Math 250: Final Review, Fall 2023

13.5) Lines and Planes in Space. 1-10, 11-26, 27-30, 31-37, 43-58, 61-64, 65-68, 71-72, 73-76, 77-80.

1. (a) $\vec{r}(t) = (1 + 2t, -1 + 3t, 2 - t)$
   (b) $\vec{r}(t) = \langle 2t, 2 - t, 3 - 2t \rangle$.
   (c) $\vec{r}(t) = \langle -1 - 2t, 2 + t, t \rangle$.
2. (a) $5(x - 2) - (y - 3) + 3(z - 1) = 0$ or $5x - y + 3z = 10$.
   (b) $(x - 1) + (y - 2) - 2z = 0$ or $x + y - 2z = 3$.
   (c) $x - y + z = 2$.

13.6) Cylinders and Quadric Surfaces. 1-6, 7-12, 15-20, 21-28, 29-51, 54-58, 60.

1. (a) Elliptic cylinder parallel to y=axis.
   (b) Parabolic cylinder parallel to x-axis.
2. (a) xy: hyperbola, xz: hyperbola, yz: no solution. Hyperboloid of two sheets.
   (b) xy: hyperbola, xz: hyperbola, yz: ellipse. Hyperboloid of one sheet.
   (c) $xy : y = \pm 3x$. $xz : (0, 0)$, $yz : y = \pm 3z$. Cone.

15.1) Graphs and Level Curves. 25-33, 34, 35, 36-43, 74-77.

1. (a) plane.
   (b) elliptic paraboloid.
   (c) hyperbolic paraboloid.
2. (a) lines.
   (b) $c = 0, y = \pm x$. $c = -1, 1, 2$, hyperbolas.
   (c) $c = 2, (0, 0)$. $c = -1, 0, 1$ circles.

15.2) Limits and Continuity. 10-12, 13-27, 29-34, 35-50, 52-53, 62-67, 71.

1. (a) Function is continuous at $(-1, 2)$ so $\displaystyle\lim_{(x,y)\to(-1,2)} f(x, y) = f(-1, 2) = 0$.
   (b) $\lim_{(x,y)\to(3,-1)} \frac{(x+3y)(x-3y)}{y(x+3y)} = \lim_{(x,y)\to(3,-1)} \frac{x-3y}{y} = -6$.

2. (a) $\lim_{(t,0)\to(0,0)} f(x,y) = \frac{1}{2}$, $\lim_{(0,t)\to(0,0)} f(x,y) = 2$.

   (b) $\lim_{(t,0)\to(0,0)} f(x,y) = 0$, $\lim_{(t,t)\to(0,0)} f(x,y) = \frac{1}{2}$.

15.3) Partial Derivatives. 1-9, 11-14, 15-30, 32-34, 38-46, 48-53, 54-59.

1. (a) $\lim_{h\to 0} \frac{(3(x+h)-y)-(3x-y)}{h} = 3$, $\lim_{h\to 0} \frac{(3x-(y+h))-(3x-y)}{h} = -1$.

   (b) $\lim_{h\to 0} \frac{(x+h)y^2-xy^2}{h} = y^2$, $\lim_{h\to 0} \frac{x(y+h)^2-xy^2}{h} = 2xy$.

2. (a) $f_{xx} = -y^2\sin(xy)$, $f_{xy} = cos(xy) - xy\sin(xy)$, $f_{yx} = \cos(xy) - xy\sin(xy)$, $f_{yy} = -x^2\sin(xy)$.

   (b) $f_{xx} = \frac{2(y^3-x^2)}{(x^2+y^3)^2}$, $f_{xy} = f_{yx} = \frac{-6xy^2}{(x^2+y^3)^2}$, $f_{yy} = \frac{6x^2y-3y^4}{(x^2+y^3)^2}$

15.4) The Chain Rule. 9-18, 19-26, 27-28, 29-30, 35-40, 57-59, 65, 67-69, 72-73, 75.

1. $\frac{dw}{dt} = 3x^2y^2(2t) + 2x^3y(1+e^{2-t}) = 3(25)(1)(4) + 2(5^3)(1)(2) = 800$.

2. $\frac{\partial w}{\partial u} = \frac{1}{x}(2) - \frac{1}{y}\frac{v^2}{u^2} = 2 - \frac{1}{2} = \frac{3}{2}$.
   $\frac{\partial w}{\partial v} = \frac{1}{x}(3) + \frac{1}{y}\frac{2v}{u} = 3 - 2 = 1$.

3. (a) $\frac{\partial F}{\partial x}(0) + \frac{\partial F}{\partial y}(\frac{\partial y}{\partial z}) + \frac{\partial F}{\partial z}(1) = 0 \Rightarrow \frac{\partial y}{\partial z} = -\frac{\partial F}{\partial z}/\frac{\partial F}{\partial y}$.

   (b) $\frac{\partial y}{\partial z} = \frac{-xye^{xz}-xye^{yz}}{e^{xz}+xze^{yz}}$

4. $\frac{\partial z}{\partial x} = \frac{-2xy-z^3}{y^2+3xz^2}$, $\frac{\partial z}{\partial y} = \frac{-2yz}{y^2+3xz^2}$.

5. $\frac{dE}{dt} = \frac{1}{2}2v\frac{dv}{dt} + g\sin(\theta)\frac{d\theta}{dt} = -g\sin(\theta)v + g\sin(\theta)v = 0$.

15.5) Directional Derivatives and Gradient. 1-10, 11-12. 13-20, 21-30, 31-36, 43-44, 47-50, 59-64, 69-72, 74, 75-78, 81, 82, 85.

1. $(\nabla f)(-1,2,1) = (-4,1,3)$. $(D_{\vec{u}}f)(-1,2,1) = \frac{8}{3} + \frac{2}{3} + 1$.

2. (a) $\sqrt{145}$, $\langle \frac{8}{\sqrt{145}}, \frac{-9}{\sqrt{145}} \rangle$.

   (b) $\langle 9, 8 \rangle$.

3. $f(x,y) = xy$.

   (a) Graph the level set of f through the point $(2,1)$.

   (b) Include the vector $\nabla f(2,1)$ on your graph.

   (c) $(x-2) + 2(y-1) = 0 \Rightarrow y = -\frac{1}{2}x + 2$.

   (d) The vector $\nabla f(2,1)$ is perpendicular to the level set of f at $(2,1)$.

15.6) Tangent Planes. 3-4, 9, 11, 13-28, 29-32, 54-56.

1. Find the tangent plane to the surface at the given point.
   (a) $-4(x+1) - 1(x-2) + 4(x-2) = 0$.
   (b) $1(x-3) + 6(y-0) + 0(z-2) = 0$ or $x + 6y = 3$.
   (c) $\frac{1}{2}(x-3) - 2(y-1) - \frac{1}{2}(z+1) = 0$.
2. Find the tangent plane to the surface at the given point.
   (a) $\frac{5}{3}(x-5) - \frac{4}{3}(x-4) - (z-3) = 0$.
   (b) $0(x-2) + 2(y-0) + 1(z-3) = 0$ or $2y - 3z = 3$.
   (c) $3(x-2) - (y-1) + 4(z+1) = 0$.

15.7) Maximum/Minimum Problems. 9-12, 13-22, 23-37, 41-42, 43-46, 62-66, 71.

1. Find any critical points and classify each as relative maximum, relative minimum, or saddle.
   (a) Local max at $(0,0)$.
   (b) local min at $(0,1)$
   (c) Saddle at $(1,2)$.
   (d) Local min $(\pm 1, 0)$. Local max $(0,0)$.
2. min $x^2 + y^2 + z^2 = 2$ at $(x,y) = (1,0)$.
3. max $V = 2$ at $(x,y) = (1,1)$.

15.8) Lagrange Multipliers. 3-4, 5-6, 7-23, 26, 27-36.

1. max $= 3\sqrt{3}$, min $= -3\sqrt{3}$.
2. Area $= 32$.
3. $D = \sqrt{30}$.

16.1) Double Integrals, Rectangular Regions. 1-3, 5-6, 7-24, 25-35, 36-39, 40-45, 46-50, 53-54.

1. $\frac{1}{4}\int_0^2 \int_1^3 2y - x \, dydx = 3$.
2. (a) $\int_0^{\ln(3)} \int_0^1 xye^{xy^2} dydx = 1 - \frac{1}{2}\ln(3)$.

(b) $\int_0^3 \int_0^1 \frac{y}{\sqrt{1+xy}} dxdy = \frac{10}{3}$.

16.2) Double Integrals, General Regions. 5-8, 9-10, 11-27, 28-34, 35-42, 43-53, 57-62, 63-68, 69, 70, 71, 73-80, 85-90, 95-96, 99-102.

1. $\int_1^2 \int_{y^2+3}^{3y+1} 2y - 1 dxdy = \frac{1}{3}$.
2. (a) $\int_0^3 \int_0^{2y} e^{y^2}\, dxdy = e^9 - 1$.
   (b) $\int_{-2}^2 \int_{x^2}^4 e^{12x-x^3}\, dydx = \frac{1}{3}(e^{16} - e^{-16})$

16.3) Double Integrals in Polar Coordinates. 7-10, 11-14, 15-18, 19-20, 21-30, 31-40, 42, 44-46,47, 49-50, 53-54, 57-60, 65-68.

1. Evaluate the integral by converting to polar.
   (a) $\int_0^4 \int_0^{\frac{\pi}{2}} (r\cos(\theta))(r\sin(\theta))r\, drd\theta = 32$.
   (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(1+r^2)^2} r\, drd\theta = \frac{2\pi}{5}$.
   (c) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sec(\theta)}^{2\sec(\theta)} r\, drd\theta = 3$.

16.4) Triple Integrals. 4-6, 7-14, 15-29, 30-35, 36-37, 38-46, 47-50, 51-54, 57-58, 62-63, 67-70.

1. $\int_0^2 \int_0^{1-\frac{1}{4}x^2} \int_0^{4-4y-x^2} dzdydx$
   $\int_0^1 \int_0^{\sqrt{4-4y}} \int_0^{4-4y-x^2} dzdxdy$
   $\int_0^2 \int_0^{4-x^2} \int_0^{1-\frac{1}{4}z-\frac{1}{4}x^2} dydzdx$
   $\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{1-\frac{1}{4}z-\frac{1}{4}x^2} dydxdz$
   $\int_0^4 \int_0^{1-\frac{1}{4}z} \int_{\sqrt{4-4y-z}} dxdydz$
   $\int_0^1 \int_0^{4-4y} \int_{\sqrt{4-4y-z}} dxdzdy$
2. (a) $\int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} dydxdz = \frac{256}{30}$.
   (b) $\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{2-\frac{1}{2}y} dzdxdy = \frac{256}{30}$.

16.5) Cylindrical and Spherical Coordinates. 3-4, 9-10, 11-14, 15-22, 23-28, 29-34, 35-38, 41-47, 48-54, 58-61, 62-63, 64-65, 66-72, 77-79.

1. $\int_0^{2\pi} \int_1^2 \int_0^{4-r^2} \frac{z}{r^{\frac{3}{2}}} r dzdrd\theta = \frac{7}{3}\pi$.
2. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^4 \cos(\theta)\sin^3(\phi)\cos(\phi) d\rho d\phi d\theta = \frac{32}{15}$.

3. (a) $\int_0^{2\pi}\int_0^3\int_r^3 r\,dzdrd\theta = 9\pi$.
   (b) $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3\sec(\phi)} \rho^2\sin(\phi)d\rho d\phi d\theta = 9\pi$.

4. (a) $\int_0^{2\pi}\int_3^5\int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\,dzdrd\theta = \frac{256\pi}{3}$.
   (b) $2\int_0^{2\pi}\int_{\cot^{-1}(4/3)}^{\frac{\pi}{2}}\int_{3\csc(\phi)}^{5} \rho^2\sin(\phi)\,d\rho d\phi d\theta = \frac{256\pi}{3}$.

16.7) Change of Variables. 5-11, 13-16, 17-22, 23-26, 27-30, 31-36, 37-39, 41-44, 46-47, 48, 50-52, 53, 56.

1. $\int_1^2\int_1^3 \frac{1}{2v}\,dvdu = \frac{1}{2}\ln(3)$.

2. $\int_0^2\int_1^3 (v-2u)(\frac{1}{2})\,dudv = -6$.

3. $\int_0^{\pi}\int_0^1 10r^2\sin(\theta)\,drd\theta = \frac{20}{3}$.

17.1) Vector Fields. 2, 8-15, 18, 24, 25-30, 35-42, 43-45, 47-48, 49-52.

1.

2.

17.2) Line Integrals. 4-10, 12-16, 17-34, 35-36, 39-40, 41-46, 47-48, 49-56, 57-60, 62, 64-65, 68, 70-72, 73.

1. Evaluate the line integrals.
   (a) $\int_0^2 t^3(0) + t(2t)\,dt = \frac{16}{3}$.
   (b) $\int_0^1 (2t)(-1+2t)(3)\,dt = 1$.
   (c) $\int_{-1}^2 (t^4-t)(4t^3) + 2t\,dt = 104.1$.

17.3) Conservative Vector Fields. 7, 9-16, 17-30, 31-34, 39-42, 44, 45-50, 51-52, 54-56, 59-62, 63-64.

1. (a) $\phi(x,y) = \sqrt{x^2+y^2}$.
   (b) $\phi(3,4) - \phi(1,0) = 4$.

2. $\vec{F} = \langle 1+\cos(y), 2y - x\sin(y)\rangle$.
   (a) $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} = -\sin(x)$.
   (b) $\phi(x,y) = x + x\cos(y) + y^2$.
   (c) $\phi(2,\pi) - \phi(1,0) = \pi^2 - 2$.

3. $\nabla \times \vec{F} = \vec{0}, \phi(x,y,z) = x^2 + xy + 2y + z^3$.

17.4) Green's Theorem. 9-14, 15-16, 17-20, 21-25, 27-30, 31-40, 41-45, 48, 53-54.

1. $\int_0^2 \int_0^1 2x - 2x \, dydx = 0$.
2. (a) $\int_{-1}^1 t - t^2 + (t+t^2)(2t)\, dt + \int_0^1 (1-2t-1)(-2)\, dt = \frac{2}{3} + 2 = \frac{8}{3}$.
   (b) $\int_{-1}^1 \int_{x^2}^1 2\, dydx = \frac{8}{3}$.
3. $\int_0^1 \int_0^{2x} 4\, dydx = 4$.
4. $\int_0^{2\pi} 16\cos^4(t) + 16\sin^4(t)\, dt = 24\pi, \int_0^{2\pi}\int_0^2 3r^3\, drd\theta = 24\pi.(The first integral is a little too hard/tedious t$

17.5) Divergence and Curl. 9-16, 17, 19, 21-22, 27-34, 41, 42, 44, 65, 67-69, 73.

1. $\vec{F}$ should be $\langle x^3, x^2y, yz^3\rangle$. $\nabla\cdot\vec{F} = 4x^2 + 3yz^2, \nabla\times\vec{F} = \langle z^3, 0, 2xy\rangle$.
2. $\nabla \times \langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\rangle = \vec{0}$.
3. $\nabla\times\vec{F} = \langle -xy, -ze^{xz}, yz - 2y\rangle \neq \vec{0}$.

17.6) Surface Integrals. 9-14, 15, 17-18, 19-24, 25-28, 29-34*, 35-38*, 43-48*, 52, 70-72, 74-75.

*: For 29-34, 35-38, 43-48, you **must** parametrize the surface by $r(u,v)$ and use the parametric form to do the integrals, rather than the 'explicit' form indicated in the book here.

1. $\int_0^{2\pi}\int_0^1 \sqrt{1+r^2}r\, drd\theta = \frac{\pi}{6}(5^{\frac{3}{2}} - 1)$
2. $\int_0^{2\pi}\int_0^2 (-\cos(\theta) + 2)r\, drd\theta = 8\pi$
3. $\int_0^3 \int_{u^2}^{3u} \sqrt{6}\, dvdu = 4.5\sqrt{6}$
4. $d\vec{S} = \langle -f(v)f'(v), f(v)\cos(u), f(v)\sin(u)\rangle\, dudv, dS = f(v)\sqrt{1+(f'(v))^2}\, dudv$
5. (a) $\int_0^\pi \int_0^1 \int u\cos(v)\sin(v) - u\cos(v)\sin(v) + uv\, dudv = \frac{\pi^2}{4}$.
   (b) $\int_0^\pi \int_0^1 u\sin(v)\sqrt{1+u^2}\, dudv = \frac{2}{3}(2^{\frac{3}{2}} - 1)$.

17.7) Stokes' Theorem. 5-10, 11-16, 17-24, 30-33, 45.

1. $\int_0^{2\pi} 4\sin^2(t) + 4\cos^2(t)\, dt = 8\pi$, $\int_0^{2\pi} \int_0^2 2\, r dr d\theta = 8\pi$.

2. $\int_0^{2\pi} 9\sin^2(t) + 9\cos^2(t)\, dt = 18\pi$.

3. $\int_0^1 \int_0^{1-u} \int 2\, dv du = 1$

17.8) Divergence Theorem. 9-12, 13-16, 17-24, 25-27, 30.

1. $\int_0^{2\pi} \int_0^2 \int_{r^2}^2 r\, dz dr d\theta = \frac{128\pi}{3}$. $-\int_0^{2\pi} \int_0^2 r^5 dr d\theta + \int_0^{2\pi} \int_0^2 16 r dr d\theta = \frac{128\pi}{3}$.

2. Use the divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$. The surfaces have outward pointing normal.

   (a) $\int_0^2 \int_0^{3-1.5z} \int_0^{6-2y-3z} 3 dx dy dz = 18$.

   (b) $\int_0^{2\pi} \int_0^{\pi} \int_0^3 3\rho^5 \sin^3(\phi)\cos(\phi)\, d\rho d\phi d\theta = 0$.