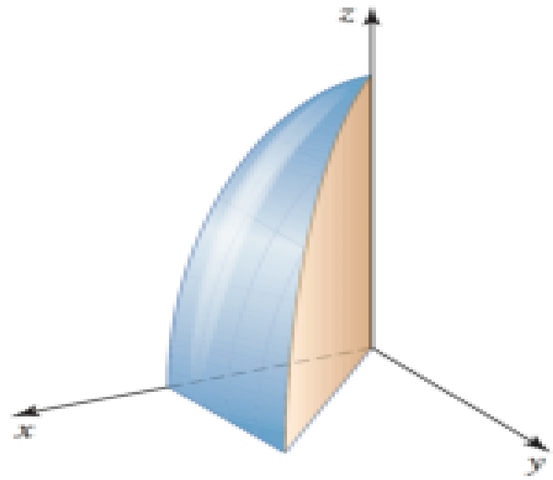
Triple integrals problems – Changing the order of integration
Math 250, Spring 2024 – Jacek Polewczak

Problem 1.

Let D be the solid in the first octant bounded by the planes $y = 0$, $z = 0$, and $y = x$, and the cylinder $4x^2 + z^2 = 4$. Write the triple integral of $f(x, y, z)$ over D in the given order of integration.



Solution

order of integration: dzdydx

From $4x^2 + z^2 = 4$, we obtain $z = \pm 2\sqrt{1 - x^2}$, also since $z \geq 0$ we must have $z = 2\sqrt{1 - x^2}$. Thus, the lower limit of integration for z is $z = 0$ and the upper limit of integration is $z = 2\sqrt{1 - x^2}$. Also, from $4x^2 + z^2 = 4$, we see that $x = 1$ when $z = 0$, and since $x \geq 0$, we must have $0 \leq x \leq 1$. Similarly for y variable ($y \geq 0$ and for $x = 1$, $y = x = 1$): $0 \leq y \leq 1$. Finally, from $4x^2 + z^2 = 4$ when $x = 0$, we have $z = \pm 2$ and since $z \geq 0$, we have $0 \leq z \leq 2$.

Next, the projection of D on xy-plane (i.e., when $z = 0$) is the region $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$.

$$\int_0^1 \int_0^x \int_0^{2\sqrt{1-x^2}} f(x, y, z)\, dz\, dy\, dx$$

order of integration: dzdxdy

The limits of integration for z are as before. Now, from $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$, we see that $x \geq y$ and since $x \leq 1$, the lower limit of integration for variable x is $x = y$ and the upper limit of integration for variable x is $x = 1$. Finally, $0 \leq y \leq 1$, thus

$$\int_0^1 \int_y^1 \int_0^{2\sqrt{1-x^2}} f(x, y, z)\, dz\, dx\, dy$$

order of integration: dydxdz

Variable y is between the plane $y = 0$ and the plane $y = x$, thus the lower limit of integration for variable y is $y = 0$ and the upper limit of integration is for variable y is $y = x$. The projection of D on the xz plane (i.e., when $y = 0$) is the region $R = \{(x, z) : 0 \leq x \leq \frac{1}{2}\sqrt{4 - z^2}, 0 \leq z \leq 2\}$, This region is the inside of the quarter of the elipse $4x^2 + z^2 = 4$ in the first quadrant of xz-coordinate system. Therefore

$$\int_0^2 \int_0^{\frac{1}{2}\sqrt{4-z^2}} \int_0^x f(x, y, z)\, dy\, dx\, dz$$

order of integration: dydzdx

The limits of integration for y are the same as in the last case. The quarter of the elipse $4x^2 + z^2 = 4$ in the first quadrant of xz-coordinate system (i.e., the projection of D on the xz plane) can be also described as $R = \{(x, z) : 0 \le x \le 1, 0 \le z \le 2\sqrt{1 - x^2}\}$. Therefore,

$$\int_0^1 \int_0^{2\sqrt{1-x^2}} \int_0^x f(x, y, z)\, dy\, dz\, dx$$

order of integration: dxdydz

x is between the plane $x = y$ and the cylinder $4x^2 + z^2 = 4$. In other words, $y \le x \le \frac{1}{2}\sqrt{4 - z^2}$. Thus, the lower limit of integration for variable x is $x = y$ and the upper limit of integration for variable x is $x = \frac{1}{2}\sqrt{4 - z^2}$. For each such x, variable y also varies between $y = 0$ and $y = \frac{1}{2}\sqrt{4 - z^2}$, with $0 \le z \le 2$. Thus

$$\int_0^2 \int_0^{\frac{1}{2}\sqrt{4-z^2}} \int_y^{\frac{1}{2}\sqrt{4-z^2}} f(x, y, z)\, dx\, dy\, dz$$

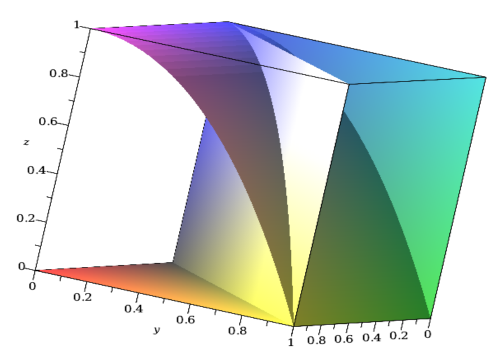
order of integration: dxdzdy

As in the last case, the lower limit of integration for variable x is $x = y$ and the upper limit of integration for variable x is $x = \frac{1}{2}\sqrt{4 - z^2}$. Also z is between $z = 0$ and $z = 2\sqrt{1 - x^2}$. Since $0 \le y \le x \le 1$, we have $y^2 \le x^2$, consequently, $1 - y^2 \ge 1 - x^2$, implying that $z = 2\sqrt{1 - x^2} \le 2\sqrt{1 - y^2}$. Therefore, the lower limit of integration for variable z is $z = 0$ and the upper limit of integration for variable z is $z = 2\sqrt{1 - y^2}$. Since y varies between $y = 0$ and $y = 1$, we have

$$\int_0^1 \int_0^{2\sqrt{1-y^2}} \int_y^{\frac{1}{2}\sqrt{4-z^2}} f(x, y, z)\, dx\, dz\, dy$$

Problem 2.

Let D be the solid in the first octant bounded by $y = x$, $z = 1 - y^2$, $x = 0$, and $z = 0$. Write triple integrals over D in all six possible orders of integration.



Solution

Since $z = 1 - y^2 \ge 0$, we have $0 \le y \le 1$, and $0 \le z \le 1$, Additionally, x is between the plane $x = 0$ and the plane $x = y$, and since $0 \le y \le 1$, we must have $0 \le x \le 1$.

order of integration: dzdydx

z is between $z = 0$ and $z = 1 - y^2$, therefore, the lower limit of integration for variable z is $z = 0$ and the upper limit of integration for variable z is $z = 1 - y^2$. Variable y (see the graph) is between $y = x$ and $y = 1$. Thus, the lower limit of integration for variable y is $y = x$ and the upper limit of integration for variable y is $y = 1$. And since $0 \le x \le 1$, we have

$$\int_0^1 \int_x^1 \int_0^{1-y^2} 1 \, dz \, dy \, dx = \frac{1}{4}.$$

order of integration: dzdxdy

In this case the limits of integration for variable z are the same as above: $0 \le z \le 1 - y^2$. Variable x is between $x = 0$ and $x = y$ (see the graph). And $0 \le y \le 1$. Thus

$$\int_0^1 \int_0^y \int_0^{1-y^2} 1 \, dz \, dx \, dy = \frac{1}{4}.$$

order of integration: dydzdx

Variable y is between the plane $y = x$ and the cylinder $z = 1 - y^2$. From $z = 1 - y^2$, we have $y = \pm\sqrt{1-z}$, however, since $y \ge 0$, we must have $y = \sqrt{1-z}$. Therefore, the lower limit of integration for variable y is $y = x$ and the upper limit of integration for variable y is $y = \sqrt{1-z}$.

For each such y, variable z varies between $z = 0$ and the intersection of the parabolic cylinder $z = 1 - y^2$ with the plane $y = x$, which is $z = 1 - x^2$. The lower limit of integration for variable z is $z = 0$ and the upper limit of integration for variable z is parabola $z = 1 - x^2$. Since $0 \le x \le 1$, we have

$$\int_0^1 \int_0^{1-x^2} \int_x^{\sqrt{1-z}} 1 \, dy \, dz \, dx = \frac{1}{4}.$$

order of integration: dydxdz

In this case the limits of integration for variable y are the same as above: $x \le y \le \sqrt{1-z}$. In xz plane, $R = \{(x, z) : 0 \le x \le 1, 0 \le z \le 1 - x^2\}$, has another representation: $R = \{(x, z) : 0 \le x \le \sqrt{1-z}, 0 \le z \le 1\}$. (*Check it!*) Therefore

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_x^{\sqrt{1-z}} 1 \, dy \, dx \, dz = \frac{1}{4}.$$

order of integration: dxdydz

Variable x varies between the plane $x = 0$ and the plane $x = y$. Therefore the lower limit of integration for variable x is $x = 0$ and the upper limit of integration for variable y is $x = y$. The region R in yz plane is $R = \{(y, z) : 0 \le y \le 1, 0 \le z \le 1 - y^2\}$. It has equivalent representation $R = \{(y, z) : 0 \le y \le \sqrt{1-z}, 0 \le z \le 1\}$. Thus

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^y 1 \, dx \, dy \, dz = \frac{1}{4}.$$

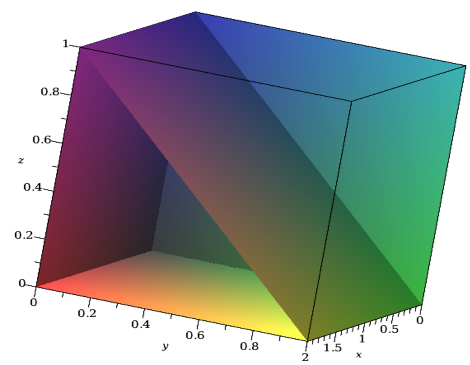
order of integration: dxdzdy

In this case the limits of integration for variable x are the same as above: $0 \le x \le y$. The region R in yz plane is $R = \{(y,z) : 0 \le y \le 1, 0 \le z \le 1-y^2\}$. Therefore

$$\int_0^1 \int_0^{1-y^2} \int_0^y 1\,dx\,dz\,dy = \frac{1}{4}.$$

Problem 3.

Write the integral $\int_0^2 \int_0^1 \int_0^{1-y} dz\,dy\,dx = 1$ in the other five possible orders of integration.



Solution

order of integration: dzdxdy

$$\int_0^1 \int_0^2 \int_0^{1-y} dz\,dx\,dy = 1$$

order of integration: dydzdx

$$\int_0^2 \int_0^1 \int_0^{1-z} dy\,dz\,dx = 1$$

order of integration: dydxdz

$$\int_0^1 \int_0^2 \int_0^{1-z} dy\,dx\,dz = 1$$

order of integration: dxdydz

$$\int_0^1 \int_0^{1-z} \int_0^2 dx\,dy\,dz = 1$$

order of integration: dxdzdy

$$\int_0^1 \int_0^{1-y} \int_0^2 dx\,dz\,dy = 1$$