13.6 Cylinders and Quadric Surfaces

In Section 13.5, we discovered that lines in three-dimensional space are described by parametric equations (or vector equations) that are linear in the variable. We also saw that planes are described with linear equations in three variables. In this section, we take this progression one step further and investigate the geometry of three-dimensional objects described by quadratic equations in three variables. The result is a collection of *quadric surfaces* that you will encounter frequently throughout the remainder of the text. You saw one such surface in Section 13.2: A sphere with radius a centered at the origin with an equation of $x^2 + y^2 + z^2 = a^2$ is an example of a quadric surface. We also introduce a family of surfaces called *cylinders*, some of which are quadric surfaces.

Cylinders and Traces »

In everyday language, we use the word *cylinder* to describe the surface that forms, say, the curved wall of a paint can. In the context of three-dimensional surfaces, the term *cylinder* refers to a surface that is parallel to a line. In this text, we focus on cylinders that are parallel to one of the coordinate axes. Equations for such cylinders are easy to identify: The variable corresponding to the coordinate axis parallel to the cylinder is missing from the equation.

For example, working in $\mathbb{R}^3$, the equation $y = x^2$ does not include z, which means that z is arbitrary and can take on all values. Therefore, $y = x^2$ describes the cylinder consisting of all lines parallel to the z-axis that pass through the parabola $y = x^2$ in the xy-plane (**Figure 13.79**). In a similar way, the equation $y = z^2$ in $\mathbb{R}^3$ is missing the variable x, so it describes a cylinder parallel to the x-axis. The cylinder consists of lines parallel to the x-axis that pass through the curve $y = z^2$ in the yz-plane.

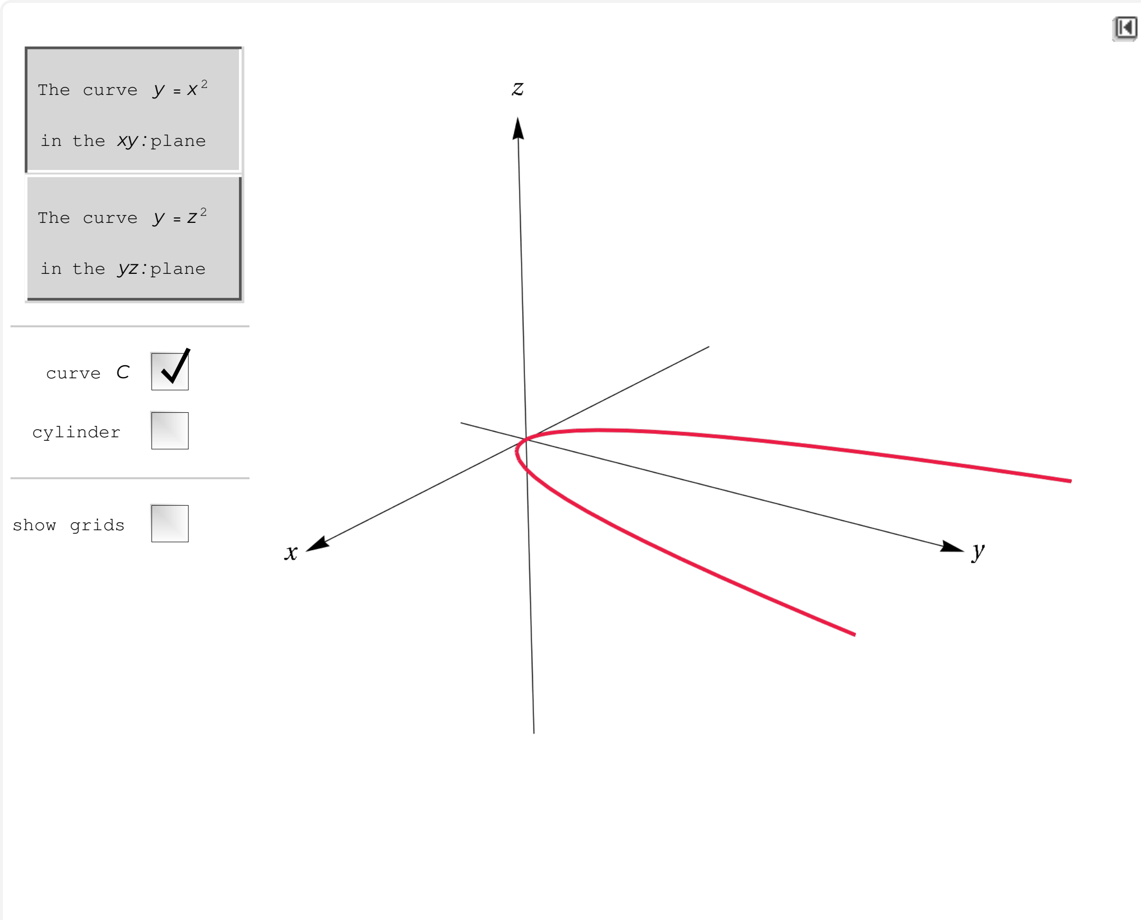


Figure 13.79

Note »

Quick Check 1 To which coordinate axis in $\mathbb{R}^3$ is the cylinder $z - 2 \ln x = 0$ parallel? To which coordinate axis in $\mathbb{R}^3$ is the cylinder $y = 4z^2 - 1$ parallel? ◆

Answer »

Graphing surfaces—and cylinders in particular—is facilitated by identifying the *traces* of the surface.

DEFINITION Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the ***xy*-trace**, the ***xz*-trace**, and the ***yz*-trace** (**Figure 13.80**).

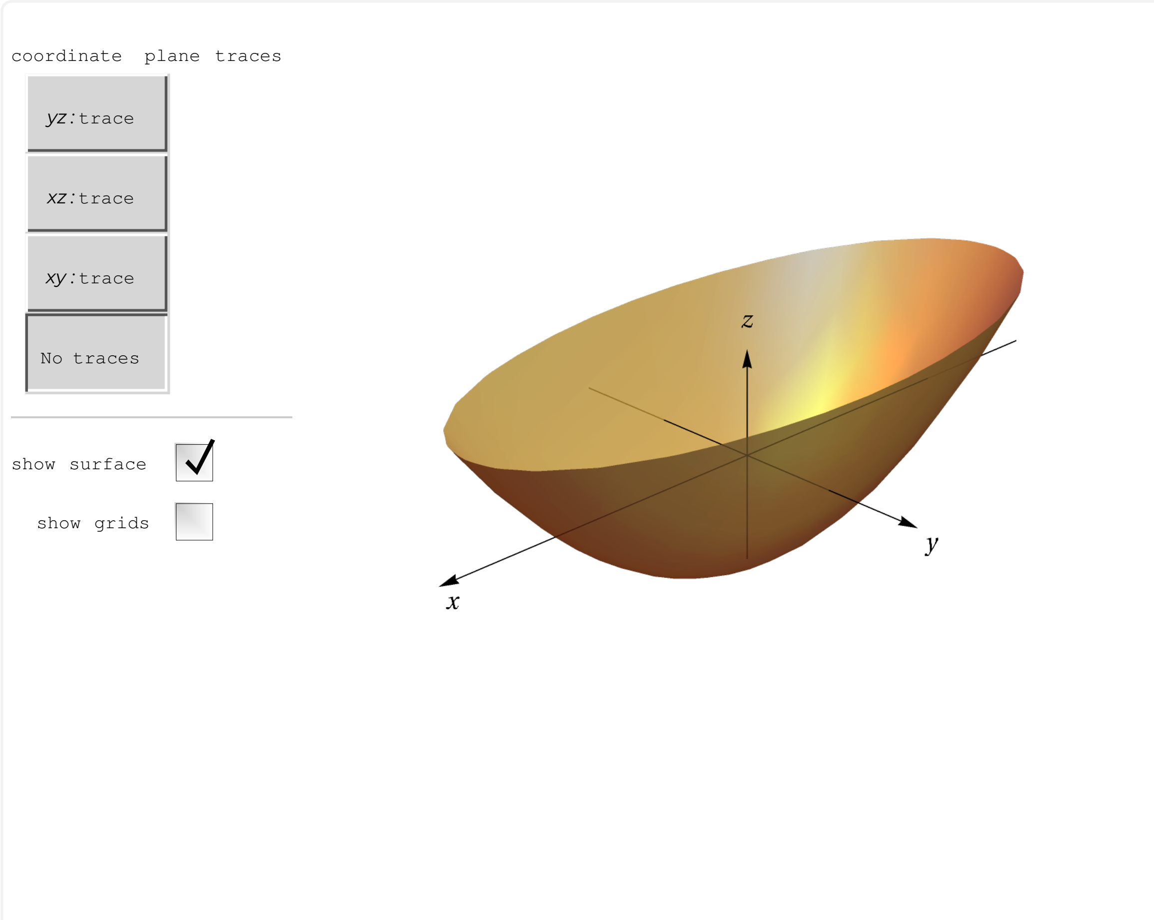


Figure 13.80

EXAMPLE 1 Graphing cylinders

Sketch the graphs of the following cylinders in $\mathbb{R}^3$. Identify the axis to which each cylinder is parallel.

a. $x^2 + 4y^2 = 16$

b. $x - \sin z = 0$

SOLUTION »

a. As an equation in $\mathbb{R}^3$, the variable z is absent. Therefore, z assumes all real values and the graph is a cylinder consisting of lines parallel to the z-axis passing through the ellipse $x^2 + 4y^2 = 16$ in the xy-plane. You can sketch the cylinder in the following steps.

1. Rewriting the given equation as $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$, we see that the trace of the cylinder in the xy-plane (the xy-trace) is an ellipse. We begin by drawing this ellipse.
2. Next draw a second trace (a copy of the ellipse in Step 1) in a plane parallel to the xy-plane.
3. Now draw lines parallel to the z-axis through the two traces to fill out the cylinder (**Figure 13.81**).

The resulting surface, called an *elliptic cylinder*, runs parallel to the z-axis.

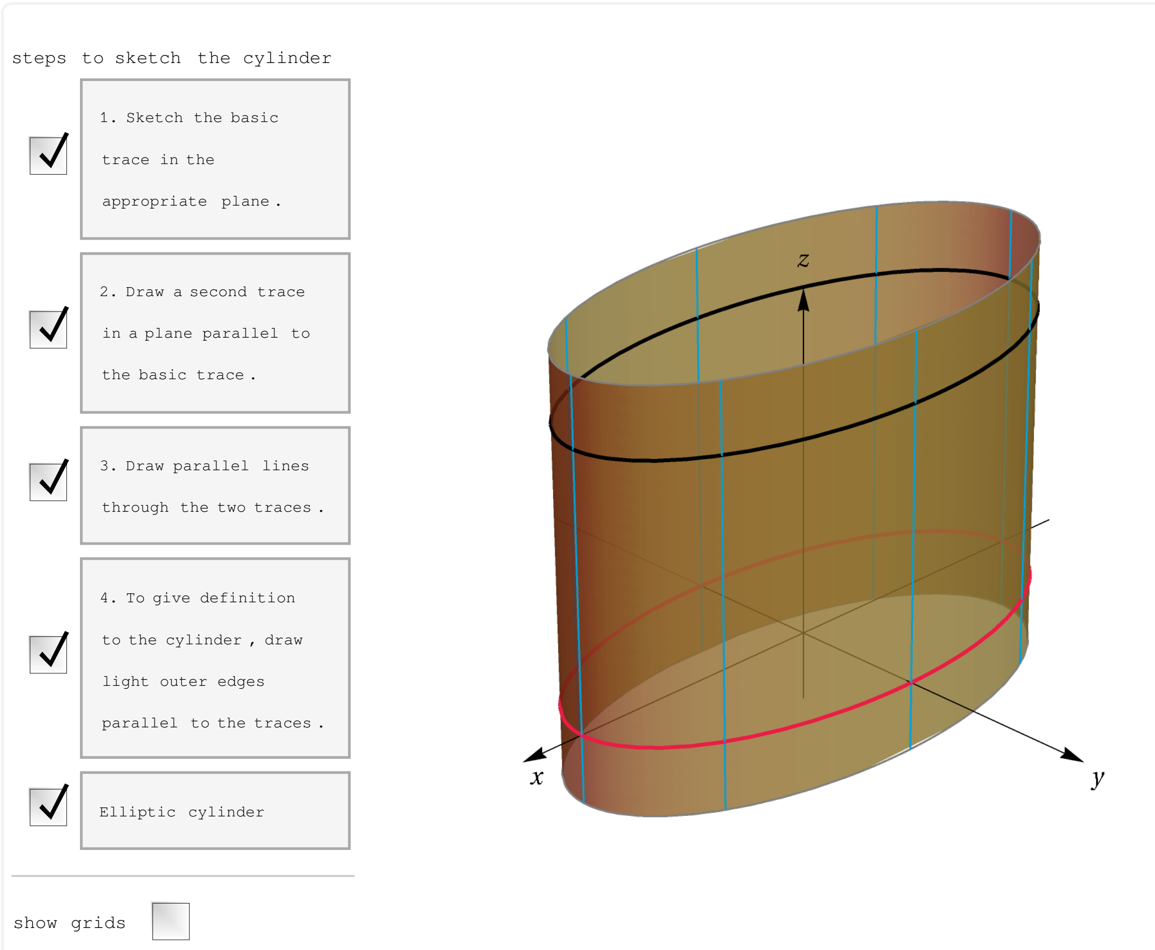


Figure 13.81

b. As an equation in $\mathbb{R}^3$, $x - \sin z = 0$ is missing the variable y. Therefore, y assumes all real values and the graph is a cylinder consisting of lines parallel to the y-axis passing through the curve $x = \sin z$ in the xz-plane. You can sketch the cylinder in the following steps.

1. Graph the curve $x = \sin z$ in the xz-plane, which is the xz-trace of the surface.

The result is a cylinder, running parallel to the y-axis, consisting of copies of the curve $x = \sin z$.

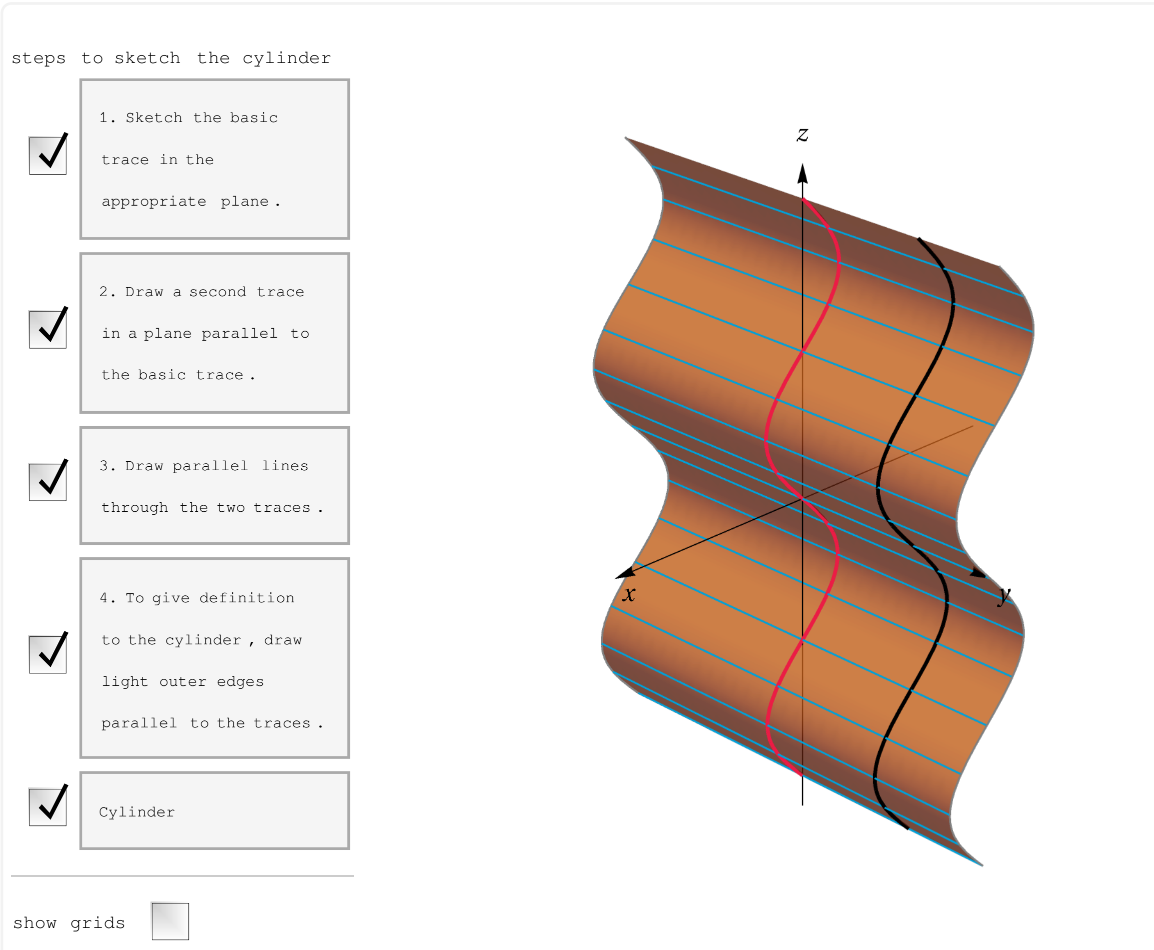


Figure 13.82

Related Exercises 8, 13 ◆

Quadric Surfaces »

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$A x^2 + B y^2 + C z^2 + D x y + E x z + F y z + G x + H y + I z + J = 0,$$

where the coefficients $A, \ldots, J$ are constants and not all of A, B, C, D, E, and F are zero. We do not attempt a detailed study of this large family of surfaces. However, a few standard surfaces are worth investigating.

Apart from their mathematical interest, quadric surfaces have a variety of practical uses. Paraboloids (defined in Example 3) share the reflective properties of their two-dimensional counterparts (Section 12.4) and are used to design satellite dishes, headlamps, and mirrors in telescopes. Cooling towers for nuclear power plants have the shape of hyperboloids of one sheet. Ellipsoids appear in the design of water tanks and gears.

Note »

> Working with quadric surfaces requires familiarity with conic sections (Section 12.4).

Making hand sketches of quadric surfaces can be challenging. Here are a few general features of quadric surfaces to keep in mind as you sketch their graphs.

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x, y, and z equal to zero in pairs in the equation of the surface and solve for the third coordinate.
2. **Traces** As illustrated in the following examples, finding traces of the surface helps visualize the surface. For example, setting $z = 0$ or $z = z_0$ (a constant) gives the traces in planes parallel to the xy-plane.
3. **Completing the figure** Sketch at least two traces in parallel planes (for example, traces with $z = 0$ and $z = \pm 1$). Then draw smooth curves that pass through the traces to fill out the surface.

Quick Check 2 Explain why the elliptic cylinder discussed in Example 1a is a quadric surface. ◆

Answer »

The equation $x^2 + 4y^2 = 16$ is a special case of the general equation for quadric surfaces; all the coefficients except A, B, and J are zero.

EXAMPLE 2 An ellipsoid

The surface defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an *ellipsoid*. Graph the ellipsoid with $a = 3$, $b = 4$, and $c = 5$.

SOLUTION »

Setting x, y, and z to zero in pairs gives the intercepts $(\pm 3, 0, 0)$, $(0, \pm 4, 0)$, and $(0, 0, \pm 5)$. Note that points in $\mathbb{R}^3$ with $|x| > 3$ or $|y| > 4$ or $|z| > 5$ do not satisfy the equation of the surface (because the left side of the equation is the sum of nonnegative terms, which cannot exceed 1). Therefore, the entire surface is contained in the rectangular box defined by $|x| \le 3$, $|y| \le 4$, and $|z| \le 5$.

The trace in the horizontal plane $z = z_0$ is found by substituting $z = z_0$ into the equation of the ellipsoid, which gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{{z_0}^2}{25} = 1 \quad \text{or} \quad \frac{x^2}{9} + \frac{y^2}{16} = 1 - \frac{{z_0}^2}{25}.$$

If $|z_0| < 5$, then $1 - \frac{{z_0}^2}{25} > 0$, and the equation describes an ellipse in the horizontal plane $z = z_0$. The largest ellipse parallel to the xy-plane occurs with $z_0 = 0$; it is the xy-trace, which is the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ with axes of length 6 and 8 (**Figure 13.83**). You can check that the yz-trace, found by setting $x = 0$, is the ellipse $\frac{y^2}{16} + \frac{z^2}{25} = 1$. The xz-trace (set $y = 0$) is the ellipse $\frac{x^2}{9} + \frac{z^2}{25} = 1$. By sketching the xy-, xz-, and yz-traces, an outline of the ellipsoid emerges.

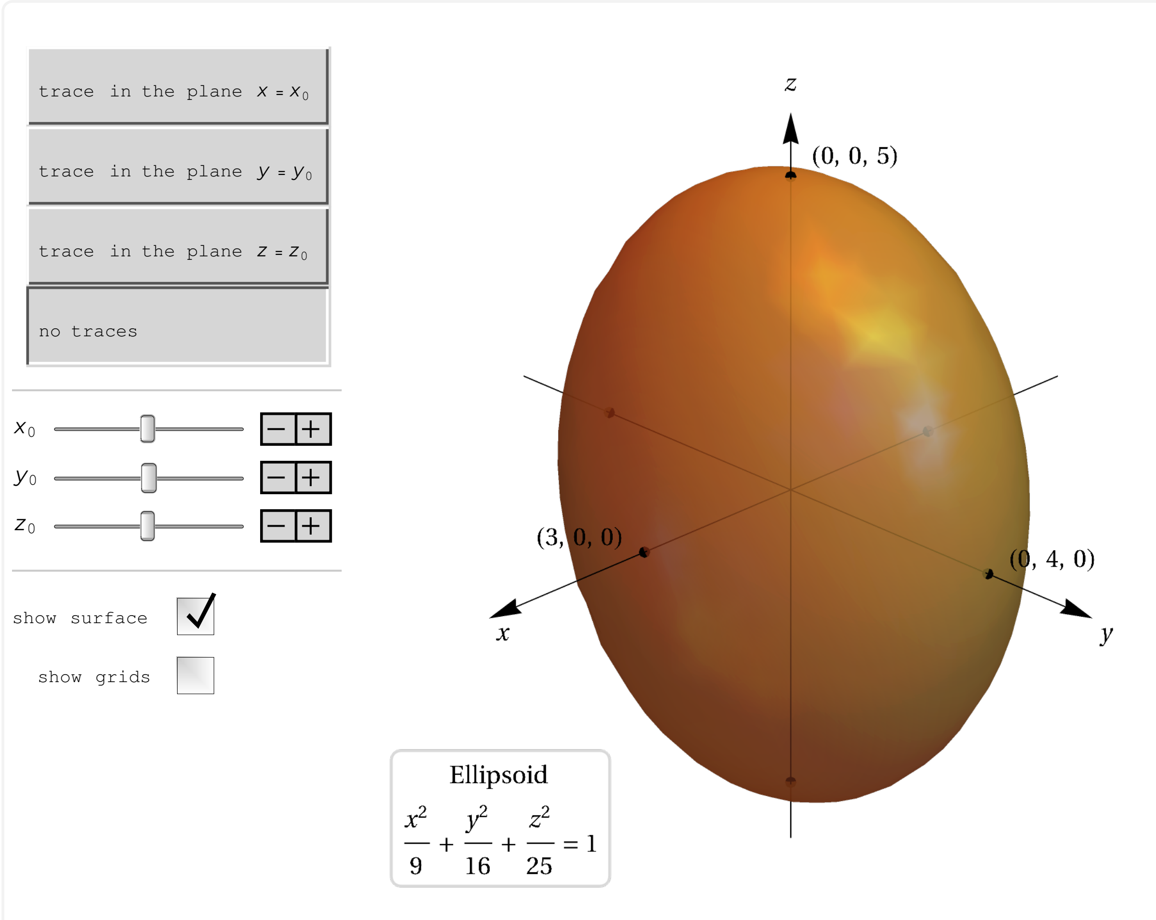


Figure 13.83

Note »

Related Exercise 29 ◆

Quick Check 3 Assume $0 < c < b < a$ in the general equation of an ellipsoid. Along which coordinate axis does the ellipsoid have its longest axis? Its shortest axis? ◆

Answer »

x-axis; z-axis

EXAMPLE 3 An elliptic paraboloid

The surface defined by the equation $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is an *elliptic paraboloid*. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

SOLUTION »

Note that the only intercept of the coordinate axes is (0, 0, 0), which is the *vertex* of the paraboloid. The trace in the horizontal plane $z = z_0$, where $z_0 > 0$, satisfies the equation $\frac{x^2}{16} + \frac{y^2}{4} = z_0$, which describes an ellipse; there are no horizontal traces when $z_0 < 0$ (**Figure 13.84**). The trace in the vertical plane $x = x_0$ is the parabola

$z = \frac{x_0^2}{16} + \frac{y^2}{4}$; the trace in the vertical plane $y = y_0$ is the parabola $z = \frac{x^2}{16} + \frac{y_0^2}{4}$.

To graph the surface, we sketch the xz-trace $z = \frac{x^2}{16}$ (setting $y = 0$) and the yz-trace $z = \frac{y^2}{4}$ (setting $x = 0$). When these traces are combined with an elliptical trace $\frac{x^2}{16} + \frac{y^2}{4} = z_0$ in a plane $z = z_0$, an outline of the surface appears.

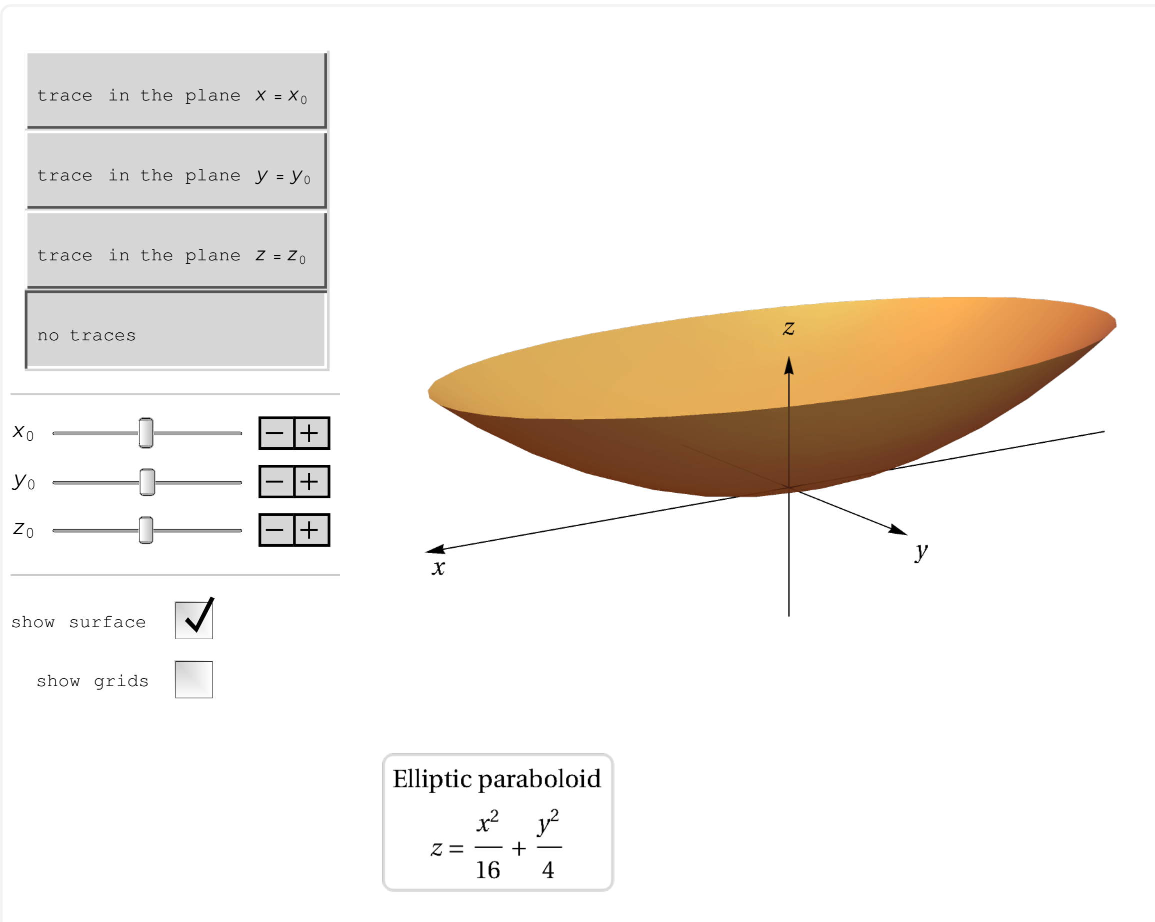


Figure 13.84

Note »

The name *elliptic paraboloid* says that the traces of this surface are parabolas and ellipses. Two of the three traces in the coordinate planes are parabolas, so it is called a paraboloid rather than an ellipsoid.

Related Exercise 32 ◆

Quick Check 4 The elliptic paraboloid $x = \frac{y^2}{3} + \frac{z^2}{7}$ is a bowl-shaped surface. Along which axis does the bowl open? ◆

Answer »

> Positive x-axis

EXAMPLE 4 A hyperboloid of one sheet

Graph the surface defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$.

SOLUTION »

The intercepts of the coordinate axes are $(0, \pm 3, 0)$ and $(\pm 2, 0, 0)$. Setting $z = z_0$, the traces in horizontal planes are ellipses of the form $\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0{}^2$. This equation has solutions for all choices of z_0, so the surface has traces in all horizontal planes. These elliptical traces increase in size as $|z_0|$ increases (**Figure 13.85**), with the smallest trace being the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the xy-plane. Setting $y = 0$, the xz-trace is the hyperbola $\frac{x^2}{4} - z^2 = 1$; with $x = 0$, the yz-trace is the hyperbola $\frac{y^2}{9} - z^2 = 1$. In fact, traces in all vertical planes are hyperbolas. The resulting surface is a *hyperboloid of one sheet*.

Note »

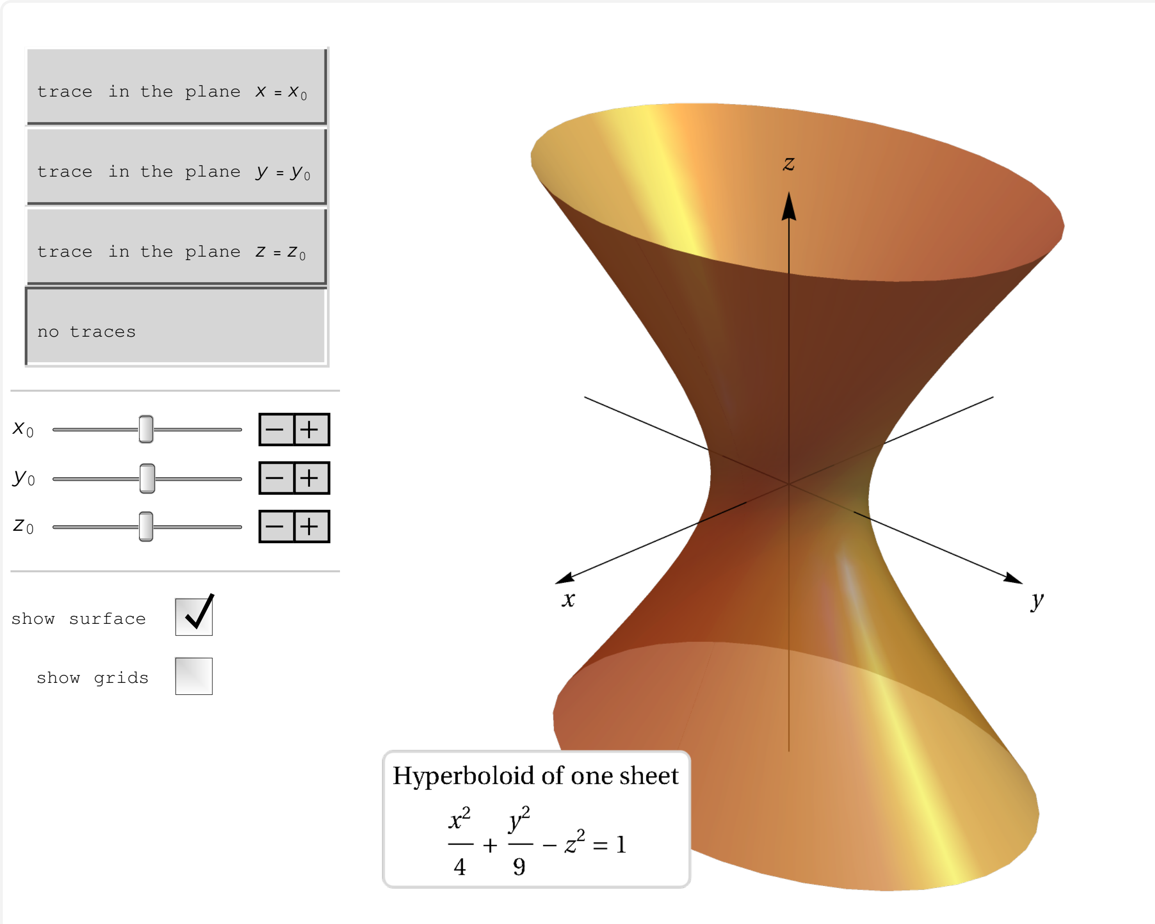


Figure 13.85

Related Exercise 33 ◆

Quick Check 5 Which coordinate axis is the axis of the hyperboloid $\frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{x^2}{c^2} = 1$? ◆

Answer »

x-axis

EXAMPLE 5 A hyperbolic paraboloid

Graph the surface defined by the equation $z = x^2 - \frac{y^2}{4}$.

SOLUTION »

Setting $z = 0$ in the equation of the surface, we see that the xy-trace consists of the two lines $y = \pm 2x$. However, slicing the surface with any other horizontal plane $z = z_0$ produces a hyperbola $x^2 - \frac{y^2}{4} = z_0$. If $z_0 > 0$, then the axis of the hyperbola is parallel to the x-axis. On the other hand, if $z_0 < 0$, then the axis of the hyperbola is parallel to the y-axis (**Figure 13.86**). Setting $x = x_0$ produces the trace $z = {x_0}^2 - \frac{y^2}{4}$, which is the equation of a parabola that opens downward in a plane parallel to the *yz*-plane. You can check that traces in planes parallel to the *xz*-plane are parabolas that open upward. The resulting surface is a *hyperbolic paraboloid*.

Note »

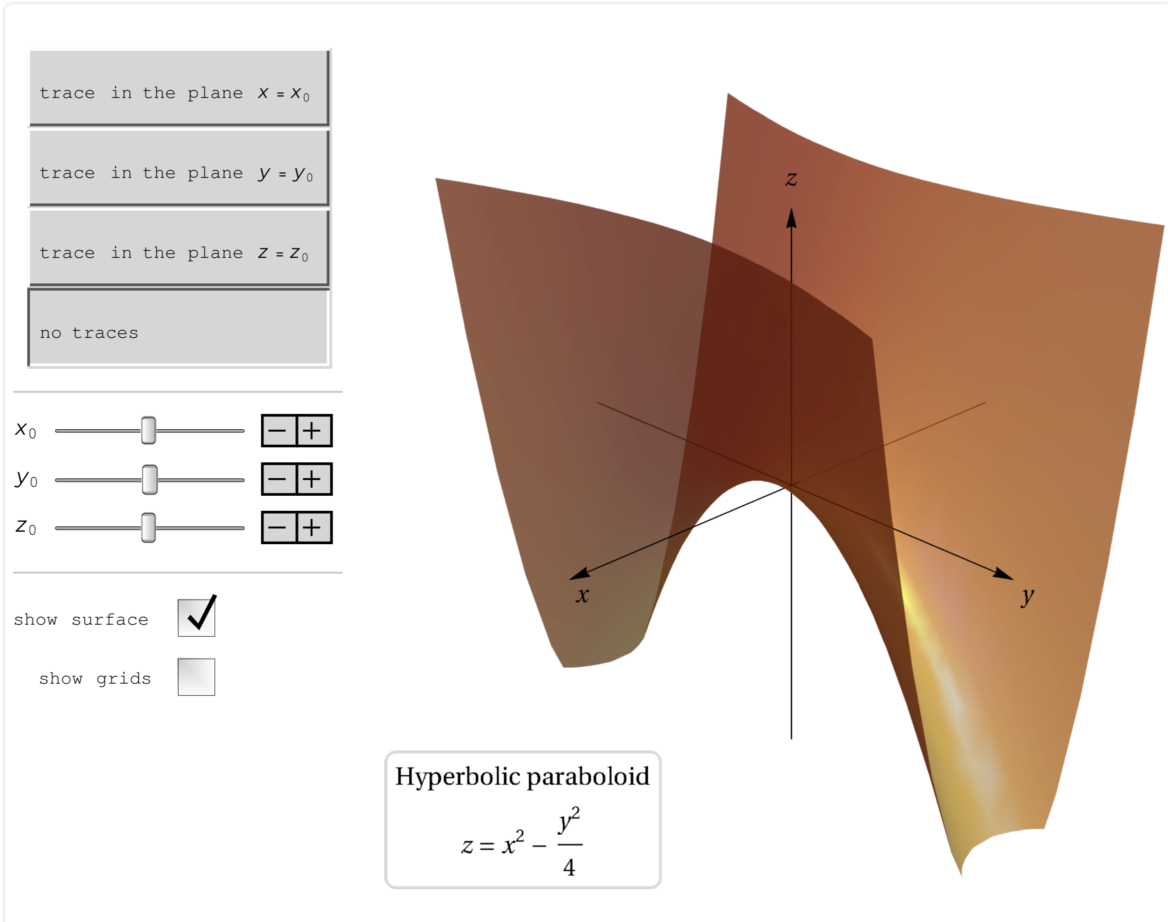


Figure 13.86

Note »

Related Exercise 35 ◆

EXAMPLE 6 Elliptic cones

Graph the surface defined by the equation $\frac{y^2}{4} + z^2 = 4x^2$.

SOLUTION »

The only point at which the surface intersects the coordinate axes is (0, 0, 0). Traces in the planes $x = x_0$ are ellipses of the form $\frac{y^2}{4} + z^2 = 4x_0{}^2$ that shrink in size as x_0 approaches 0. Setting $y = 0$, the *xz*-trace satisfies the equation $z^2 = 4x^2$ or $z = \pm 2x$, which are equations of two lines in the *xz*-plane that intersect at the origin. Setting $z = 0$, the *xy*-trace satisfies $y^2 = 16x^2$ or $y = \pm 4x$, which describe two lines in the *xy*-plane that intersect at the origin (**Figure 13.87**). The complete surface consists of two *cones* opening in opposite directions along the *x*-axis with a common vertex at the origin.

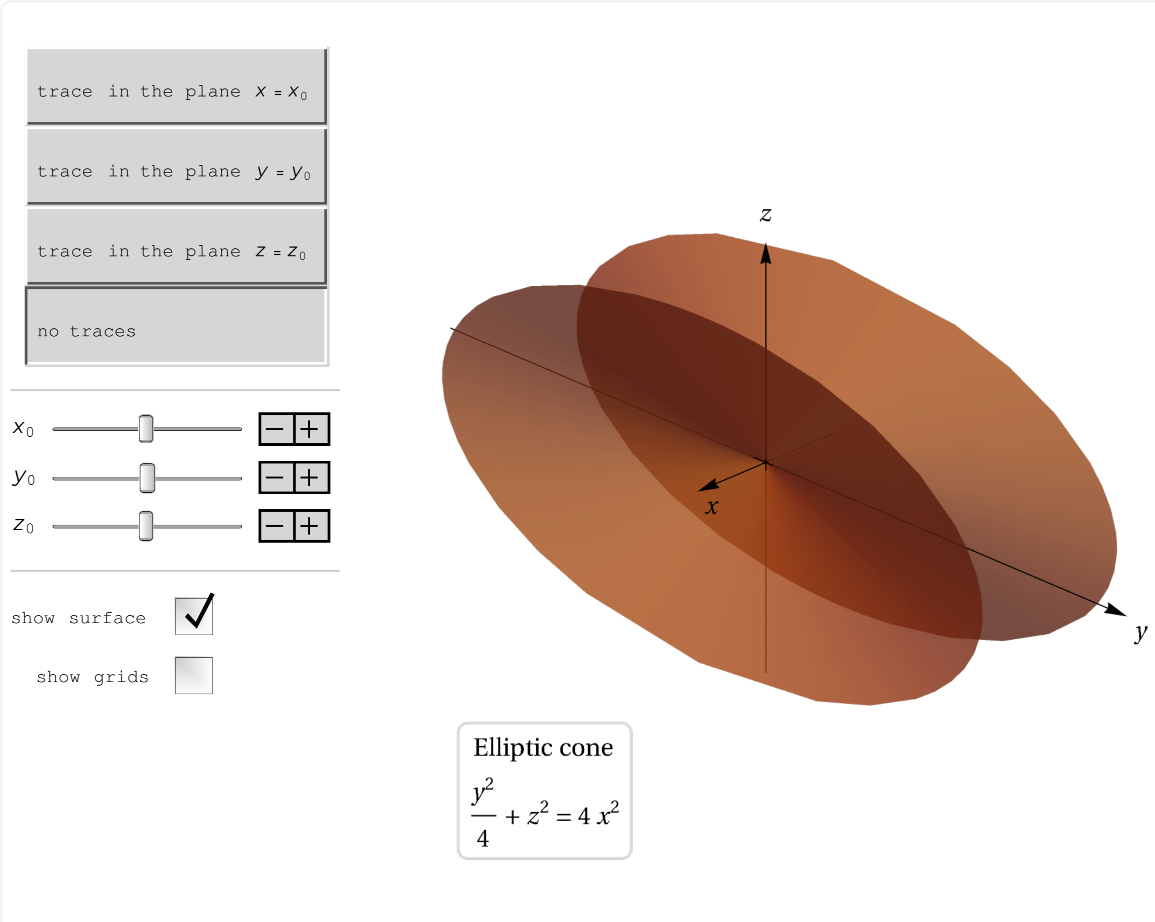


Figure 13.87

Related Exercise 38 ◆

EXAMPLE 7 A hyperboloid of two sheets

Graph the surface defined by the equation

$$-16x^2 - 4y^2 + z^2 + 64x - 80 = 0\,.$$

SOLUTION »

We first regroup terms, which yields

$$-16(\underbrace{x^2 - 4x}_{\text{complete the square}}) - 4y^2 + z^2 - 80 = 0,$$

and then complete the square in x:

$$-16\left(\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4\right) - 4y^2 + z^2 - 80 = 0.$$

Collecting terms and dividing by 16 gives the equation

$$-(x-2)^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1.$$

Note »

The equation $-x^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1$ describes a hyperboloid of two sheets with its axis on the z-axis. Therefore, the equation in Example 7 describes the same surface shifted 2 units in the positive x-direction.

Notice that if $z = 0$, the equation has no solution, so the surface does not intersect the xy-plane. The traces in planes parallel to the xz- and yz-planes are hyperbolas. If $|z_0| \geq 4$, the trace in the plane $z = z_0$ is an ellipse. This equation describes a *hyperboloid of two sheets*, with its axis parallel to the z-axis and shifted 2 units in the positive x-direction (**Figure 13.88**).

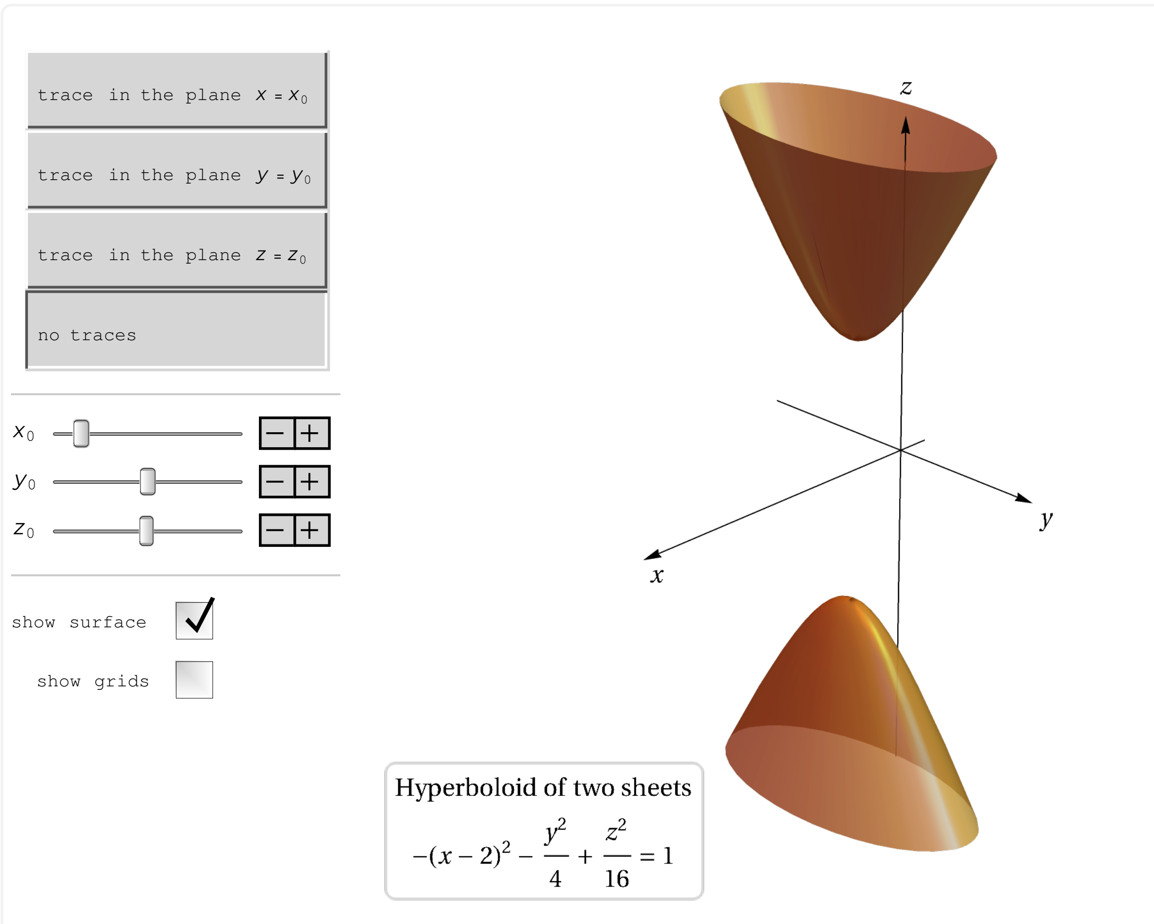


Figure 13.88

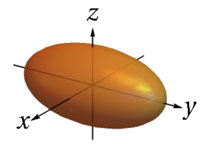
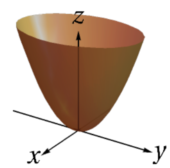
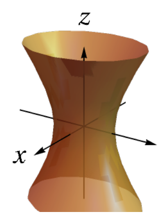
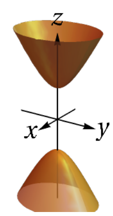
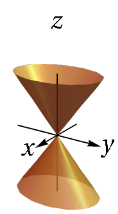
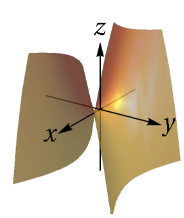
Related Exercise 56 ◆

Quick Check 6 In which variable(s) should you complete the square to identify the surface $x = y^2 + 2y + z^2 - 4z + 16$? Name and describe the surface. ◆

Answer »

Table 13.1 (where a, b, and c are nonzero real numbers) summarizes the standard quadric surfaces. It is important to note that the same surfaces with different orientations are obtained when the roles of the variables are interchanged. For this reason, Table 13.1 summarizes many more surfaces than those listed.

Table 13.1

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.	
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0\rvert>\lvert c\rvert$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.	

Exercises »

Getting Started »

Practice Exercises »

7–14. Cylinders in $\mathbb{R}^3$ *Consider the following cylinders in $\mathbb{R}^3$.*

a. *Identify the coordinate axis to which the cylinder is parallel.*

b. *Sketch the cylinder.*

7. $z = y^2$

8. $x^2 + 4y^2 = 4$

9. $x^2 + z^2 = 4$

10. $x = z^2 - 4$

11. $y - x^3 = 0$

12. $x - 2z^2 = 0$

13. $z - \ln y = 0$

14. $x - \frac{1}{y} = 0$

15–20. Identifying quadric surfaces *Identify the following quadric surfaces by name. Find and describe the xy-, xz-, and yz-traces, when they exist.*

15. $25x^2 + 25y^2 + z^2 = 25$

16. $25x^2 + 25y^2 - z^2 = 25$

17. $25x^2 + 25y^2 - z = 0$

18. $25x^2 - 25y^2 - z = 0$

19. $-25x^2 - 25y^2 + z^2 = 25$

20. $-25x^2 - 25y^2 + z^2 = 0$

21–28. Identifying surfaces *Identify the following surfaces by name.*

21. $y = 4z^2 - x^2$

22. $-y^2 - 9z^2 + \frac{x^2}{4} = 1$

23. $y = \frac{x^2}{6} + \frac{z^2}{16}$

24. $z^2 + 4y^2 - x^2 = 1$

25. $y^2 - z^2 = 2$

26. $x^2 + 4y^2 = 1$

27. $9x^2 + 4z^2 - 36y = 0$

28. $9y^2 + 4z^2 - 36x^2 = 0$

29–51. Quadric surfaces *Consider the following equations of quadric surfaces.*

- ***a.*** *Find the intercepts with the three coordinate axes, when they exist.*
- ***b.*** *Find the equations of the xy-, xz-, and yz-traces, when they exist.*
- ***c.*** *Identify and sketch a graph of the surface.*

29. $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$

30. $4x^2 + y^2 + \frac{z^2}{2} = 1$

31. $x = y^2 + z^2$

32. $z = \frac{x^2}{4} + \frac{y^2}{9}$

33. $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 1$

34. $\frac{y^2}{4} + \frac{z^2}{9} - \frac{x^2}{16} = 1$

35. $z = \frac{x^2}{9} - y^2$

36. $y = \frac{x^2}{16} - 4z^2$

37. $x^2 + \frac{y^2}{4} = z^2$

38. $4y^2 + z^2 = x^2$

39. $\frac{x^2}{3} + 3y^2 + \frac{z^2}{12} = 3$

40. $\frac{x^2}{6} + 24y^2 + \frac{z^2}{24} - 6 = 0$

41. $9x - 81y^2 - \frac{z^2}{4} = 0$

42. $2y - \frac{x^2}{8} - \frac{z^2}{18} = 0$

43. $\frac{y^2}{16} + 36z^2 - \frac{x^2}{4} - 9 = 0$

44. $9z^2 + x^2 - \frac{y^2}{3} - 1 = 0$

45. $5x - \frac{y^2}{5} + \frac{z^2}{20} = 0$

46. $6y + \frac{x^2}{6} - \frac{z^2}{24} = 0$

47. $\frac{z^2}{32} + \frac{y^2}{18} = 2x^2$

48. $\frac{x^2}{3} + \frac{z^2}{12} = 3y^2$

49. $-x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$

50. $-\frac{x^2}{6} - 24y^2 + \frac{z^2}{24} - 6 = 0$

51. $-\frac{x^2}{3} + 3y^2 - \frac{z^2}{12} = 1$

52. Describe the relationship between the graphs of quadric surfaces $x^2 + y^2 - z^2 + 2z = 1$ and $x^2 + y^2 - z^2 = 0$, and state the names of the surfaces.

53. Describe the relationship between the graphs of $x^2 + 4y^2 + 9z^2 = 100$ and $x^2 + 4y^2 + 9z^2 + 54z = 19$, and state the names of the surfaces.

54–58. Identifying surfaces *Identify and briefly describe the surfaces defined by the following equations.*

54. $x^2 + y^2 + 4z^2 + 2x = 0$

55. $9x^2 + y^2 - 4z^2 + 2y = 0$

56. $-x^2 - y^2 + \frac{z^2}{9} + 6x - 8y = 26$

57. $\frac{x^2}{4} + y^2 - 2x - 10y - z^2 + 41 = 0$

58. $z = -x^2 - y^2$

59. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The graph of the equation $y = z^2$ in $\mathbb{R}^3$ is both a cylinder and quadric surface.

b. The xy-traces of the ellipsoid $x^2 + 2y^2 + 3z^2 = 16$ and the cylinder $x^2 + 2y^2 = 16$ are identical.

c. Traces of the surface $y = 3x^2 - z^2$ in planes parallel to the xy-plane are parabolas.

d. Traces of the surface $y = 3x^2 - z^2$ in planes parallel to the xz-plane are parabolas.

e. The graph of the ellipsoid $x^2 + 2y^2 + 3(z-4)^2 = 25$ is obtained by shifting the graph of the ellipsoid $x^2 + 2y^2 + 3z^2 = 25$ down 4 units.

60. Matching graphs with equations Match equations a–f with surfaces A–F.

a. $y - z^2 = 0$

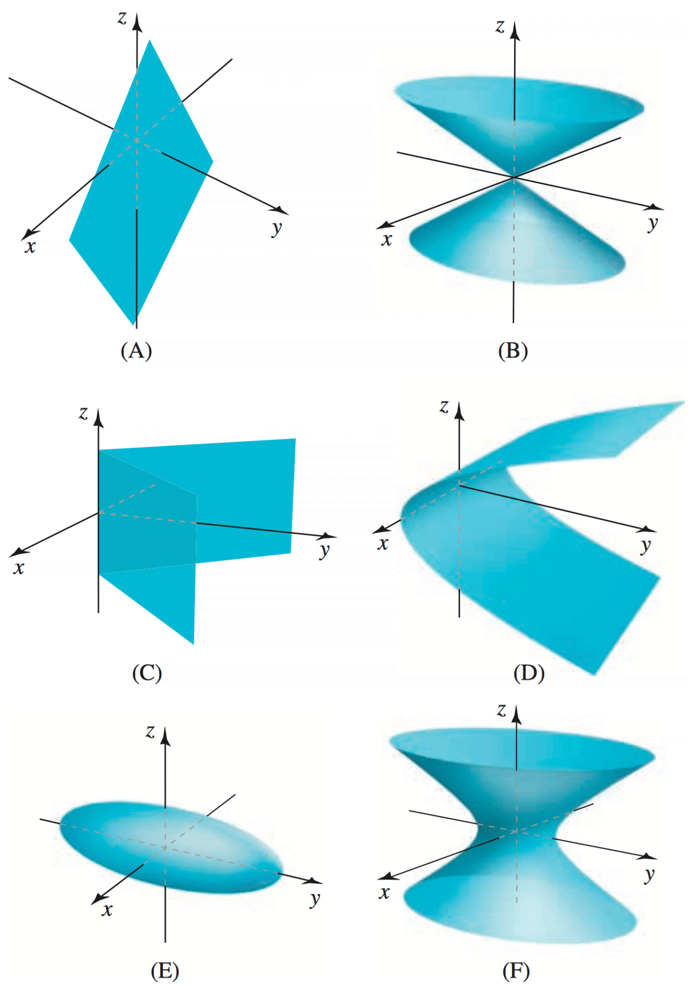
b. $2x + 3y - z = 5$

c. $4x^2 + \dfrac{y^2}{9} + z^2 = 1$

d. $x^2 + \dfrac{y^2}{9} - z^2 = 1$

e. $x^2 + \dfrac{y^2}{9} = z^2$

f. $y = |x|$



Explorations and Challenges »

61. Solids of revolution Which of the quadric surfaces in Table 13.1 can be generated by revolving a curve in one of the coordinate planes about a coordinate axis, assuming $a = b = c \neq 0$?

62. Solids of revolution Consider the ellipse $x^2 + 4y^2 = 1$ in the xy-plane.

a. If this ellipse is revolved about the x-axis, what is the equation of the resulting ellipsoid?

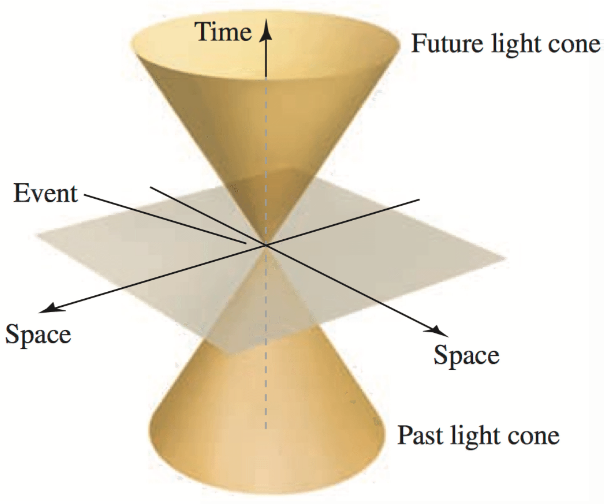
b. If this ellipse is revolved about the y-axis, what is the equation of the resulting ellipsoid?

63. Volume Find the volume of the solid that is bounded between the planes $z = 0$ and $z = 3$ and the cylinders $y = x^2$ and $y = 2 - x^2$.

64. Light cones The idea of a *light cone* appears in the Special Theory of Relativity. The xy-plane (see figure) represents all of three-dimensional space, and the z-axis is the time axis (t-axis). If an event E occurs at the origin, the interior of the future light cone ($t > 0$) represents all events in the future that could be affected by E, assuming that no signal travels faster than the speed of light. The interior of the past light cone ($t < 0$) represents all events in the past that could have affected E, again assuming no signal travels faster than the speed of light.

a. If time is measured in seconds and distance (x and y) is measured in light-seconds (the distance light travels in 1 s), the light cone makes a $45°$ angle with the xy-plane. Write the equation of the light cone in this case.

b. Suppose distance is measured in meters and time is measured in seconds. Write the equation of the light cone in this case, given that the speed of light is 3×10^8 m/s.



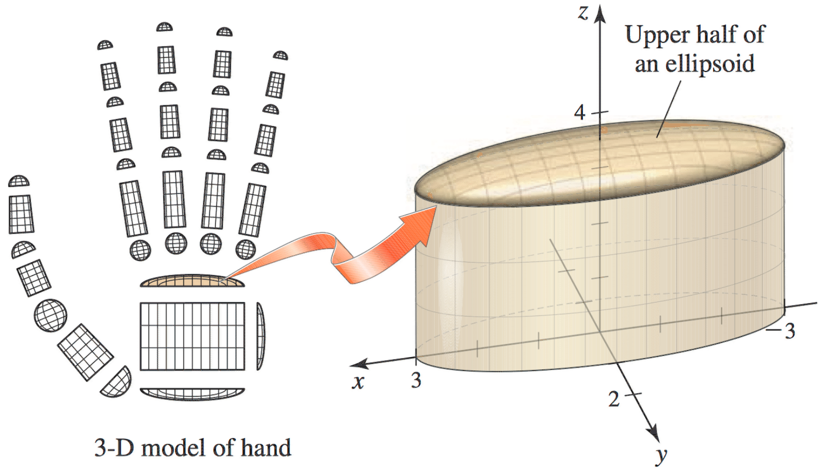
65. Designing an NFL football A *prolate spheroid* is a surface of revolution obtained by rotating an ellipse about its major axis.

a. Explain why one possible equation for a prolate spheroid is $\dfrac{x^2+z^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $b > a > 0$.

b. According to the National Football League (NFL) rulebook, the shape of an NFL football is required to be a prolate spheroid with a long axis between 11 and 11.25 inches long and a short circumference (i.e. the circumference of the xz-trace) between 21 and 21.25 inches. Find an equation for the shape of the football if the long axis is 11.1 inches and the short circumference is 21.1 inches.

66. Hand tracking Researchers are developing hand tracking software that will allow computers to track and recognize detailed hand movements for better human-computer interaction. One three-dimensional hand model under investigation is constructed from a set of truncated quadrics (see figure). For example, the palm of the hand consists of a truncated elliptic cylinder, capped off by the upper half of an ellipsoid. Suppose the palm of the hand is modeled by the truncated cylinder $\dfrac{4x^2}{9}+4y^2=1$, for $0 \le z \le 3$. Find an equation of the upper half of an ellipsoid, whose bottom corresponds with the top of the cylinder if the distance from the top of the truncated cylinder to the top of the ellipsoid is $1/2$.

(*Source: Computer Vision and Pattern Recognition,* 2, Dec 2001)



67. Designing a snow cone A surface, having the shape of an oblong snow cone, consists of a truncated cone, $\frac{x^2}{2} + y^2 = \frac{z^2}{8}$, for $0 \le z \le 3$, capped off by the upper half of an ellipsoid. Find an equation for the upper half of the ellipsoid so that the bottom edge of the truncated ellipsoid and the top edge of the cone coincide, and the distance from the top of the cone to the top of the ellipsoid is $3/2$.

68. Designing a glass The outer, lateral side of a 6-inch-tall glass has the shape of the truncated hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, for $0 \le z \le 6$. If the base of the glass has a radius of 1 inch and the top of the glass has a radius of 2 inches, find the values of a^2, b^2, and c^2 that satisfy these conditions. Assume horizontal traces of the glass are circular.