Additional Problems
Math 250, Spring 2024 – Jacek Polewczak

Lagrange's multipliers

Problem 1.
Use the method of Lagrange's multipliers to find the minimum of $f(x,y) = x^2 + 4xy + y^2$, subject to the constraint $x - y - 6 = 0$.

Solution

$f(x,y) = x^2 + 4xy + y^2$, $\quad g(x,y) = x - y - 6$. $\quad \nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = x - y - 6 = 0$ is equivalent to

$$< 2x + 4y, 4x + 2y >= \lambda < 1, -1 >, \quad x - y = 6 \quad \Longrightarrow \quad 2x + 4y = \lambda, \quad 4x + 2y = -\lambda, \quad x - y = 6 \Longrightarrow$$
$$2x + 4y = -4x - 2y, \quad x - y = 6 \quad \Longrightarrow \quad 6x + 6y = 0, \quad x - y = 6 \quad \Longrightarrow \quad 6x + 6(x-6) = 0 \quad \Longrightarrow \quad x = 3$$

and $y = 3 - 6 = -3$, with the corresponding $\lambda = -6$. Critical point is $(3, -3)$ (with the corresponding $\lambda = -6$) and the minimum is $f(3, -3) = -18$.

Problem 2.
Use the method of Lagrange's multipliers to find the least distance between the origin and the plane $x + 3y - 2z = 4$.

Solution

Minimize the square of the distance to the plane, $f(x,y,z) = x^2 + y^2 + z^2$, subject to $g(x,y,z) = x + 3y - 2z - 4 = 0$.
$\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = x + 3y - 2z - 4 = 0$ is equivalent to

$$< 2x, 2y, 2z >= \lambda < 1, 3, -2 >, \quad x + 3y - 2z - 4 = 0 \quad \Longrightarrow \quad 2x = \lambda, \quad 2y = 3\lambda, \quad 2z = -2\lambda, \quad x + 3y - 2z = 4.$$

Eliminating $\lambda = -z$, we solve the linear system for x, y, z:

$$2x = -z, \quad 2y = -3z, x + 3y - 2z = 4$$

The solution is $(2/7, 6/7, -4/7)$ (with the corresponding $\lambda = 4/7$). The nature of the problem indicates that this will give a minimum rather than a maximum (**WHY ???**). The least distance to the plane is

$$\left[f\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right) \right]^{\frac{1}{2}} = \left(\frac{8}{7}\right)^{\frac{1}{2}} \approx 1.0690.$$

Double integrals over general regions, section 16.2

Problem 1.

Find the volumes of the indicated solids by an iterated integration.

(a) The tetrahedron bounded by the coordinate planes and the plane $3x + 4y + z - 12 = 0$.

(b) The solid bounded by the parabolic cylinder $x^2 = 4y$ and the planes $z = 0$ and $5y + 9z - 45 = 0$.

Solution

$$\text{(a)} \quad \text{Volume} = \int_0^4 \left[\int_0^{(-3/4)x+3} (12 - 3x - 4y)\, dy \right] dx = 24.$$

(b)

Solution 1

The plane $5y + 9z = 45$ intersects the xy-plane in the line $y = 9$, so the region E (in xy plane) is

$$E = \{(x,y,z) : -6 \le x \le 6,\ x^2/4 \le y \le 9,\ 0 \le z \le 5 - (5/9)y\}$$

and

$$\text{Volume} = \int_{-6}^{6} \int_{x^2/4}^{9} \int_{0}^{5-(5/9)y} 1 \cdot dz dy dx = \int_{-6}^{6} \int_{x^2/4}^{9} \left(5 - \frac{5}{9}y\right) dy dx = 144.$$

Solution 2

The plane $5y + 9z = 45$ intersects the xy-plane in the line $y = 9$, so the region E (in xy plane) is

$$E = \{(x,y,z) : -2\sqrt{y} \le x \le 2\sqrt{y},\ 0 \le y \le 9,\ 0 \le z \le 5 - (5/9)y\}$$

and

$$\text{Volume } = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \int_0^{5-(5/9)y} 1 \cdot \, dz dx dy = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \left(5 - \frac{5}{9}y\right) \, dx dy = 144.$$

Triple integrals section 16.4

Problem 1.

Evaluate the iterated integral

$$\int_0^{\frac{\pi}{2}} \int_0^z \int_0^y \sin(x+y+z) \, dx dy dz$$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^z \int_0^y \sin(x+y+z) \, dx dy dz = \int_0^{\pi/2} \int_0^z \Big[-\cos(2y+z) + \cos(y+z) \Big] dy dz = \int_0^{\pi/2} \left[-\frac{\sin(3z)}{2} + \sin(2z) - \frac{\sin z}{2} \right] dz = \frac{1}{3}.$$