Math 250: Final Review, Fall 2023

13.5) Lines and Planes in Space. 1-10, 11-26, 27-30, 31-37, 43-58, 61-64, 65-68, 71-72, 73-76, 77-80.

1. (a) Find the equation of the line that is perpendicular to the plane $2x + 3y - z = 5$ and contains the point $\langle 1, -1, 2\rangle$.
   (b) Find an equation of the line that goes through the point $(0, 2, 3)$ and is perpendicular to the vectors $\vec{v} = \langle 1, 0, 1\rangle$ and $\vec{w} = \langle 1, 2, 0\rangle$.
   (c) Find the equation of the line contained in the planes and $x + y + z = 1$ and $2x + 3y + z = 4$.
2. (a) Find the equation of the plane that contains the point $(2, 3, 1)$, and is perpendicular to the line $\vec{r}(t) = \langle -1 + 5t, 7 - t, 3t\rangle$.
   (b) Find the equation of the plane that contains the points $(1, 2, 0)$,$(2, 3, 1)$, and $(3, 2, 1)$.
   (c) Find the equation of the plane that contains the point $(1, 2, 3)$ and contains the line $\vec{r}(t) = \langle 3 - t, 2 + t, 1 + 2t\rangle$.

13.6) Cylinders and Quadric Surfaces. 1-6, 7-12, 15-20, 21-28, 29-51, 54-58, 60.

1. Sketch the surfaces.
   (a) $4x^2 + z^2 = 4$.
   (b) $z = 4 - y^2$.
2. Sketch the xy, xz, and yz traces. Then sketch the surface.
   (a) $x^2 - y^2 - z^2 = 1$.
   (b) $-x^2 + y^2 + 4z^2 = 4$.
   (c) $9x^2 - y^2 + 9z^2 = 0$.

15.1) Graphs and Level Curves. 25-33, 34, 35, 36-43, 74-77.

1. Sketch each function $f(x, y)$.
   (a) $f(x, y) = 6 - 2x - 3y$
   (b) $f(x, y) = x^2 + \frac{1}{4}y^2$
   (c) $f(x, y) = y^2 - x^2$
2. Sketch the level curves $f(x, y) = c$ for $c = -1, 0, 1, 2$.

(a) $f(x,y) = 2x - y$.
(b) $f(x,y) = x^2 - y^2$.
(c) $f(x,y) = 3 - e^{x^2+y^2}$.

15.2) Limits and Continuity. 10-12, 13-27, 29-34, 35-50, 52-53, 62-67, 71.

1. (a) Find $\lim_{(x,y)\to(-1,2)} \frac{\ln(x+y)}{x^2+y^2}$.

   (b) Find $\lim_{(x,y)\to(3,-1)} \frac{x^2-9y^2}{xy+3y^2}$.

2. (a) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{2x^2+y^2}$ does not exist.

   (b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+xy^2}$ does not exist.

15.3) Partial Derivatives. 1-9, 11-14, 15-30, 32-34, 38-46, 48-53, 54-59.

1. Use the **limit definition** to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   (a) $f(x,y) = 3x - y$.
   (b) $f(x,y) = xy^2$.

2. Find all first and second partials.

   (a) $f(x,y) = \sin(xy)$.
   (b) $f(x,y) = \ln(x^2 + y^3)$.

15.4) The Chain Rule. 9-18, 19-26, 27-28, 29-30, 35-40, 57-59, 65, 67-69, 72-73, 75.

1. $w = x^3y^2$, $x = t^2 + 1$, $y = t - e^{2-t}$. Use the **chain rule** to find $\frac{dw}{dt}$ at $t = 2$.

2. $w = \ln(xy)$, $x = 2u + 3v$, $y = \frac{v^2}{u}$. Use the **chain rule** to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at $(u,v) = (2,-1)$.

3. (a) If $F(x,y,z) = c$ where c is constant and $y = y(x,z)$, use the chain rule to show that $\frac{\partial y}{\partial z} = -\frac{\partial F}{\partial z} / \frac{\partial F}{\partial y}$.

   (b) Use this formula to find $\frac{\partial y}{\partial z}$, when $ye^{xz} + xe^{yz} = 1$.

4. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by implicit differentiation. $x^2y + y^2z + xz^3 = 1$.

5. For a unit-length pendulum, if θ is the angular position and $v = \frac{d\theta}{dt}$, then $\frac{dv}{dt} = -g\sin(\theta)$, where g is constant. Use the chain rule to show that $\frac{dE}{dt} = 0$, where $E(\theta, v) = \frac{1}{2}v^2 - g\cos(\theta)$.

15.5) Directional Derivatives and Gradient. 1-10, 11-12. 13-20, 21-30, 31-36, 43-44, 47-50, 59-64, 69-72, 74, 75-78, 81, 82, 85.

1. Find the gradient of f at the point P. Then find $D_{\vec{u}}f$ in the direction $\vec{u}$. $f(x,y,z) = x^2y + z^3$, $P = (-1,2,1)$, $\vec{u} = \langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

2. $f(x,y) = x^2 + xy - y^3$.

   (a) Find the maximum rate of change of f (steepest ascent), and the direction of the maximum rate of change, at $P = (3,2)$.

   (b) Find a vector that points in a direction of no change of f, at $P = (3,2)$.

3. $f(x,y) = xy$.

   (a) Graph the level set of f through the point $(2,1)$.

   (b) Include the vector $\nabla f(2,1)$ on your graph.

   (c) Find the tangent line to the level set at the point $(2,1)$ and include it on your graph.

   (d) How are the answers to parts b and c related?

15.6) Tangent Planes. 3-4, 9, 11, 13-28, 29-32, 54-56.

1. Find the tangent plane to the surface at the given point.

   (a) $3x^2 + xy + z^2 = 5$, $P = (-1,2,2)$.

   (b)

   (c) $xe^{yz} = 3$, $P = (3,0,2)$.

   (d) $\frac{x-y}{3y+z} = 1$, $P = (3,1,-1)$.

2. Find the tangent plane to the surface at the given point.

   (a) $z = \sqrt{x^2 - y^2}$, $P = (5,4,3)$.

   (b) $z = 3 - \sin(xy)$, $P = (2,0,3)$.

   (c) $y = xe^{x+2z}$, $P = (2,1,-1)$.

15.7) Maximum/Minimum Problems. 9-12, 13-22, 23-37, 41-42, 43-46, 62-66, 71.

1. Find any critical points and classify each as relative maximum, relative minimum, or saddle.

   (a) $f(x,y) = e^{-x^2} + e^{-y^2}$

   (b) $f(x,y) = x^2 + y^2 + xy - x - 2y$

(c) $f(x, y) = xy - 2x - y$

(d) $f(x, y) = x^4 - 2x^2 + y^2$

2. Find the minimum of $x^2 + y^2 + z^2$ if (x, y, z) is on the plane $x - z = 2$. Use the second derivative test to prove your answer is a local minimum.

3. Find the maximum volume of a box $V = xyz$ if the point (x, y, z) is on the paraboloid $z = 4 - x^2 - y^2$. $(x, y, z > 0.)$ Use the second derivative test to prove your answer is a local maximum.

15.8) Lagrange Multipliers. 3-4, 5-6, 7-23, 26, 27-36.

1. Use Lagrange multipliers to find the maximum and minimum of $f(x, y) = xy^3$ if $x^2 + y^2 = 4$.

2. The area of a rectangle with vertices $(\pm x, \pm y)$ is $4xy$. Use Langrange multipliers to find the maximum area of such a rectangle with vertices on the ellipse $4x^2 + y^2 = 32$.

3. Use Lagrange multipliers to find the minimum distance between the plane $3x + y - z = 18$ and the point $(2, 0, -1)$. (Hint: To find (x, y, z) you can minimize the square of the distance, $f(x, y, z) = (x - 2)^2 + y^2 + (z + 1)^2$.)

16.1) Double Integrals, Rectangular Regions. 1-3, 5-6, 7-24, 25-35, 36-39, 40-45, 46-50, 53-54.

1. Find the average value of $f(x) = 2y - x$ for $0 \leq x \leq 2$, $1 \leq y \leq 3$.

2. Choose the most convenient order of integration and evaluate the integral.

   (a) $\iint_R xye^{xy^2}\, dA,\ \ 0 \leq x \leq \ln(3),\ 0 \leq y \leq 1.$

   (b) $\iint_R \frac{y}{\sqrt{1+xy}}\, dA,\ \ 0 \leq x \leq 1,\ 0 \leq y \leq 3.$

16.2) Double Integrals, General Regions. 5-8, 9-10, 11-27, 28-34, 35-42, 43-53, 57-62, 63-68, 69, 70, 71, 73-80, 85-90, 95-96, 99-102.

1. Find the volume under $f(x, y) = 2y - 1$, on the region between $x = y^2 + 3$ and $x = 3y + 1$.

2. Evaluate the integrals by changing the order of integration.

   (a) $\int_0^6 \int_{\frac{1}{2}x}^3 e^{y^2}\, dydx.$

   (b) $\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} e^{12x - x^3}\, dxdy$

16.3) Double Integrals in Polar Coordinates. 7-10, 11-14, 15-18, 19-20, 21-30, 31-40, 42, 44-46,47, 49-50, 53-54, 57-60, 65-68.

1. Evaluate the integral by converting to polar.

   (a) $\int_0^4 \int_0^{\sqrt{16-x^2}} xy\,dydx$.

   (b) $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \frac{1}{(1+x^2+y^2)^2}\,dxdy$.

   (c) $\int_1^2 \int_{-x}^{x} 1\,dydx$.

16.4) Triple Integrals. 4-6, 7-14, 15-29, 30-35, 36-37, 38-46, 47-50, 51-54, 57-58, 62-63, 67-70.

1. Express the volume under $x^2 + 4y + z = 4$, with $x, y, z \geq 0$ by six different triple integrals.

2. V is the region between $y = x^2$, $z = 0$, and $y + 2z = 4$.

   (a) Find the volume using an integral $dydxdz$.

   (b) Find the volume using an integral $dzdxdy$.

16.5) Cylindrical and Spherical Coordinates. 3-4, 9-10, 11-14, 15-22, 23-28, 29-34, 35-38, 41-47, 48-54, 58-61, 62-63, 64-65, 66-72, 77-79.

1. Evaluate $\iiint_V \frac{z}{(x^2+y^2)^{\frac{3}{2}}}\,dxdydz$ where V is the region with $1 \leq x^2 + y^2 \leq 4$ and $0 \leq z \leq 4 - x^2 - y^2$.

2. Evaluate $\iiint_V xz\,dxdydz$ where V is the region inside the sphere of radius 2 in the first octant.

3. (a) Find the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and below $z = 3$ by an integral in cylindrical coordinates.

   (b) Find the same volume using an integral in spherical coordinates.

4. (a) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 25$ and outside the cylinder $x^2 + y^2 = 9$ by an integral in cylindrical coordinates.

   (b) Find the same volume using an integral in spherical coordinates.

16.7) Change of Variables. 5-11, 13-16, 17-22, 23-26, 27-30, 31-36, 37-39, 41-44, 46-47, 48, 50-52, 53, 56.

1. Let R be the region between $xy = 1$, $xy = 2$, $y = x$, $y = 3x$. Use the change of variables $u = xy$, $v = \frac{y}{x}$ to find the area of R by a double integral.

2. Let R be the region with $1 \leq x + 2y \leq 3$, $0 \leq 3x + 4y \leq 2$. Use a change of variables to evaluate $\iint_R x\,dA$.

3. Let R be the region inside the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$, with $y \geq 0$. Evaluate $\int_R y\, dxdy$ by making the change of variables $x = 2u$,$y = 5v$.

17.1) Vector Fields. 2, 8-15, 18, 24, 25-30, 35-42, 43-45, 47-48, 49-52.

1. Sketch the vector fields: $\vec{F}(x, y) = \langle x, y\rangle$ and $V(x, y) = \langle y, -x\rangle$.
2. Plot the vector field at the points $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(-1, 1)$. $\vec{F}(x, y) = \langle 2x + y, -x + 2y\rangle$.

17.2) Line Integrals. 4-10, 12-16, 17-34, 35-36, 39-40, 41-46, 47-48, 49-56, 57-60, 62, 64-65, 68, 70-72, 73.

1. Evaluate the line integrals.
   (a) $\int_c yzdx + xydz$ $\vec{r}(t) = \langle 1, t, t^2\rangle$, from $(1, 0, 0)$ to $(1, 2, 4)$.
   (b) $\int_c xz\, ds$, where c is the line from $(0, 1, -1)$ to $(2, 0, 1)$.
   (c) $\int_c \langle x-y, 2y\rangle \cdot d\vec{r}$, where c is the curve along $x = y^4$ that connects $(1, -1)$ to $(16, 2)$.

17.3) Conservative Vector Fields. 7, 9-16, 17-30, 31-34, 39-42, 44, 45-50, 51-52, 54-56, 59-62, 63-64.

1. $\vec{F}(x, y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.
   (a) Find a function ϕ such that $\nabla\phi = \vec{F}$.
   (b) Use this to find $\int_c \vec{F} \cdot d\vec{r}$, where $\vec{r}(t) = \langle 1 + t, 2t\rangle$, $0 \leq t \leq 2$.
2. $\vec{F} = \langle 1 + \cos(y), 2y - x\sin(y)\rangle$.
   (a) Show $\vec{F}$ is conservative.
   (b) Find a function ϕ such that $\nabla\phi = \vec{F}$.
   (c) Use this to evaluate $\int_c \vec{F} \cdot d\vec{r}$, $\vec{r}(t) = \langle 2^t, \pi t\rangle$, $0 \leq t \leq 1$.
3. Show $\vec{F}$ is conservative, and find a function ϕ such that $\nabla f = \vec{F}$.
   $\vec{F} = \langle 2x + y, x + 2, 3z^2\rangle$.

17.4) Green???s Theorem. 9-14, 15-16, 17-20, 21-25, 27-30, 31-40, 41-45, 48, 53-54.

1. Use Green's theorem to compute $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 + y^2)\vec{\imath} + 2xy\vec{\jmath}$, and c along the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$, $(0, 1)$ oriented counterclockwise.
2. Let R be the region $x^2 \leq y \leq 1$, $\vec{F} = \langle x - y, x + y\rangle$.

(a) Compute $\int_c \vec{F} \cdot d\vec{r}$ directly, where c is the boundary oriented counterclockwise. (The boundary has two components!)

(b) Use Green's theorem to compute $\int_c \vec{F} \cdot d\vec{r}$. Show that you get the same answer.

3. Use Green's theorem to compute $\int_c \langle x - y, 2x + 3y \rangle \cdot \vec{n} ds$, where c is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

4. Verify Green's Theorem (flux form) by computing both sides. $\vec{F} = \langle x^3, y^3 \rangle$ with c the circle which bounds the region $x^2 + y^2 \leq 4$, oriented counterclockwise.

17.5) Divergence and Curl. 9-16, 17, 19, 21-22, 27-34, 41, 42, 44, 65, 67-69, 73.

1. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$. $\vec{F}(x, y, z) = \langle x^3, x^2 y, yz^3 \rangle$.

2. Show that if $\vec{F} = \nabla \phi$ for some $\phi(x, y, z)$, then $\nabla \times \vec{F} = \vec{0}$.

3. Show there is no function $\phi(x, y, z)$ with $\nabla \phi = \langle x^2 + y^2, xyz, e^{xz} \rangle$.

17.6) Surface Integrals. 9-14, 15, 17-18, 19-24, 25-28, 29-34*, 35-38*, 43-48*, 52, 70-72, 74-75.

*: For 29-34, 35-38, 43-48, you **must** parametrize the surface by $r(u, v)$ and use the parametric form to do the integrals, rather than the ???explicit??? form indicated in the book here.

1. Evaluate $\iint_S 1 \, dS$, where S is the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$.

2. Evaluate $\iint_S \langle 1, 0, 2 \rangle \cdot d\vec{S}$, where S is the cone $z = \sqrt{x^2 + y^2}$ with $0 < z < 2$. Upward pointing normal.

3. Use a surface integral to find the area of the region of the plane $z = x + 2y + 3$ with $x^2 \leq y \leq 3x$.

4. A surface of revolution given by $y = f(x)$ revolved around the x-axis can be written $\vec{r}(u, v) = \langle v, f(v) \cos(u), f(v) \sin(u) \rangle$, with $0 \leq u \leq 2\pi$, $a \leq v \leq b$. Use this to derive the formula Area $= 2\pi \int_a^b f(v) \sqrt{1 + (f'(v))^2} \, dv$.

5. A helicoid is given by the parametric surface $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), v \rangle$, with $0 \leq u \leq 1$, $0 \leq v \leq \pi$. Evaluate the integrals.

   (a) $\iint_S \langle x, y, z \rangle \cdot d\vec{S}$. Upward pointing normal.

   (b) $\iint_S y \, dS$.

17.7) Stokes??? Theorem. 5-10, 11-16, 17-24, 30-33, 45.

1. Compute both sides in Stokes' theorem and show that they are equal, for the surface $z = 4 - x^2 - y^2$, $z \geq 0$. $\vec{F} = \langle -y, x, z^2 \rangle$.

2. Use Stokes' theorem to evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where S is the hemisphere $x^2+y^2+z^2 = 9$, $y \geq 0$, and $\vec{F} = \langle x - z, e^{xy}, x + z \rangle$. Right pointing normal.

3. Use Stokes' theorem to evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ oriented counterclockwise, and $\vec{F} = \langle x-y, x+y, z \rangle$. (Hint: $z = 1-x-y$ gives the surface, with $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$).

17.8) Divergence Theorem. 9-12, 13-16, 17-24, 25-27, 30.

1. $\vec{F}(x, y, z) = \langle y, -x, z^2 \rangle$. Evaluate both sides of the divergence theorem and show that they are equal, for the region $x^2 + y^2 \leq z \leq 4$. (The boundary has two pieces, $z = x^2 + y^2$ and $z = 4$. Outward pointing normals.)

2. Use the divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$. The surfaces have outward pointing normal.

   (a) $\vec{F} = \langle x, y, z \rangle$, S the boundary of the tetrahedron $x, y, z \geq 0$, $x + 2y + 3z \leq 6$.

   (b) $\vec{F} = \langle x^3 z, y^3 z, xy \rangle$. S the sphere $x^2 + y^2 + z^2 = 9$.