Practice Problems II
Math 250, Spring 2026 – Jacek Polewczak

Problem 1.
Evaluate the indicated double integrals.

$$\text{(a)}\quad \iint_R xy\sqrt{1+x^2}\,dA;\quad R=\{(x,y): 0\le x\le\sqrt{3},\ 1\le y\le 2\}$$

$$\text{(b)}\quad \int_{-2}^{2}\int_{-1}^{1}|x^2y^3|\,dydx,$$

$$\text{(c)}\quad \iint_S x\,dA;\quad S \text{ is the region between } y=x \text{ and } y=x^3.\quad (\textit{Note that } S \textit{ has two parts.})$$

Solution

(a) $$\iint_R xy\sqrt{1+x^2}\,dA=\int_1^2\int_0^{\sqrt{3}}xy\sqrt{1+x^2}\,dxdy=\int_1^2\left[\frac{1}{3}y(1+x^2)^{\frac{3}{2}}\right]_{x=0}^{x=\sqrt{3}}dy=\int_1^2\frac{7}{3}y\,dy=\left[\frac{7}{6}y^2\right]_1^2=3.5.$$

(b) $$\int_{-2}^{2}\int_{-1}^{1}|x^2y^3|\,dydx=\int_{-2}^{2}\int_{-1}^{0}|x^2y^3|\,dydx+\int_{-2}^{2}\int_{0}^{1}|x^2y^3|\,dydx=-\int_{-2}^{2}\int_{-1}^{0}x^2y^3\,dydx+\int_{-2}^{2}\int_{0}^{1}x^2y^3\,dydx$$
$$=-\left(\int_{-2}^{2}x^2\,dx\right)\left(\int_{-1}^{0}y^3\,dy\right)+\left(\int_{-2}^{2}x^2\,dx\right)\left(\int_{0}^{1}y^3\,dy\right)=2\left(\int_0^2x^2\,dx\right)2\left(\int_0^1y^3\,dy\right)=2\left(\frac{8}{3}\right)2\left(\frac{1}{4}\right)=\frac{8}{3}.$$

(c) Since S is symmetric with respect to the origin and the integrand is an odd function in x, the value of the integral is 0.
We can check it by the direct computation. Indeed,

$$\iint_S x\,dA=\int_{-1}^{0}\left(\int_x^{x^3}x\,dy\right)dx+\int_0^1\left(\int_{x^3}^{x}x\,dy\right)dx=\int_{-1}^{0}x\Big[y\Big]_{y=x}^{y=x^3}dx+\int_0^1x\Big[y\Big]_{y=x^3}^{y=x}dx$$
$$=\int_{-1}^{0}(x^4-x^2)\,dx+\int_0^1(x^2-x^4)\,dx=\left[\frac{1}{5}x^5-\frac{1}{3}x^3\right]_{-1}^{0}+\left[\frac{1}{3}x^3-\frac{1}{5}x^5\right]_0^1=\left(\frac{1}{5}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)=0.$$

Problem 2.
Find the volumes of the indicated solids by an iterated integration.
(a) The tetrahedron bounded by the coordinate planes and the plane $3x+4y+z-12=0$.
(b) The solid bounded by the parabolic cylinder $x^2=4y$ and the planes $z=0$ and $5y+9z-45=0$.

Solution

$$\text{(a)}\quad \text{Volume}=\int_0^4\left[\int_0^{(-3/4)x+3}(12-3x-4y)\,dy\right]dx=24.$$

(b)

Solution 1

The plane $5y+9z=45$ intersects the xy-plane in the line $y=9$, so the region E (in xy plane) is

$$E=\{(x,y,z): -6\le x\le 6,\ x^2/4\le y\le 9,\ 0\le z\le 5-(5/9)y\}$$

and

$$\text{Volume}=\int_{-6}^{6}\int_{x^2/4}^{9}\int_{0}^{5-(5/9)y}1\cdot\,dzdydx=\int_{-6}^{6}\int_{x^2/4}^{9}\left(5-\frac{5}{9}y\right)dydx=144.$$

Solution 2

The plane $5y + 9z = 45$ intersects the xy-plane in the line $y = 9$, so the region E (in xy plane) is

$$E = \{(x, y, z) : -2\sqrt{y} \le x \le 2\sqrt{y},\, 0 \le y \le 9,\, 0 \le z \le 5 - (5/9)y\}$$

and

$$\text{Volume} = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \int_0^{5-(5/9)y} 1 \cdot dz dx dy = \int_0^9 \int_{-2\sqrt{y}}^{2\sqrt{y}} \left(5 - \frac{5}{9}y\right) dx dy = 144.$$

Problem 3.

S is the smaller region bounded by $\theta = \pi/6$ and $r = 4\sin\theta$. Find the area of the region S by calculating $\iint_S r\, dr d\theta$.

Solution

$$\begin{aligned}\text{Area } &= \int_0^{\pi/6} \left[\int_0^{4\sin\theta} r\, dr\right] d\theta = \int_0^{\pi/6} \left[\frac{r^2}{2}\right]_{r=0}^{r=4\sin\theta} d\theta \\ &= \int_0^{\pi/6} 8\sin^2\theta\, d\theta = \int_0^{\pi/6} 4(1 - \cos 2\theta)\, d\theta \\ &= \Big[4\theta - 2\sin 2\theta\Big]_0^{\pi/6} = \frac{2}{3}\pi - \sqrt{3} \approx 0.3623.\end{aligned}$$

Problem 4.

Evaluate the following double integral by using polar coordinates

$$\iint_S \sqrt{4 - x^2 - y^2}\, dA,$$

where S is the first quadrant sector of the circle $x^2 + y^2 = 4$ between $y = 0$ and $y = x$.

Solution

$$\begin{aligned}&\int_0^{\pi/4} \left[\int_0^2 (4 - r^2)^{1/2} r\, dr\right] d\theta = \int_0^{\pi/4} \left[-\frac{(4 - r^2)^{3/2}}{3}\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{\pi/4} \frac{8}{3}\, d\theta = \left[\frac{8\theta}{3}\right]_0^{\pi/4} = \frac{2\pi}{3} \approx 2.0944.\end{aligned}$$

Problem 5.

Find the volume of the solid lying under the graph of $z = f(x, y) = x^3 + 4y$ and above the region R in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution

The region R is

$$R = \{(x, y) : 0 \le x \le 2,\, x^2 \le y \le 2x\}.$$

Therefore

$$\begin{aligned}V = \iint_R f(x, y)\, dA = \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\, dy dx = \int_0^2 \left[x^3 y + 2y^2\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[(2x^4 + 8x^2) - (x^5 + 2x^4)\right] dx \\ = \int_0^2 (8x^2 - x^5)\, dx = \left[\frac{8}{3}x^3 - \frac{1}{6}x^5\right]_0^2 = \frac{32}{3}\end{aligned}$$

Alternative Solution

One can also look at the region R as::

$$R = \{(x, y) : 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y}\}.$$

Then

$$V = \iint_R f(x, y)\, dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^3 + 4y)\, dxdy = \int_0^4 \left[\frac{1}{4}x^4 + 4xy\right]_{x=y/2}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left[\left(\frac{1}{4}y^2 + 4y^{3/2}\right) - \left(\frac{1}{64}y^4 + 2y^2 + 2x^4\right)\right] dx$$

$$= \int_0^4 \left(-\frac{7}{4}y^2 + 4y^{3/2} - \frac{1}{64}y^4\right) dy = \left[-\frac{7}{12}y^3 + \frac{8}{5}y^{5/2} - \frac{1}{320}y^5\right]_0^4 = \frac{32}{3}.$$

Problem 6.
Evaluate $\iint_R (2x - y)\, dA$, when R is the region bounded by the parabola $x = y^2$ and the line $x - y = 2$.

Solution

The region R is

$$R = \{(x, y) : -1 \le y \le 2,\ y^2 \le x \le y + 2\}$$

and

$$\iint_R (2x - y)\, dA = \int_{-1}^{2} \int_{y^2}^{y+2} (2x - y)\, dxdy = \int_{-1}^{2} \left[x^2 - xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^{2} \left[(y+2)^2 - y(y+2)\right] - \left[y^4 - y^3\right] dx$$

$$= \int_{-1}^{2} (4 + 2y + y^3 - y^4)\, dy = \left[4y + y^2 + \frac{1}{4}y^4 - \frac{1}{5}y^5\right]_{-1}^{2} = \frac{233}{20}.$$

Alternative Solution

We can also write

$$\iint_R (2x - y)\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (2x - y)\, dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} (2x - y)\, dydx.$$

Problem 7.
Evaluate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy.$$

Solution

The integral

$$\int \frac{\sin x}{x}\, dx$$

cannot be evaluated in terms of elementary functions. We attempt to evaluate the original integral by reversing the order of integration. We have

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy = \iint_R \frac{\sin x}{x}\, dA,$$

where

$$R = \{(x, y) : 0 \le y \le 1,\ y \le x \le 1\}.$$

However, by viewing the region R as

$$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\},$$

we have

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy = \iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dydx = \int_0^1 \left[\frac{y \sin x}{x}\right]_{y=0}^{y=x} dx = \int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + 1.$$

Problem 8.

Evaluate

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx.$$

Solution

The integral

$$\int \sin(y^2)\, dy$$

cannot be evaluated in terms of elementary functions. We attempt to evaluate the original integral by reversing the order of integration. We have

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx = \iint_R \sin(y^2)\, dA,$$

where

$$R = \{(x, y) : 0 \le x \le 1,\ x \le y \le 1\}.$$

However, by viewing the region R as

$$R = \{(x, y) : 0 \le y \le 1,\ 0 \le x \le y\},$$

we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dydx = \iint_R \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dxdy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy = \left[-\frac{1}{2}\cos(y^2)\right]_0^1 = \frac{1}{2}(1 - \cos 1).$$

Problem 9.

Evaluate $\iint_R (2x + 3y)\, dA$, where R is the region in the first quadrant bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

the region R is a polar rectangular

$$R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi/2\}.$$

Thus,

$$\iint_R (2x + 3y)\, dA = \int_0^{\pi/2} \int_1^2 (2r\cos\theta + 3r\sin\theta) r\, dr d\theta = \int_0^{\pi/2} \left[\frac{2}{3}r^3 \cos\theta + r^3 \sin\theta\right]_{r=1}^{r=2} d\theta = \int_0^{\pi/2} \left(\frac{14}{3}\cos\theta + 7\sin\theta\right) d\theta$$

$$= \left[\frac{14}{3}\sin\theta - 7\cos\theta\right]_0^{\pi/2} = \frac{35}{3}.$$

Problem 10.
Use a double integral to find the area enclosed by one loop of the three-leaved rose $r = \sin(3\theta)$.

Solution

A loop of the rose is described by the region

$$R = \{(r, \theta) : 0 \le \theta \le \pi/3,\ 0 \le r \le \sin(3\theta)\}.$$

Thus

$$A = \iint_R 1\,dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r\,dr\theta = \int_0^{\pi/3} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\sin(3\theta)} d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/3} (1 - \cos(6\theta))\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{6}\sin(6\theta)\right]_{\theta=0}^{\theta=\pi/3} = \frac{\pi}{12}.$$

Problem 11.
Evaluate the integral

$$\int_{-\infty}^{\infty} \exp(-x^2)\,dx.$$

The integral

$$\int \exp(-x^2)\,dx$$

cannot be evaluated in terms of elementary functions. Switching to the polar coordinates will help. First, we notice that if $A = \int_{-\infty}^{\infty} \exp(-x^2)\,dx$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x^2 - y^2)\,dxdy = A^2. \qquad \textit{(Do you know why?)}$$

Next,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x^2 - y^2)\,dxdy = \lim_{a\to\infty} \int_0^{a}\int_0^{2\pi} \exp(-r^2)r\,drd\theta = \lim_{a\to\infty} 2\pi\left[-\frac{1}{2}\exp(-r^2)\right]_{r=0}^{r=a} = 2\pi \lim_{a\to\infty}\left[-\frac{1}{2}exp(-a^2) + \frac{1}{2}\right]$$

$$= 2\pi\frac{1}{2} = \pi.$$

Thus

$$\int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi}.$$

Problem 12.
Evaluate $\iiint_E yz\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

Solution

The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$, or $z = 1 - x - y$. Next, we observe that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E onto xy-plane is the region R

$$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1 - x\}.$$

Therefore

$$\iiint_E yz\,dV = \iint_R \left[\int_0^{1-x-y} yz\,dz \right] dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\,dzdydx = \int_0^1 \int_0^{1-x} \left[\frac{1}{2} yz^2 \right]_{z=0}^{z=1-x-y} dydx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} y(1-x-y)^2\,dydx = \int_0^1 \int_0^{1-x} \left(\frac{1}{2}x^2 y + xy^2 + \frac{1}{2}y^3 - xy - y^2 + \frac{1}{2}y \right) dydx$$

$$= \int_0^1 \left[-\frac{5}{24}(1-x)^4 + (1-x)^2 \left(\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} \right) \right] dx = \frac{1}{120}.$$