. Solve this equation for V in terms of P and T.

a. Determine the rate of change of the volume with respect to the pressure at constant temperature. Interpret the result.

b. Determine the rate of change of the volume with respect to the temperature at constant pressure. Interpret the result.

c. Assuming $k = 1$, draw several level curves of the volume function and interpret the results as in Example 6.

70. Body mass index The body mass index (BMI) for an adult human is given by the function $B = \dfrac{w}{h^2}$, where w is the weight measured in kilograms and h is the height measured in meters.

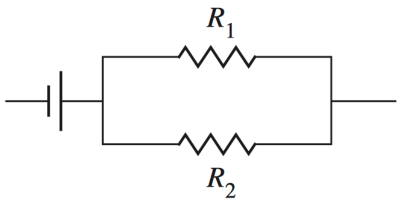
a. Find the rate of change of the BMI with respect to weight at a constant height.

b. For fixed h, is the BMI an increasing or decreasing function of w? Explain.

c. Find the rate of change of the BMI with respect to height at a constant weight.

d. For fixed w, is the BMI an increasing or decreasing function of h? Explain.

71. Resistors in parallel Two resistors in an electrical circuit with resistance R_1 and R_2 wired in parallel with a constant voltage give an effective resistance of R, where $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.



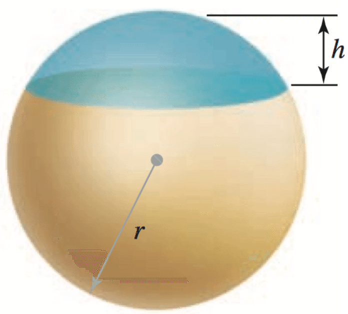
a. Find $\dfrac{\partial R}{\partial R_1}$ and $\dfrac{\partial R}{\partial R_2}$ by solving for R and differentiating.

b. Find $\dfrac{\partial R}{\partial R_1}$ and $\dfrac{\partial R}{\partial R_2}$ by differentiating implicitly.

c. Describe how an increase in R_1 with R_2 constant affects R.

d. Describe how a decrease in R_2 with R_1 constant affects R.

72. **Spherical caps** The volume of the cap of a sphere of radius r and thickness h is $V = \dfrac{\pi}{3} h^2(3\,r - h)$, for $0 \le h \le 2\,r$.



$V = \frac{\pi}{3}h^2(3r - h)$

a. Compute the partial derivatives V_h and V_r.

b. For a sphere of any radius, is the rate of change of volume with respect to r greater when $h = 0.2\,r$ or when $h = 0.8\,r$?

c. For a sphere of any radius, for what value of h is the rate of change of volume with respect to r equal to 1?

d. For a fixed radius r, for what value of h $(0 \le h \le 2\,r)$ is the rate of change of volume with respect to h the greatest?

73–76. Heat equation *The flow of heat along a thin conducting bar is governed by the one-dimensional heat equation (with analogs for thin plates in two dimensions and for solids in three dimensions),*

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},$$

where u is a measure of the temperature at a location x on the bar at time t and the positive constant k is related to the conductivity of the material. Show that the following functions satisfy the heat equation with $k = 1$.

73. $u(x,\ t) = 4\,e^{-4\,t} \cos 2\,x$

74. $u(x,\ t) = 10\,e^{-t} \sin x$

75. $u(x,\ t) = A\,e^{-a^2\,t}\cos a x$, for any real numbers a and A

76. $u(x,\ t) = e^{-t}(2\sin x + 3\cos x)$

77–78. Nondifferentiability? *Consider the following functions f.*

a. *Is f continuous at $(0,\ 0)$?*

b. *Is f differentiable at $(0,\ 0)$?*

c. *If possible, evaluate $f_x(0,\ 0)$ and $f_y(0,\ 0)$.*

d. *Determine whether f_x and f_y are continuous at $(0,\ 0)$.*

e. *Explain why Theorems 15.5 and 15.6 are consistent with the results in parts (a)–(d).*

77. $f(x,\ y) = \begin{cases} -\dfrac{x\,y}{x^2 + y^2} & \text{if } (x,\ y) \neq (0,\ 0) \\ 0 & \text{if } (x,\ y) = (0,\ 0) \end{cases}$

78. $f(x,\ y) = \begin{cases} \dfrac{2\,x\,y^2}{x^2 + y^4} & \text{if } (x,\ y) \neq (0,\ 0) \\ 0 & \text{if } (x,\ y) = (0,\ 0) \end{cases}$

79. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\dfrac{\partial}{\partial x}\left(y^{10}\right) = 10\,y^9$.

b. $\dfrac{\partial^2}{\partial x\,\partial y}\left(\sqrt{x\,y}\right) = \dfrac{1}{\sqrt{x\,y}}$.

c. If f has continuous partial derivatives of all orders, then $f_{xxy} = f_{yxx}$.

80. Mixed partial derivatives

a. Consider the function $w = f(x,\ y,\ z)$. List all possible second partial derivatives that could be computed.

b. Let $f(x,\ y,\ z) = x^2\,y + 2\,x\,z^2 - 3\,y^2\,z$ and determine which second partial derivatives are equal.

c. How many second partial derivatives does $p = g(w,\ x,\ y,\ z)$ have?

Explorations and Challenges »

81. Partial derivatives and level curves Consider the function $z = \dfrac{x}{y^2}$.

a. Compute z_x and z_y.

b. Sketch the level curves for $z = 1, 2, 3,$ and 4.

c. Move along the horizontal line $y = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_x as computed in part (a).

d. Move along the vertical line $x = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_y as computed in part (a).

82. Volume of a box A box with a square base of length x and height h has a volume $V = x^2\,h$.

a. Compute the partial derivatives V_x and V_h.

b. For a box with $h = 1.5$ m, use linear approximation to estimate the change in volume if x increases from $x = 0.5$ m to $x = 0.51$ m.

c. For a box with $x = 0.5$ m, use linear approximation to estimate the change in volume if h decreases from $h = 1.5$ m to $h = 1.49$ m.

d. For a fixed height, does a 10% change in x always produce (approximately) a 10% change in V? Explain.

e. For a fixed base length, does a 10% change in h always produce (approximately) a 10% change in V? Explain.

83. Electric potential function The electric potential in the xy-plane associated with two positive charges, one at $(0, 1)$ with twice the magnitude of the charge at $(0, -1)$, is

$$\phi(x, y) = \frac{2}{\sqrt{x^2 + (y-1)^2}} + \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

a. Compute ϕ_x and ϕ_y.

b. Describe how ϕ_x and ϕ_y behave as $x, y \to \pm\infty$.

c. Evaluate $\phi_x(0, y)$, for all $y \neq \pm 1$. Interpret this result.

d. Evaluate $\phi_y(x, 0)$, for all x. Interpret this result.

T **84. Cobb-Douglas production function** The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = c\,L^a\,K^b$. Suppose $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = 1$.

a. Evaluate the partial derivatives Q_L and Q_K.

b. Suppose $L = 10$ is fixed and K increases from $K = 20$ to $K = 20.5$. Use linear approximation to estimate the change in Q.

c. Suppose $K = 20$ is fixed and L decreases from $L = 10$ to $L = 9.5$. Use linear approximation to estimate the change in Q.

d. Graph the level curves of the production function in the first quadrant of the LK-plane for $Q = 1, 2$, and 3.

e. Use the graph of part (d). If you move along the vertical line $L = 2$ in the positive K-direction, how does Q change? Is this consistent with Q_K computed in part (a)?

f. Use the graph of part (d). If you move along the horizontal line $K = 2$ in the positive L-direction, how does Q change? Is this consistent with Q_L computed in part (a)?

85. An identity Show that if $f(x, y) = \dfrac{a\,x + b\,y}{c\,x + d\,y}$, where a, b, c, and d are real numbers with $a\,d - b\,c = 0$, then $f_x = f_y = 0$, for all x and y in the domain of f. Give an explanation.

86. Wave on a string Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x, t) = 2\sin(\pi\,x)\sin\dfrac{\pi\,t}{2}$, for $0 \le x \le 1$ and $t \ge 0$ (see figure). At a fixed point in time, the string forms a wave on $[0, 1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

a. What is the period of the motion in time?

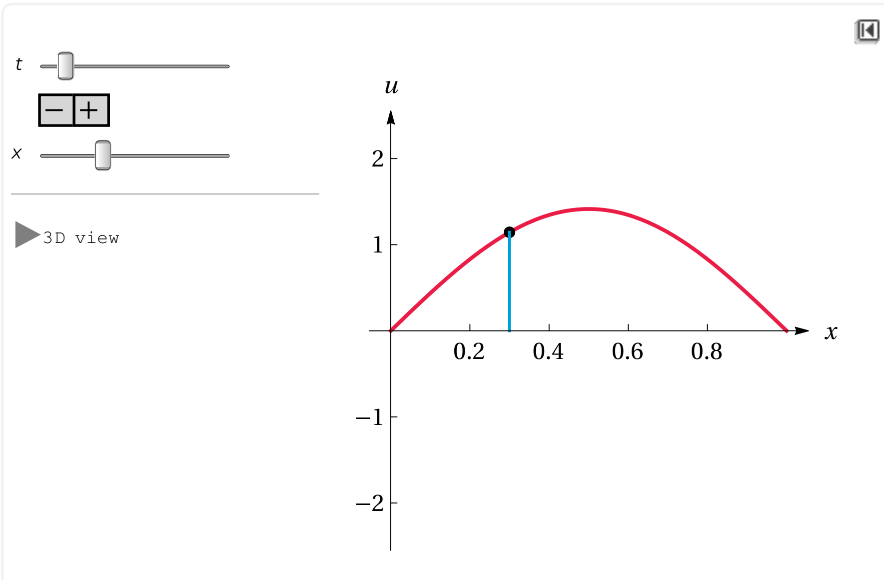
b. Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).

c. At a fixed time, what point on the string is moving fastest?

d. At a fixed position on the string, when is the string moving fastest?

e. Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).

f. At a fixed time, where is the slope of the string greatest?



87–89. Wave equation *Traveling waves (for example, water waves or electromagnetic waves) exhibit periodic motion in both time and position. In one dimension, some types of wave motion are governed by the one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the height or displacement of the wave surface at position x and time t, and c is the constant speed of the wave. Show that the following functions are solutions of the wave equation.

87. $u(x, t) = \cos(2(x + c\,t))$

88. $u(x, t) = 5\cos(2(x + c\,t)) + 3\sin(x - c\,t)$

89. $u(x, t) = A\,f(x + c\,t) + B\,g(x - c\,t)$, where A and B are constants, and f and g are twice differentiable functions of one variable.

90–93. Laplace's equation *A classical equation of mathematics is Laplace's equation, which arises in both theory and applications. It governs ideal fluid flow, electrostatic potentials, and the steady-state distribution of heat in a conducting medium. In two dimensions, Laplace's equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that the following functions are ***harmonic****; that is, they satisfy Laplace's equation.*

90. $u(x, y) = e^{-x} \sin y$

91. $u(x, y) = x\left(x^2 - 3\,y^2\right)$

92. $u(x, y) = e^{a x} \cos a y$, for any real number a

93. $u(x, y) = \tan^{-1}\left(\dfrac{y}{x-1}\right) - \tan^{-1}\left(\dfrac{y}{x+1}\right)$

94–95. Differentiability *Use the definition of differentiability to prove that the following functions are differentiable at* $(0, 0)$. *You must produce functions ε_1 and ε_2 with the required properties.*

94. $f(x, y) = x + y$

95. $f(x, y) = x y$

96–97. Nondifferentiability? *Consider the following functions* f.

a. *Is f continuous at* $(0, 0)$?

b. *Is f differentiable at* $(0, 0)$?

c. *If possible, evaluate $f_x(0, 0)$ and $f_y(0, 0)$.*

d. *Determine whether f_x and f_y are continuous at* $(0, 0)$.

e. *Explain why Theorems 15.5 and 15.6 are consistent with the results in parts (a)—(d).*

96. $f(x, y) = 1 - |x y|$

97. $f(x, y) = \sqrt{|x y|}$

98. Cauchy-Riemann equations In the advanced subject of complex variables, a function typically has the form $f(x, y) = u(x, y) + i\, v(x, y)$, where u and v are real-valued functions and $i = \sqrt{-1}$ is the imaginary unit. A function $f = u + i\, v$ is said to be *analytic* (analogous to differentiable) if it satisfies the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

a. Show that $f(x, y) = \left(x^2 - y^2\right) + i\,(2 x y)$ is analytic.

b. Show that $f(x, y) = x\left(x^2 - 3 y^2\right) + i\, y\left(3 x^2 - y^2\right)$ is analytic.

c. Show that if $f = u + i\, v$ is analytic, then $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. Assume u and v satisfy the conditions in Theorem 15.4.

99. Derivatives of an integral Let h be continuous for all real numbers. Find f_x and f_y when $f(x, y) = \int_1^{xy} h(s)\, ds$.