15.2 Limits and Continuity

You have now seen examples of functions of several variables, but calculus has not yet entered the picture. In this section we revisit topics encountered in single-variable calculus and see how they apply to functions of several variables. We begin with the fundamental concepts of limits and continuity.

Limit of a Function of Two Variables »

A function f of two variables has a limit L as $P(x, y)$ approaches a fixed point $P_0(a, b)$ if $|f(x, y) - L|$ can be made arbitrarily small for all P in the domain that are sufficiently close to P_0. If such a limit exists, we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L.$$

To make this definition more precise, *close to* must be defined carefully.

A point x on the number line is close to another point a provided the distance $|x - a|$ is small (**Figure 15.19a**). In $\mathbb{R}^2$, a point $P(x, y)$ is close to another point $P_0(a, b)$ if the distance between them $|PP_0| = \sqrt{(x-a)^2 + (y-b)^2}$ is small (**Figure 15.19b**). When we say *for all P close to* P_0, it means that $|PP_0|$ is small for points P *on all sides of* P_0.

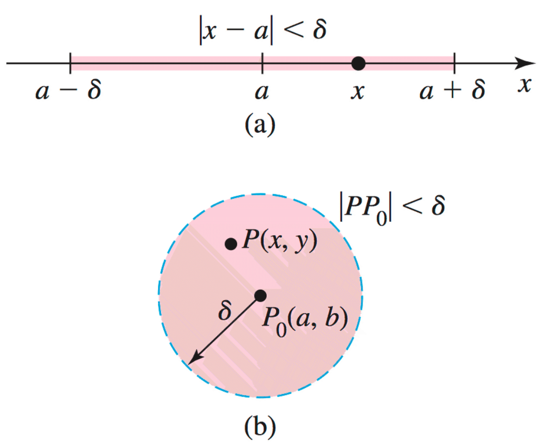


Figure 15.19

With this understanding of closeness, we can give a formal definition of a limit with two independent variables. This definition parallels the formal definition of a limit given in Section 2.7 (**Figure 15.20**).

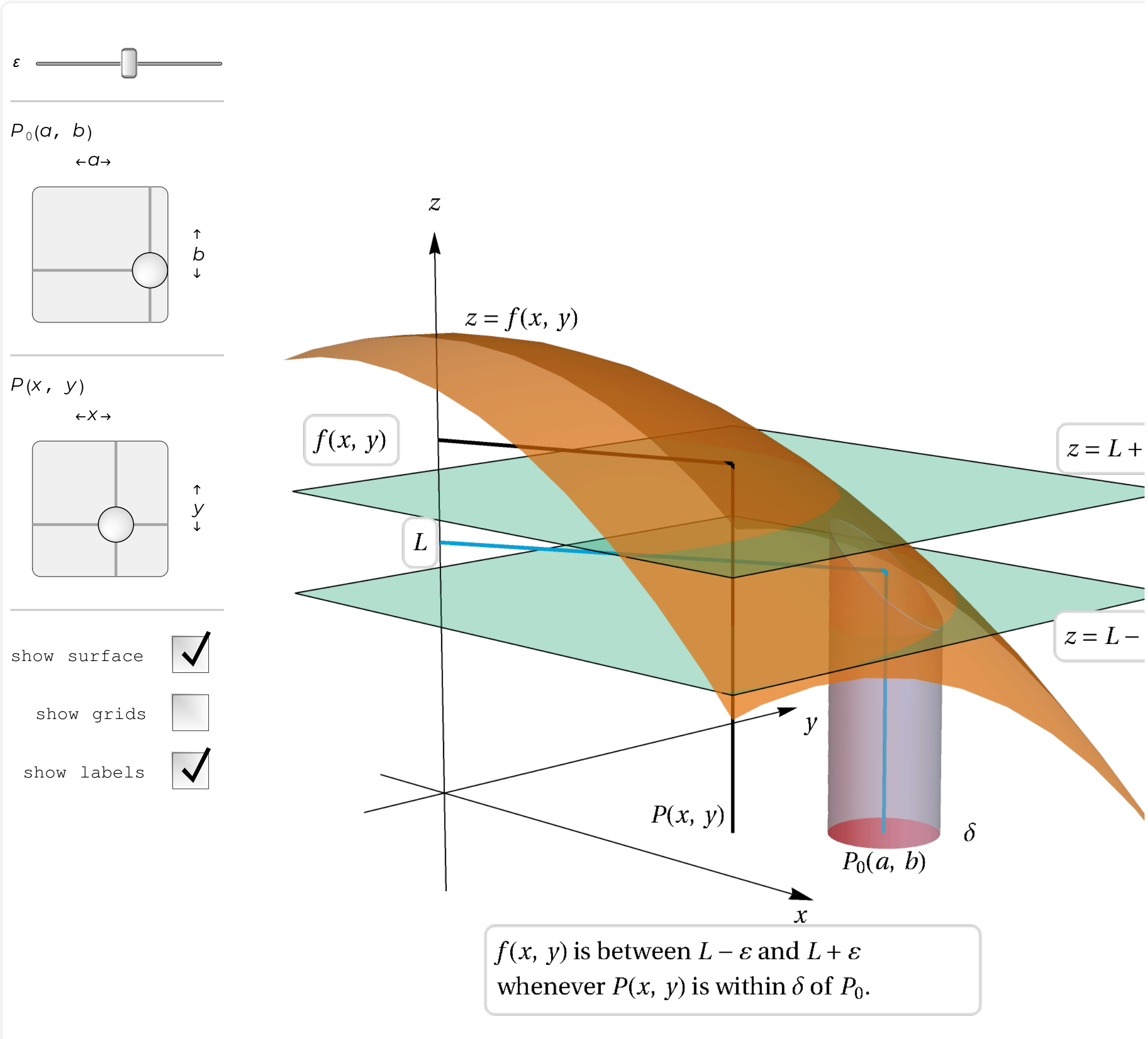


Figure 15.20

DEFINITION **Limit of a Function of Two Variables**

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Note »

> The formal definition extends naturally to any number of variables. With n variables, the limit point is $P_0(a_1, \ldots, a_n)$, the variable point is $P(x_1, \ldots, x_n)$, and $|PP_0| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$.

The condition $|PP_0| < \delta$ means that the distance between $P(x, y)$ and $P_0(a, b)$ is less than δ as P approaches P_0 from all possible directions (**Figure 15.21**). Therefore, the limit exists only if $f(x, y)$ approaches L as P approaches P_0 *along all possible paths* in the domain of f. As shown in upcoming examples, this interpretation is critical in determining whether a limit exists.

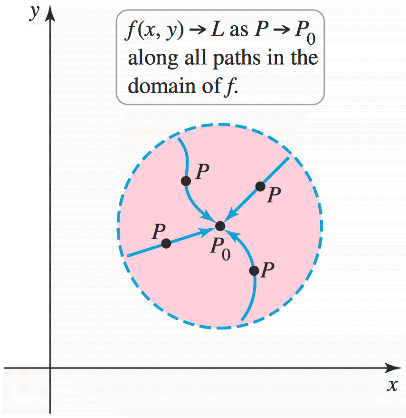


Figure 15.21

As with functions of one variable, we first establish limits of the simplest functions.

THEOREM 15.1 **Limits of Constants and Linear Functions**

Let a, b, and c be real numbers.

1. Constant function $f(x, y) = c$: $\lim_{(x,y)\to(a,b)} c = c$

2. Linear function $f(x, y) = x$: $\lim_{(x,y)\to(a,b)} x = a$

3. Linear function $f(x, y) = y$: $\lim_{(x,y)\to(a,b)} y = b$

Proof:

1. Consider the constant function $f(x, y) = c$ and assume $\varepsilon > 0$ is given. To prove that the value of the limit is $L = c$, we must produce a $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. For constant functions, we may use *any* constant $\delta > 0$. Then, for every (x, y) in the domain of f,

$$|f(x, y) - L| = |f(x, y) - c| = |c - c| = 0 < \varepsilon$$

whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$.

2. Assume $\varepsilon > 0$ is given and take $\delta = \varepsilon$. The condition $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon \quad \delta = \varepsilon$$
$$\sqrt{(x-a)^2} < \varepsilon \quad (x-a)^2 \le (x-a)^2 + (y-b)^2$$
$$|x-a| < \varepsilon. \quad \sqrt{x^2} = |x| \text{ for real numbers } x$$

Because $f(x, y) = x$ and $L = a$, we have shown that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Therefore, $\lim_{(x,y)\to(a,b)} f(x, y) = L$, or $\lim_{(x,y)\to(a,b)} x = a$. The proof that $\lim_{(x,y)\to(a,b)} y = b$ is similar (Exercise 86). ◆

Using the three basic limits in Theorem 15.1, we can compute limits of more complicated functions. The only tools needed are limit laws analogous to those given in Theorem 2.3. The proofs of these laws are examined in Exercises 88–89.

THEOREM 15.2 **Limit Laws for Functions of Two Variables**

Let L and M be real numbers and suppose $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$. Assume c is a constant, and $n > 0$ is an integer.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} c\, f(x, y) = c\, L$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x, y)\, g(x, y) = L\, M$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, provided $M \ne 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$
7. **Root** $\lim_{(x,y)\to(a,b)} (f(x, y))^{1/n} = L^{1/n}$, where we assume $L > 0$ if n is even.

Combining Theorems 15.1 and 15.2 allows us to find limits of polynomial, rational, and algebraic functions in two variables.

Note »

Recall that a polynomial in two variables consists of sums and products of polynomials in x and polynomials in y. A rational function is the quotient of two polynomials.

EXAMPLE 1 Limits of two-variable functions

Evaluate $\lim_{(x,y)\to(2,8)} \left(3\, x^2\, y + \sqrt{x\, y}\right)$.

SOLUTION »

All the operations in this function appear in Theorem 15.2. Therefore, we can apply the limit laws directly.

$$\lim_{(x,y)\to(2,8)} \left(3x^2 y + \sqrt{xy}\right) = \lim_{(x,y)\to(2,8)} 3x^2 y + \lim_{(x,y)\to(2,8)} \sqrt{xy} \quad \text{Law 1}$$

$$= 3 \lim_{(x,y)\to(2,8)} x^2 \cdot \lim_{(x,y)\to(2,8)} y + \sqrt{\lim_{(x,y)\to(2,8)} x \cdot \lim_{(x,y)\to(2,8)} y} \quad \text{Laws 3, 4, 7}$$

$$= 3 \cdot 2^2 \cdot 8 + \sqrt{2 \cdot 8} = 100 \quad \text{Law 6 and Theorem 15.1}$$

Related Exercise 16 ◆

In Example 1, the value of the limit equals the value of the function at (a, b); in other words $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, and the limit can be evaluated by substitution. This is a property of *continuous* functions, discussed later in this section.

Quick Check 1 Which of the following limits exist?

a. $\lim_{(x,y)\to(1,1)} 3x^{12} y^2$ **b.** $\lim_{(x,y)\to(0,0)} 3x^{-2} y^2$ **c.** $\lim_{(x,y)\to(1,2)} \sqrt{x - y^2}$ ◆

Answer »

The limit exists only for (a).

Limits at Boundary Points »

This is an appropriate place to make some definitions that will be used in the remainder of the text.

DEFINITION **Interior and Boundary Points**

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R (**Figure 15.22**).

A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Note »

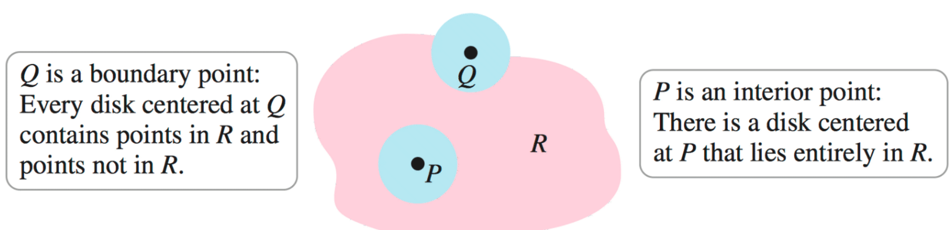


Figure 15.22

For example, let R be the points in $\mathbb{R}^2$ satisfying $x^2 + y^2 < 9$. The boundary points of R lie on the circle $x^2 + y^2 = 9$. The interior points lie inside that circle and satisfy $x^2 + y^2 < 9$. Notice that the boundary points of a set need not lie in the set.

DEFINITION **Open and Closed Sets**

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Note »

An example of an open region in $\mathbb{R}^2$ is the open disk $\{(x, y): x^2 + y^2 < 9\}$. An example of a closed region in $\mathbb{R}^2$ is the square $\{(x, y): |x| \leq 1, |y| \leq 1\}$. Later in the text, we encounter interior and boundary points of three-dimensional sets such as balls, boxes, and pyramids.

Quick Check 2 Give an example of a set that contains none of its boundary points. ◆

Answer »

Suppose $P_0(a, b)$ is a boundary point of the domain of f. The limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, even if P_0 is not in the domain of f, provided $f(x, y)$ approaches the same value as (x, y) approaches (a, b) *along all paths that lie in the domain* (**Figure 15.23**).

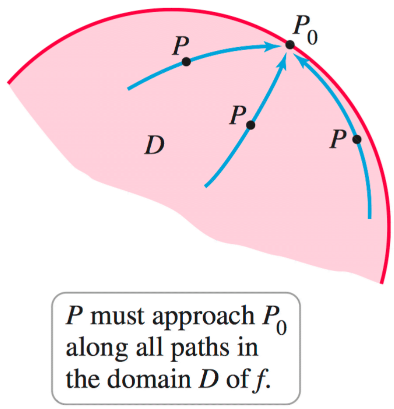


Figure 15.23

Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x - y}$ whose domain is $\{(x, y): x \neq y\}$. Provided $x \neq y$, we may cancel the factor $(x - y)$ from the numerator and denominator and write

$$f(x, y) = \frac{x^2 - y^2}{x - y} = \frac{(x - y)(x + y)}{x - y} = x + y.$$

The graph of f (**Figure 15.24**) is the plane $z = x + y$, with points corresponding to the line $x = y$ removed.

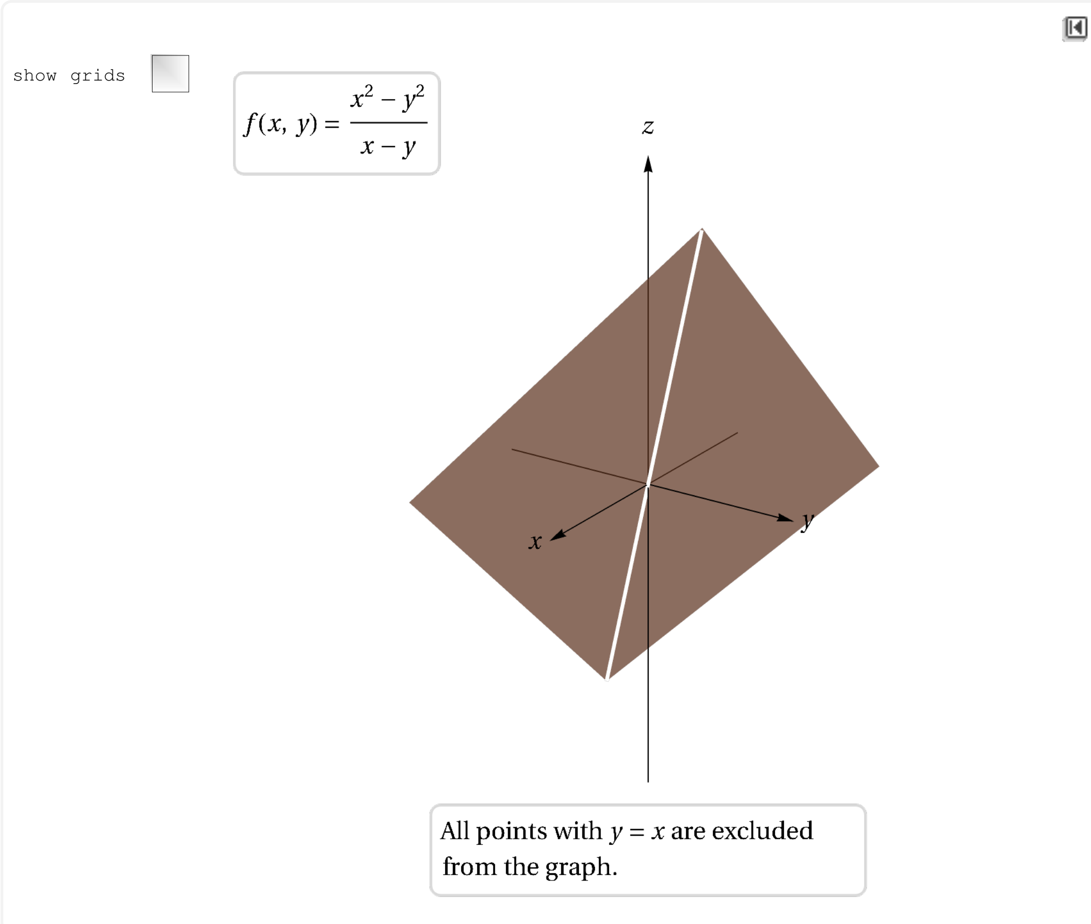


Figure 15.24

Note »

Now we examine $\lim_{(x,y)\to(4,4)} \frac{x^2 - y^2}{x - y}$, where $(4, 4)$ is a boundary point of the domain of f but does not lie in the domain. For this limit to exist, $f(x, y)$ must approach the same value along all paths to $(4, 4)$ that lie in the domain of f—that is, all paths approaching $(4, 4)$ except the path $x = y$. To evaluate the limit, we proceed as follows:

$$\lim_{(x,y)\to(4,4)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(4,4)} (x + y) \quad \text{Assume } x \neq y \text{, cancel } x - y.$$

$$= 4 + 4 = 8. \quad \text{Same limit along all paths in the domain}$$

To emphasize, we let $(x, y) \to (4, 4)$ along all paths except the path $x = y$, which lies outside the domain of f. Along all admissible paths, the function approaches 8.

Quick Check 3 Can the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{x}$ be evaluated by direct substitution? ◆

Answer »

If a factor of x is first canceled, then the limit may be evaluated by substitution.

EXAMPLE 2 Limits at boundary points

Evaluate $\lim\limits_{(x,y)\to(4,1)} \dfrac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$.

SOLUTION »

Points in the domain of this function satisfy $x \ge 0$ and $y \ge 0$ (because of the square roots) and $x \ne 4y$ (to ensure the denominator is nonzero). We see that the point $(4, 1)$ lies on the boundary of the domain. Multiplying the numerator and denominator by the algebraic conjugate of the denominator, the limit is computed as follows:

$$\begin{aligned}
\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} &= \lim_{(x,y)\to(4,1)} \frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})} && \text{Multiply by conjugate.}\\
&= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y} && \text{Simplify.}\\
&= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y}) && \text{Cancel } x-4y\text{, assumed to be nonzero.}\\
&= 4. && \text{Evaluate limit.}
\end{aligned}$$

Because points on the line $x = 4y$ are outside the domain of the function, we assume $x - 4y \ne 0$. Along all other paths to $(4, 1)$, the function values approach 4 (**Figure 15.25**).

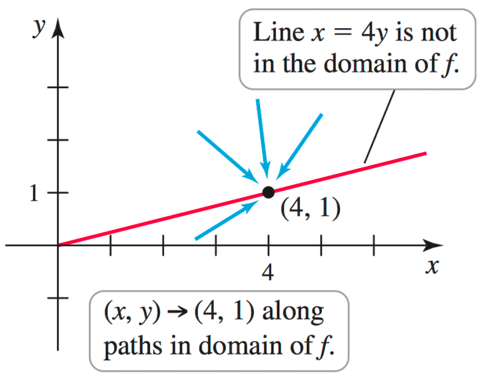


Figure 15.25

Related Exercises 26–27 ◆

EXAMPLE 3 Nonexistence of a limit

Investigate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$.

SOLUTION »

The domain of the function is $\{(x, y): (x, y) \ne (0, 0)\}$; therefore, the limit is at a boundary point outside the domain. Suppose we let (x, y) approach $(0, 0)$ along the line $y = mx$ for a fixed constant m. Substituting $y = mx$ and noting that $y \to 0$ as $x \to 0$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y = m\,x)}} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+m\,x)^2}{\left(x^2+m^2\,x^2\right)} = \lim_{x\to 0} \frac{x^2(1+m)^2}{x^2\left(1+m^2\right)} = \frac{(1+m)^2}{1+m^2}.$$

Note »

The constant m determines the direction of approach to (0, 0). Therefore, depending on m, the function approaches different values as (x, y) approaches (0, 0) (**Figure 15.26**).

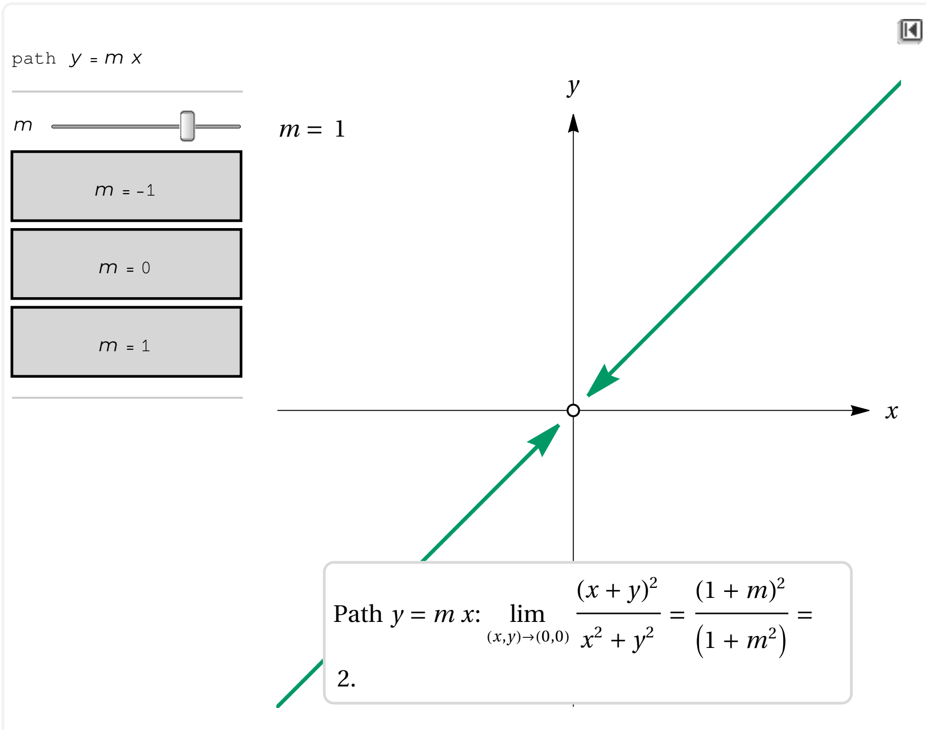


Figure 15.26

For example, if $m = 0$, the corresponding limit is 1 and if $m = -1$, the limit is 0. The reason for this behavior is revealed if we plot the surface and look at two level curves. The lines $y = x$ and $y = -x$ (excluding the origin) are level curves of the function for $z = 2$ and $z = 0$, respectively (**Figure 15.27**). Therefore, as $(x, y) \to (0, 0)$ along $y = x$, $f(x, y) \to 2$, and as $(x, y) \to (0, 0)$ along $y = -x$, $f(x, y) \to 0$. Because the function approaches different values along different paths, we conclude that the *limit does not exist*.

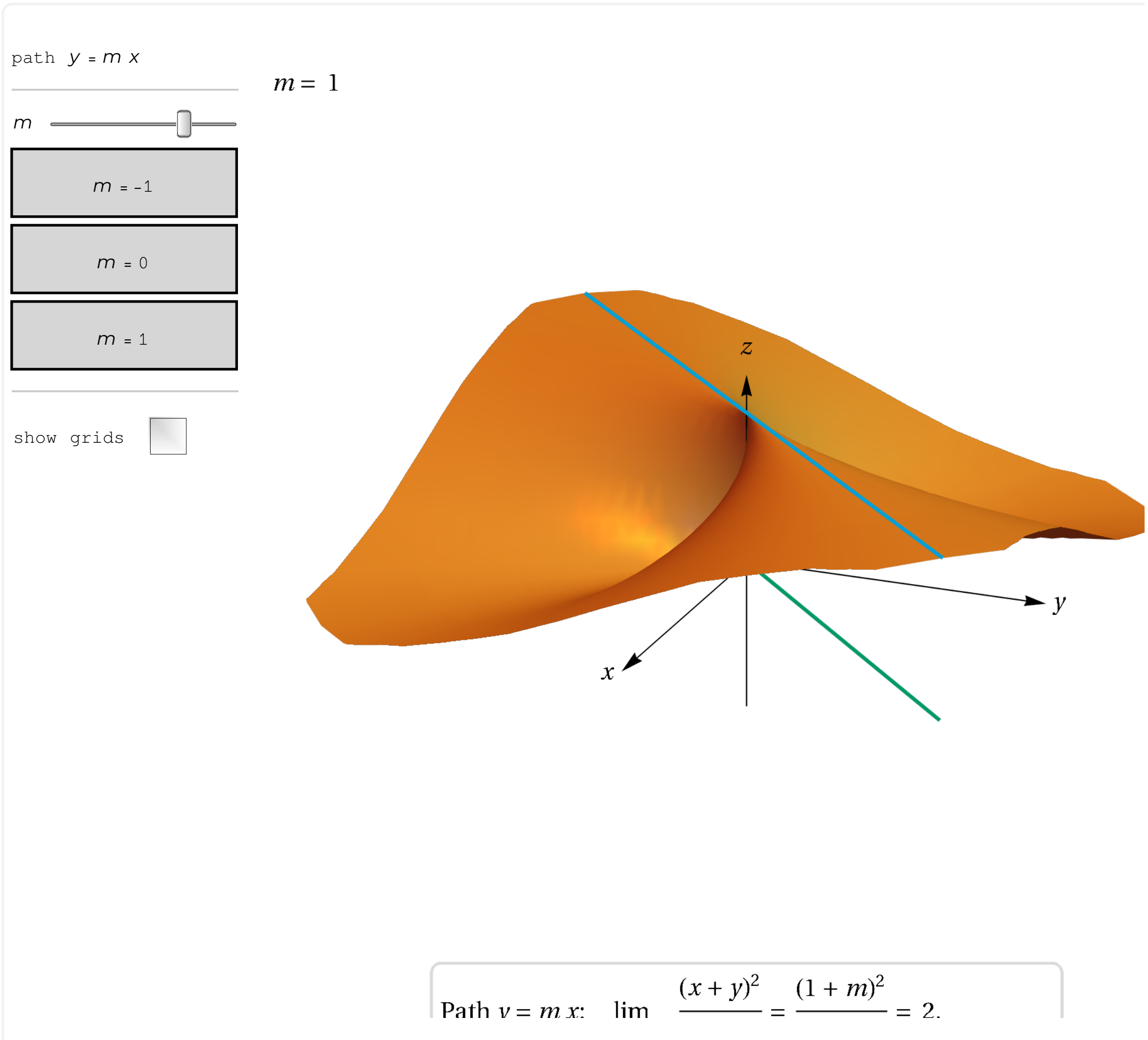


Figure 15.27

Related Exercise 30 ◆

The strategy used in Example 3 is one of the most effective ways to prove the nonexistence of a limit.

> **PROCEDURE** **Two-Path Test for Nonexistence of Limits**
>
> If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Quick Check 4 What is the analog of the Two-Path Test for functions of a single variable? ◆

Answer »

If the left and right limits at a point are not equal, then the two-sided limit does not exist.

Continuity of Functions of Two Variables »

The following definition of continuity for functions of two variables is analogous to the continuity definition for functions of one variable.

DEFINITION **Continuity**

The function f is continuous at the point (a, b) provided

1. f is defined at (a, b),

2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and

3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

A function of two (or more) variables is continuous at a point, provided its limit equals its value at that point (which implies the limit and the value both exist). The definition of continuity applies at boundary points of the domain of f provided the limits in the definition are taken along paths that lie in the domain. Because limits of polynomials and rational functions can be evaluated by substitution at points of their domains (that is, $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$), it follows that polynomials and rational functions are continuous at all points of their domains.

EXAMPLE 4 **Checking continuity**

Determine the points at which the following function is continuous.

$$f(x, y) = \begin{cases} \dfrac{3xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

SOLUTION »

The function $\dfrac{3xy^2}{x^2 + y^4}$ is a rational function, so it is continuous at all points of its domain, which consists of all points of $\mathbb{R}^2$ except $(0, 0)$. In order for f to be continuous at $(0, 0)$, we must show that

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2 + y^4}$$

exists and equals $f(0, 0) = 0$ along all paths that approach $(0, 0)$.

You can verify that as (x, y) approaches $(0, 0)$ along paths of the form $y = mx$, where m is any constant, the function values approach $f(0, 0) = 0$. However, along parabolic paths of the form $x = my^2$ (where m is a nonzero constant), the limit behaves differently (**Figure 15.28**).

Note »

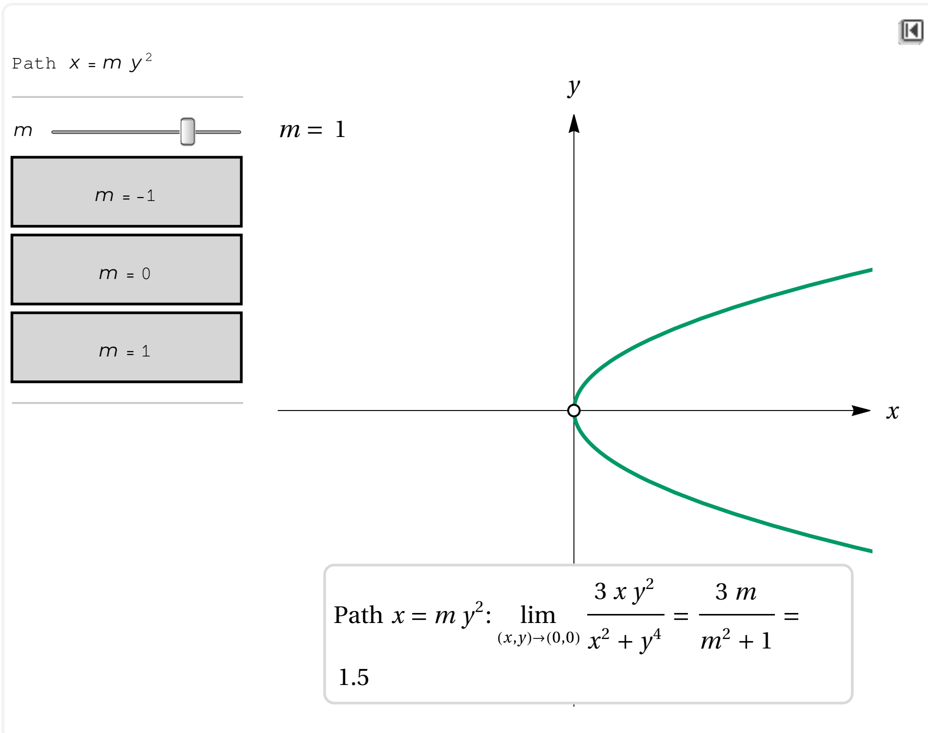


Figure 15.28

This time we substitute $x = m\,y^2$ and note that $x \to 0$ as $y \to 0$:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{3\,x\,y^2}{x^2 + y^4} &= \lim_{y\to 0} \frac{3\left(m\,y^2\right)y^2}{\left(m\,y^2\right)^2 + y^4} && \text{Substitute } x = m\,y^2. \\
&= \lim_{y\to 0} \frac{3\,m\,y^4}{m^2\,y^4 + y^4} && \text{Simplify.} \\
&= \lim_{y\to 0} \frac{3\,m}{m^2 + 1} && \text{Cancel } y^4. \\
&= \frac{3\,m}{m^2 + 1}.
\end{aligned}$$

We see that along parabolic paths, the limit depends on the approach path. For example, with $m = 1$, along the path $x = y^2$ the function values approach $\frac{3}{2}$; with $m = -1$, along the path $x = -y^2$ the function values approach $-\frac{3}{2}$ (**Figure 15.29**). Because $f(x, y)$ approaches two different numbers along two different paths, the limit at $(0, 0)$ does not exist, and f is not continuous at $(0, 0)$.

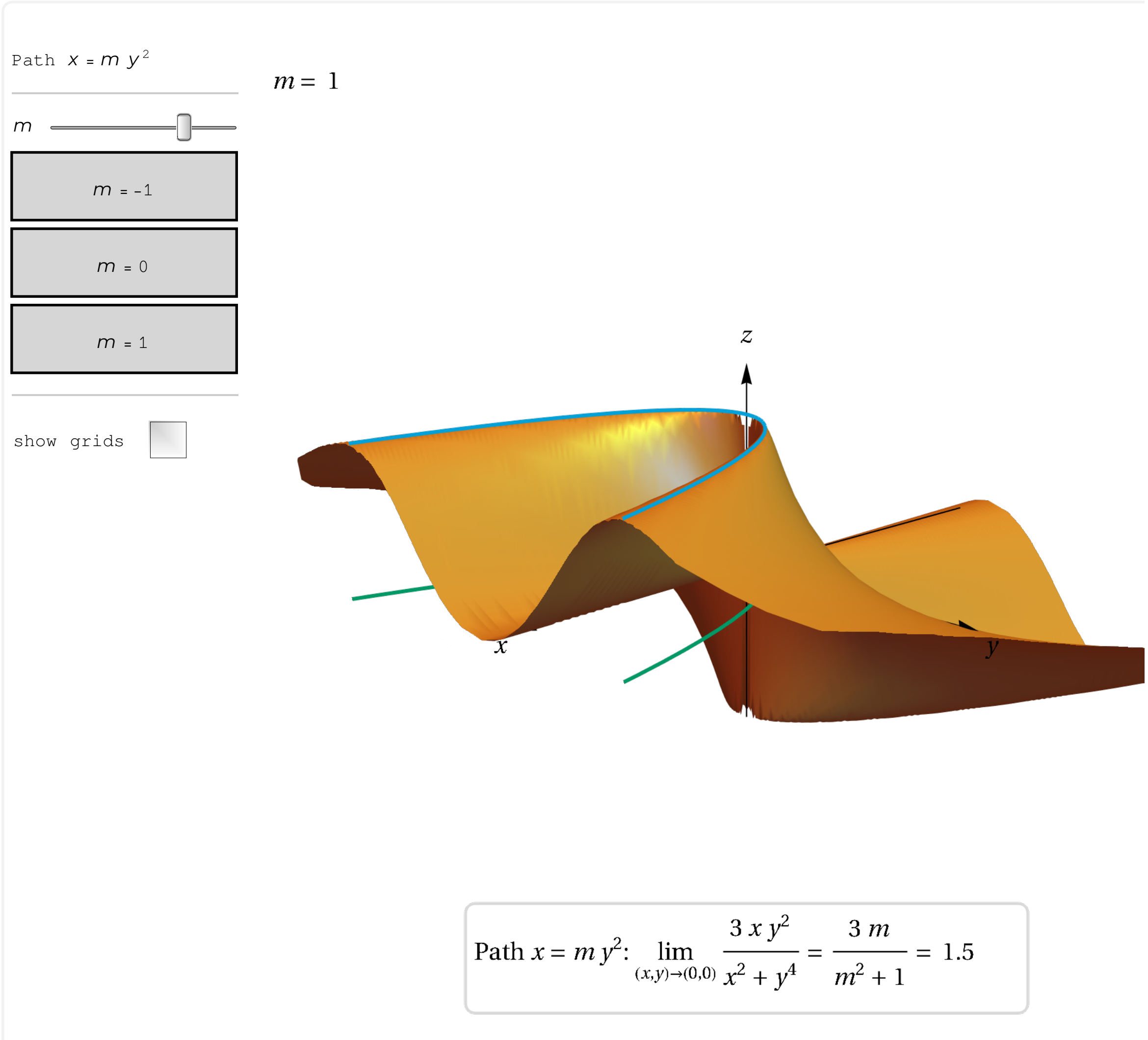


Figure 15.29

Related Exercises 41–42 ◆

Quick Check 5 Which of the following functions are continuous at (0, 0)?

a. $f(x, y) = 2\,x^2\,y^5$

b. $f(x, y) = \dfrac{2\,x^2\,y^5}{x-1}$

c. $f(x, y) = 2\,x^{-2}\,y^5$ ◆

Answer »

(a) and (b) are continuous at (0, 0).

Composite Functions

Recall that for functions of a single variable, compositions of continuous functions are also continuous. The following theorem gives the analogous result for functions of two variables; it is proved in Appendix A.

THEOREM 15.3 **Continuity of Composite Functions**

If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

With Theorem 15.3, we can easily analyze the continuity of many functions. For example, $\sin x$, $\cos x$, and e^x are continuous functions of a single variable, for all real values of x. Therefore, compositions of these functions with polynomials in x and y (for example, $\sin\left(x^2 y\right)$ and $e^{x^4 - y^2}$) are continuous for all real numbers x and y. Similarly, $\sqrt{x}$ is a continuous function of a single variable, for $x \geq 0$. Therefore, $\sqrt{u(x, y)}$ is continuous at (x, y) provided u is continuous at (x, y) and $u(x, y) \geq 0$. As long as we observe restrictions on domains, then compositions of continuous functions are also continuous.

EXAMPLE 5 **Continuity of composite functions**

Determine the points at which the following functions are continuous.

a. $h(x, y) = \ln\left(x^2 + y^2 + 4\right)$

b. $h(x, y) = e^{x/y}$

SOLUTION »

a. This function is the composition of $f(g(x, y))$, where

$$f(u) = \ln u \quad \text{and} \quad u = g(x, y) = x^2 + y^2 + 4.$$

As a polynomial, g is continuous for all (x, y) in $\mathbb{R}^2$. The function f is continuous for $u > 0$. Because $u = x^2 + y^2 + 4 > 0$ for all (x, y), it follows that h is continuous at all points of $\mathbb{R}^2$.

b. Letting $f(u) = e^u$ and $u = g(x, y) = \dfrac{x}{y}$, we have $h(x, y) = f(g(x, y))$. Note that f is continuous at all points of $\mathbb{R}$ and g is continuous at all points of $\mathbb{R}^2$ provided $y \neq 0$. Therefore, h is continuous on the set $\{(x, y) : y \neq 0\}$.

Related Exercises 48–49 ◆

Functions of Three Variables »

The work we have done with limits and continuity of functions of two variables extends to functions of three or more variables. Specifically, the limit laws of Theorem 15.2 apply to functions of the form $w = f(x, y, z)$. Polynomials and rational functions are continuous at all points of their domains, and limits of these functions may be evaluated by direct substitution at all points of their domains. Compositions of continuous functions of the form $f(g(x, y, z))$ are also continuous at points at which $g(x, y, z)$ is in the domain of f.

EXAMPLE 6 **Functions of three variables**

a. Evaluate $\displaystyle\lim_{(x,y,z)\to(2,\pi/2,0)} \frac{x^2 \sin y}{z^2 + 4}$.

b. Find the points at which $h(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 1}$ is continuous.

SOLUTION »

a. This function consists of products and quotients of functions that are continuous at $\left(2, \frac{\pi}{2}, 0\right)$. Therefore, the limit is evaluated by direct substitution:

$$\lim_{(x,y,z)\to(2,\pi/2,0)} \frac{x^2 \sin y}{z^2 + 4} = \frac{2^2 \sin(\pi/2)}{0^2 + 4} = 1.$$

b. This function is a composition in which the outer function $f(u) = \sqrt{u}$ is continuous for $u \geq 0$. The inner function

$$g(x, y, z) = x^2 + y^2 + z^2 - 1$$

is nonnegative provided $x^2 + y^2 + z^2 \geq 1$. Therefore, the function is continuous at all points on or outside the unit sphere in $\mathbb{R}^3$.

Related Exercise 55 ◆

Exercises »

Getting Started »

Practice Exercises »

13–28. Limits of functions *Evaluate the following limits.*

13. $\lim_{(x,y)\to(2,9)} 101$

14. $\lim_{(x,y)\to(1,-3)} (3x + 4y - 2)$

15. $\lim_{(x,y)\to(-3,3)} \left(4x^2 - y^2\right)$

16. $\lim_{(x,y)\to(2,-1)} \left(x y^8 - 3x^2 y^3\right)$

17. $\lim_{(x,y)\to(0,\pi)} \frac{\cos xy + \sin xy}{2y}$

18. $\lim_{(x,y)\to(e^2,4)} \ln \sqrt{xy}$

19. $\lim_{(x,y)\to(2,0)} \frac{x^2 - 3xy^2}{x + y}$

20. $\lim_{(u,v)\to(1,-1)} \frac{10uv - 2v^2}{u^2 + v^2}$

21. $\lim_{(x,y)\to(6,2)} \frac{x^2 - 3xy}{x - 3y}$

22. $\lim_{(x,y)\to(1,-2)} \frac{y^2 + 2xy}{y + 2x}$

23. $\lim_{(x,y)\to(3,1)} \frac{x^2 - 7xy + 12y^2}{x - 3y}$

24. $\lim_{(x,y)\to(-1,1)} \frac{2x^2 - xy - 3y^2}{x + y}$

25. $\lim_{(x,y)\to(2,2)} \frac{y^2 - 4}{xy - 2x}$

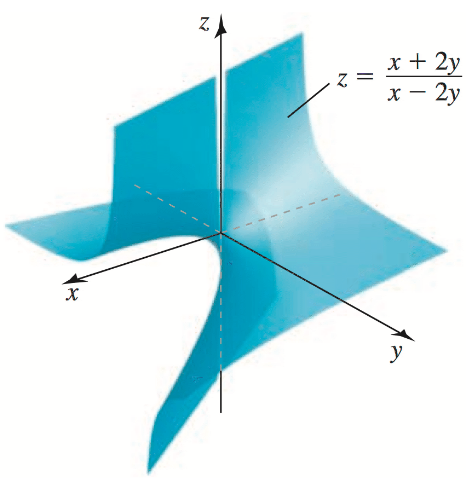
26. $\lim_{(x,y)\to(4,5)} \frac{\sqrt{x + y} - 3}{x + y - 9}$

27. $\lim_{(x,y)\to(1,2)} \frac{\sqrt{y} - \sqrt{x + 1}}{y - x - 1}$

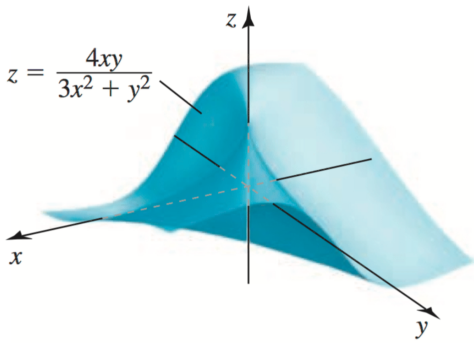
28. $\lim_{(u,v)\to(8,8)} \frac{u^{1/3} - v^{1/3}}{u^{2/3} - v^{2/3}}$

29–34. Nonexistence of limits *Use the Two-Path Test to prove that the following limits do not exist.*

29. $\lim_{(x,y)\to(0,0)} \frac{x + 2y}{x - 2y}$



30. $\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2 + y^2}$



31. $\lim_{(x,y)\to(0,0)} \frac{y^4 - 2x^2}{y^4 + x^2}$

32. $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^2}{x^3 + y^2}$

33. $\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{x y^2}$

34. $\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2 - y^2}}$

35–54. Continuity *At what points of* $\mathbb{R}^2$ *are the following functions continuous?*

35. $f(x, y) = x^2 + 2xy - y^3$

36. $f(x, y) = \frac{xy}{x^2 y^2 + 1}$

37. $p(x, y) = \frac{4x^2 y^2}{x^4 + y^2}$

38. $S(x, y) = \frac{2xy}{x^2 - y^2}$

39. $f(x, y) = \frac{2}{x\left(y^2 + 1\right)}$

40. $f(x, y) = \frac{x^2 + y^2}{x\left(y^2 - 1\right)}$

41. $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

42. $f(x, y) = \begin{cases} \dfrac{y^4 - 2x^2}{y^4 + x^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

43. $f(x, y) = \sqrt{x^2 + y^2}$

44. $f(x, y) = e^{x^2 + y^2}$

45. $f(x, y) = \sin xy$

46. $g(x, y) = \ln (x - y)$

47. $h(x, y) = \cos (x + y)$

48. $p(x, y) = e^{x-y}$

49. $f(x, y) = \ln \left(x^2 + y^2\right)$

50. $f(x, y) = \sqrt{4 - x^2 - y^2}$

51. $g(x, y) = \sqrt[3]{x^2 + y^2 - 9}$

52. $h(x, y) = \dfrac{\sqrt{x - y}}{4}$

53. $f(x, y) = \begin{cases} \dfrac{\sin \left(x^2 + y^2\right)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

54. $f(x, y) = \begin{cases} \dfrac{1 - \cos \left(x^2 + y^2\right)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

55–60. Limits of functions of three variables *Evaluate the following limits.*

55. $\lim\limits_{(x,y,z) \to (1, \ln 2, 3)} z e^{xy}$

56. $\lim\limits_{(x,y,z) \to (0,1,0)} \ln (1 + y) e^{xz}$

57. $\lim\limits_{(x,y,z) \to (1,1,1)} \dfrac{yz - xy - xz - x^2}{yz + xy + xz - y^2}$

58. $\lim\limits_{(x,y,z) \to (1,1,1)} \dfrac{x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz}}{x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz}}$

59. $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2 + x\,y - x\,z - y\,z}{x - z}$

60. $\lim\limits_{(x,y,z)\to(1,-1,1)} \dfrac{x\,z + 5\,x + y\,z + 5\,y}{x + y}$

61. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the limits $\lim\limits_{(x,0)\to(0,0)} f(x,\ 0)$ and $\lim\limits_{(0,y)\to(0,0)} f(0,\ y)$ exist and equal L, then $\lim\limits_{(x,y)\to(0,0)} f(x,\ y) = L$.

b. If $\lim\limits_{(x,y)\to(a,b)} f(x,\ y)$ equals a finite number L, then f is continuous at $(a,\ b)$.

c. If f is continuous at $(a,\ b)$, then $\lim\limits_{(x,y)\to(a,b)} f(x,\ y)$ exists.

d. If P is a boundary point of the domain of f, then P is in the domain of f.

62–76. Miscellaneous limits *Use the method of your choice to evaluate the following limits.*

62. $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{x^8 + y^2}$

63. $\lim\limits_{(x,y)\to(0,1)} \dfrac{y \sin x}{x\,(y + 1)}$

64. $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2 + x\,y - 2\,y^2}{2\,x^2 - x\,y - y^2}$

65. $\lim\limits_{(x,y)\to(1,0)} \dfrac{y \ln y}{x}$

66. $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x\,y|}{x\,y}$

67. $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x - y|}{|x + y|}$

68. $\lim\limits_{(u,v)\to(-1,0)} \dfrac{u\,v\,e^{-v}}{u^2 + v^2}$

69. $\lim\limits_{(x,y)\to(2,0)} \dfrac{1 - \cos y}{x\,y^2}$

70. $\lim\limits_{(x,y)\to(4,0)} x^2\,y \ln xy$

71. $\lim\limits_{(x,y)\to(1,0)} \dfrac{\sin xy}{x\,y}$

72. $\lim_{(x,y)\to(0,\pi/2)} \dfrac{1-\cos xy}{4\,x^2\,y^3}$

73. $\lim_{(x,y)\to(0,2)} (2\,x\,y)^{x\,y}$

74. $\lim_{(x,y)\to(3,3)} \dfrac{x^2+2\,x\,y-6\,x+y^2-6\,y}{x+y-6}$

75. $\lim_{(x,y)\to(1,2)} \dfrac{x^2+2\,x\,y-x+y^2-y-6}{x+y-3}$

76. $\lim_{(x,y)\to(0,0)} \tan^{-1} \dfrac{\left(2+(x+y)^2+(x-y)^2\right)}{2\,e^{x^2+y^2}}$

77. **Piecewise function** Let

$$f(x,\ y)=\begin{cases}\dfrac{\sin\left(x^2+y^2-1\right)}{x^2+y^2-1} & \text{if } x^2+y^2\neq 1\\ b & \text{if } x^2+y^2=1.\end{cases}$$

Find the value of b for which f is continuous at all points in $\mathbb{R}^2$.

78. **Piecewise function** Let

$$f(x,\ y)=\begin{cases}\dfrac{1+2\,xy-\cos xy}{xy} & \text{if } x\,y\neq 0\\ a & \text{if } x\,y=0\end{cases}$$

Find the value of a for which f is continuous at all points in $\mathbb{R}^2$.

79–81. Limits using polar coordinates *Limits at* $(0,\ 0)$ *may be easier to evaluate by converting to polar coordinates. Remember that the same limit must be obtained as* $r\to 0$ *along all paths in the domain to* $(0,\ 0)$. *Evaluate the following limits or state that they do not exist.*

79. $\lim_{(x,y)\to(0,0)} \dfrac{x^2\,y}{x^2+y^2}$

80. $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x^2+y^2}}$

81. $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2+x^2\,y^2}{x^2+y^2}$

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82. **Sine limits** Verify that $\lim_{(x,y)\to(0,0)} \dfrac{\sin x+\sin y}{x+y}=1$.

83. **Nonexistence of limits** Show that $\lim_{(x,y)\to(0,0)} \frac{a\,x^m\,y^n}{b\,x^{m+n} + c\,y^{m+n}}$ does not exist when a, b, and c are nonzero real numbers and m and n are positive integers.

84. **Nonexistence of limits** Show that $\lim_{(x,y)\to(0,0)} \frac{a\,x^{2\,(p-n)}\,y^n}{b\,x^{2\,p} + c\,y^p}$ does not exist when a, b, and c are nonzero real numbers and n and p are positive integers with $p \geq n$.

85. **Filling in a function value** The domain of $f(x,\ y) = e^{-1/(x^2+y^2)}$ excludes (0, 0). How should f be defined at (0, 0) to make it continuous there?

86. **Limit proof** Use the formal definition of a limit to prove that $\lim_{(x,y)\to(a,b)} y = b$. (*Hint:* Take $\delta = \varepsilon$.)

87. **Limit proof** Use the formal definition a limit to prove that $\lim_{(x,y)\to(a,b)} (x + y) = a + b$. (*Hint:* Take $\delta = \frac{\varepsilon}{2}$.)

88. **Proof of Limit Law 1** Use the formal definition of a limit to prove that
$\lim_{(x,y)\to(a,b)} (f(x,\ y) + g(x,\ y)) = \lim_{(x,y)\to(a,.b)} f(x,\ y) + \lim_{(x,y)\to(a,b)} g(x,\ y).$

89. **Proof of Limit Law 3** Use the formal definition of a limit to prove that
$\lim_{(x,y)\to(a,b)} c\,f(x,\ y) = c \lim_{(x,y)\to(a,.b)} f(x,\ y).$