**Practice Problems II**
**Math 250, Spring 2024 – Jacek Polewczak**

**Problem 1.**
Evaluate the indicated double integrals.

$$\text{(a)}\quad \iint_R xy\sqrt{1+x^2}\,dA; \quad R=\{(x,y): 0\le x\le \sqrt{3},\ 1\le y\le 2\}$$

$$\text{(b)}\quad \int_{-2}^{2}\int_{-1}^{1}|x^2y^3|\,dydx,$$

$$\text{(c)}\quad \iint_S x\,dA; \quad S \text{ is the region between } y=x \text{ and } y=x^3. \quad (\textit{Note that } S \textit{ has two parts.})$$

**Solution**

(a) $$\iint_R xy\sqrt{1+x^2}\,dA = \int_1^2\int_0^{\sqrt{3}} xy\sqrt{1+x^2}\,dxdy = \int_1^2 \left[\frac{1}{3}y(1+x^2)^{\frac{3}{2}}\right]_{x=0}^{x=\sqrt{3}} dy = \int_1^2 \frac{7}{3}y\,dy = \left[\frac{7}{6}y^2\right]_1^2 = 3.5.$$

(b) $$\int_{-2}^{2}\int_{-1}^{1}|x^2y^3|\,dydx = \int_{-2}^{2}\int_{-1}^{0}|x^2y^3|\,dydx + \int_{-2}^{2}\int_{0}^{1}|x^2y^3|\,dydx = -\int_{-2}^{2}\int_{-1}^{0}x^2y^3\,dydx + \int_{-2}^{2}\int_{0}^{1}x^2y^3\,dydx$$
$$= -\left(\int_{-2}^{2}x^2\,dx\right)\left(\int_{-1}^{0}y^3\,dy\right) + \left(\int_{-2}^{2}x^2\,dx\right)\left(\int_{0}^{1}y^3\,dy\right) = 2\left(\int_0^2 x^2\,dx\right)2\left(\int_0^1 y^3\,dy\right) = 2\left(\frac{8}{3}\right)2\left(\frac{1}{4}\right) = \frac{8}{3}.$$

(c) Since $S$ is symmetric with respect to the origin and the integrand is an odd function in $x$, the value of the integral is 0.
We can check it by the direct computation. Indeed,

$$\iint_S x\,dA = \int_{-1}^{0}\left(\int_x^{x^3} x\,dy\right)dx + \int_0^1\left(\int_{x^3}^{x} x\,dy\right)dx = \int_{-1}^{0} x\Big[y\Big]_{y=x}^{y=x^3}dx + \int_0^1 x\Big[y\Big]_{y=x^3}^{y=x}dx$$
$$= \int_{-1}^{0}(x^4-x^2)\,dx + \int_0^1 (x^2-x^4)\,dx = \left[\frac{1}{5}x^5 - \frac{1}{3}x^3\right]_{-1}^{0} + \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \left(\frac{1}{5}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) = 0.$$

**Problem 2.**
Find the volumes of the indicated solids by an iterated integration.
(a) The tetrahedron bounded by the coordinate planes and the plane $3x+4y+z-12=0$.
(b) The solid bounded by the parabolic cylinder $x^2=4y$ and the planes $z=0$ and $5y+9z-45=0$.

**Solution**

$$\text{(a)}\quad \text{Volume} = \int_0^4\left[\int_0^{(-3/4)x+3}(12-3x-4y)\,dy\right]dx = 24.$$

(b)

**Solution 1**

The plane $5y+9z=45$ intersects the $xy$-plane in the line $y=9$, so the region E (in $xy$ plane) is

$$E=\{(x,y,z): -6\le x\le 6,\ x^2/4\le y\le 9,\ 0\le z\le 5-(5/9)y\}$$

and

$$\text{Volume} = \int_{-6}^{6}\int_{x^2/4}^{9}\int_0^{5-(5/9)y} 1\cdot dzdydx = \int_{-6}^{6}\int_{x^2/4}^{9}\left(5-\frac{5}{9}y\right)dydx = 144.$$

**Solution 2**

The plane $5y + 9z = 45$ intersects the $xy$-plane in the line $y = 9$, so the region E (in $xy$ plane) is

$$E = \{(x, y, z) : -2\sqrt{y} \le x \le 2\sqrt{y},\ 0 \le y \le 9,\ 0 \le z \le 5 - (5/9)y\}$$

and

$$\text{Volume } = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \int_0^{5-(5/9)y} 1 \cdot dz dx dy = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \left(5 - \frac{5}{9}y\right) dx dy = 144.$$

**Problem 3.**

$S$ is the smaller region bounded by $\theta = \pi/6$ and $r = 4\sin\theta$. Find the area of the region $S$ by calculating $\iint\limits_S r\, dr d\theta$.

**Solution**

$$\begin{aligned} \text{Area } &= \int_0^{\pi/6} \left[\int_0^{4\sin\theta} r\, dr\right] d\theta = \int_0^{\pi/6} \left[\frac{r^2}{2}\right]_{r=0}^{r=4\sin\theta} d\theta \\ &= \int_0^{\pi/6} 8\sin^2\theta\, d\theta = \int_0^{\pi/6} 4(1 - \cos 2\theta)\, d\theta \\ &= \Big[4\theta - 2\sin 2\theta\Big]_0^{\pi/6} = \frac{2}{3}\pi - \sqrt{3} \approx 0.3623. \end{aligned}$$

**Problem 4.**

Evaluate the following double integral by using polar coordinates

$$\iint\limits_S \sqrt{4 - x^2 - y^2}\, dA,$$

where $S$ is the first quadrant sector of the circle $x^2 + y^2 = 4$ between $y = 0$ and $y = x$.

**Solution**

$$\begin{aligned} &\int_0^{\pi/4} \left[\int_0^2 (4 - r^2)^{1/2} r\, dr\right] d\theta = \int_0^{\pi/4} \left[-\frac{(4 - r^2)^{3/2}}{3}\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{\pi/4} \frac{8}{3}\, d\theta = \left[\frac{8\theta}{3}\right]_0^{\pi/4} = \frac{2\pi}{3} \approx 2.0944. \end{aligned}$$

**Problem 5.**

Find the volume of the solid lying under the graph of $z = f(x, y) = x^3 + 4y$ and above the region $R$ in the $xy$-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

**Solution**

The region $R$ is

$$R = \{(x, y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}.$$

Therefore

$$\begin{aligned} V = \iint\limits_R f(x, y)\, dA &= \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\, dy dx = \int_0^2 \left[x^3 y + 2y^2\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[(2x^4 + 8x^2) - (x^5 + 2x^4)\right] dx \\ &= \int_0^2 (8x^2 - x^5)\, dx = \left[\frac{8}{3}x^3 - \frac{1}{6}x^5\right]_0^2 = \frac{32}{3} \end{aligned}$$

**Alternative Solution**

One can also look at the region $R$ as::

$$R = \{(x, y) : 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y}\}.$$

Then

$$V = \iint_R f(x, y)\, dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^3 + 4y)\, dxdy = \int_0^4 \left[\frac{1}{4}x^4 + 4xy\right]_{x=y/2}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left[\left(\frac{1}{4}y^2 + 4y^{3/2}\right) - \left(\frac{1}{64}y^4 + 2y^2 + 2x^4\right)\right] dx$$

$$= \int_0^4 \left(-\frac{7}{4}y^2 + 4y^{3/2} - \frac{1}{64}y^4\right) dy = \left[-\frac{7}{12}y^3 + \frac{8}{5}y^{5/2} - \frac{1}{320}y^5\right]_0^4 = \frac{32}{3}.$$

**Problem 6.**
Evaluate $\iint_R (2x - y)\, dA$, when $R$ is the region bounded by the parabola $x = y^2$ and the line $x - y = 2$.

**Solution**

The region $R$ is

$$R = \{(x, y) : -1 \le y \le 2,\ y^2 \le x \le y + 2\}$$

and

$$\iint_R (2x - y)\, dA = \int_{-1}^{2} \int_{y^2}^{y+2} (2x - y)\, dxdy = \int_{-1}^{2} \left[x^2 - xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^{2} \left[(y + 2)^2 - y(y + 2)\right] - \left[y^4 - y^3\right] dx$$

$$= \int_{-1}^{2} (4 + 2y + y^3 - y^4)\, dy = \left[4y + y^2 + \frac{1}{4}y^4 - \frac{1}{5}y^5\right]_{-1}^{2} = \frac{233}{20}.$$

**Alternative Solution**

We can also write

$$\iint_R (2x - y)\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (2x - y)\, dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} (2x - y)\, dydx.$$

**Problem 7.**
Evaluate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy.$$

**Solution**

The integral

$$\int \frac{\sin x}{x}\, dx$$

cannot be evaluated in terms of elementary functions. We attempt to evaluate the original integral by reversing the order of integration. We have

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy = \iint_R \frac{\sin x}{x}\, dA,$$

where

$$R = \{(x,y) : 0 \le y \le 1,\, y \le x \le 1\}.$$

However, by viewing the region $R$ as

$$R = \{(x,y) : 0 \le x \le 1,\, 0 \le y \le x\},$$

we have

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy = \iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dydx = \int_0^1 \left[\frac{y \sin x}{x}\right]_{y=0}^{y=x} dx = \int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + 1.$$

**Problem 8.**

Evaluate

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx.$$

**Solution**

The integral

$$\int \sin(y^2)\, dy$$

cannot be evaluated in terms of elementary functions. We attempt to evaluate the original integral by reversing the order of integration. We have

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx = \iint_R \sin(y^2)\, dA,$$

where

$$R = \{(x,y) : 0 \le x \le 1,\, x \le y \le 1\}.$$

However, by viewing the region $R$ as

$$R = \{(x,y) : 0 \le y \le 1,\, 0 \le x \le y\},$$

we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx = \iint_R \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dxdy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy = \left[-\frac{1}{2}\cos(y^2)\right]_0^1 = \frac{1}{2}(1 - \cos 1).$$

**Problem 9.**

Evaluate $\iint_R (2x + 3y)\, dA$, where $R$ is the region in the first quadrant bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

**Solution**

the region $R$ is a polar rectangular

$$R = \{(r,\theta) : 1 \le r \le 2,\, 0 \le \theta \le \pi/2\}.$$

Thus,

$$\iint_R (2x + 3y)\, dA = \int_0^{\pi/2} \int_1^2 (2r\cos\theta + 3r\sin\theta) r\, drd\theta = \int_0^{\pi/2} \left[\frac{2}{3} r^3 \cos\theta + r^3 \sin\theta\right]_{r=1}^{r=2} d\theta = \int_0^{\pi/2} \left(\frac{14}{3}\cos\theta + 7\sin\theta\right) d\theta$$

$$= \left[\frac{14}{3}\sin\theta - 7\cos\theta\right]_0^{\pi/2} = \frac{35}{3}.$$

**Problem 10.**
Use a double integral to find the area enclosed by one loop of the three-leaved rose $r = \sin(3\theta)$.

**Solution**

A loop of the rose is described by the region

$$R = \{(r, \theta) : 0 \le \theta \le \pi/3,\ 0 \le r \le \sin(3\theta)\}.$$

Thus

$$A = \iint\limits_R 1\,dA = \int_0^{\pi/3}\int_0^{\sin(3\theta)} r\,dr\theta = \int_0^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\sin(3\theta)} d\theta = \frac{1}{2}\int_0^{\pi/3}\sin^2(3\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/3}(1-\cos(6\theta))\,d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{1}{6}\sin(6\theta)\right]_{\theta=0}^{\theta=\pi/3} = \frac{\pi}{12}.$$

**Problem 11.**
Evaluate the integral

$$\int_{-\infty}^{\infty} \exp(-x^2)\,dx.$$

The integral

$$\int \exp(-x^2)\,dx$$

cannot be evaluated in terms of elementary functions. Switching to the polar coordinates will help. First, we notice that if $A = \int\limits_{-\infty}^{\infty} \exp(-x^2)\,dx$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x^2 - y^2)\,dxdy = A^2. \qquad \textit{(Do you know why?)}$$

Next,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x^2-y^2)\,dxdy = \lim_{a\to\infty}\int_0^a\int_0^{2\pi}\exp(-r^2)r\,drd\theta = \lim_{a\to\infty} 2\pi\left[-\frac{1}{2}\exp(-r^2)\right]_{r=0}^{r=a} = 2\pi\lim_{a\to\infty}\left[-\frac{1}{2}exp(-a^2) + \frac{1}{2}\right]$$
$$= 2\pi\frac{1}{2} = \pi.$$

Thus

$$\int_{-\infty}^{\infty}\exp(-x^2)\,dx = \sqrt{\pi}.$$

**Problem 12.**
Evaluate $\iiint_E yz\,dV$, where $E$ is the solid tetrahedron bounded by the four planes $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

**Solution**

The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x+y+z = 1$, or $z = 1-x-y$. Next, we observe that the planes $x+y+z = 1$ and $z = 0$ intersect in the line $x+y = 1$ (or $y = 1-x$) in the $xy$-plane. So the projection of $E$ onto $xy$-plane is the region $R$

$$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}.$$

Therefore

$$\iiint\limits_E yz\,dV = \iint\limits_R \left[\int_0^{1-x-y} yz\,dz\right] dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\,dzdydx = \int_0^1 \int_0^{1-x} \left[\frac{1}{2}yz^2\right]_{z=0}^{z=1-x-y} dydx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}y(1-x-y)^2\,dydx = \int_0^1 \int_0^{1-x} \left(\frac{1}{2}x^2y + xy^2 + \frac{1}{2}y^3 - xy - y^2 + \frac{1}{2}y\right) dydx$$

$$= \int_0^1 \left[-\frac{5}{24}(1-x)^4 + (1-x)^2\left(\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\right)\right] dx = \frac{1}{120}.$$