Practice Problems Ic
Math 250, Spring 2026 – Jacek Polewczak

Problem 1.
Find the equation of the tangent plane at $(2,-1)$ for $f(x,y) = \dfrac{x^2}{y}$.

Solution

$f_x = 2x/y,\ f_y = -x^2/y^2;\quad f_x(2,-1) = -4$ and $f_y(2,-1) = -4$.
The tangent plane is $z = -4(x-2) - 4(y+1) + f(2,-1)$, or $z = -4x - 4y$.

Problem 2.
Find the directional derivative of f at the given point P in the direction of $\mathbf{a}$.

(a) $f(x,y) = \exp(-xy);\quad P = (-1,1);\quad \mathbf{a} = -\mathbf{i} + \sqrt{3}\mathbf{j}$

(b) $f(x,y,z) = x^2 + y^2 + z^2;\quad P = (1,-1,2);\quad \mathbf{a} = \sqrt{2}\mathbf{i} - \mathbf{j} - \mathbf{k}$

Solution

(a) The directional unit vector is $\mathbf{u} = \dfrac{1}{2} < -1, \sqrt{3} >$.

Thus, $D_{\mathbf{u}} f(x,y) =< -y\exp(-xy), -x\exp(-xy) > \cdot \dfrac{< -1, \sqrt{3} >}{2}$ and

$$D_{\mathbf{u}} f(-1,1) =< -\mathrm{e}, \mathrm{e} > \cdot \frac{< -1, \sqrt{3} >}{2} = \frac{\mathrm{e} + e\sqrt{3}}{2} \approx 3.7132.$$

(b) The directional unit vector is $\mathbf{u} = \dfrac{1}{2} < \sqrt{2}, -1, -1 >$.

Thus, $D_{\mathbf{u}} f(x,y) =< 2x, 2y, 2z > \cdot \dfrac{1}{2} < \sqrt{2}, -1, -1 >$ and

$$D_{\mathbf{u}} f(1,-1,2) = \sqrt{2} - 1 \approx 0.4142.$$

Problem 3.
Use the method of Lagrange's multipliers to find the minimum of $f(x,y) = x^2 + 4xy + y^2$, subject to the constraint $x - y - 6 = 0$.

Solution

$f(x,y) = x^2 + 4xy + y^2,\quad g(x,y) = x - y - 6.\quad \nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = x - y - 6 = 0$ is equivalent to

$$< 2x + 4y, 4x + 2y >= \lambda < 1, -1 >,\quad x - y = 6 \quad \Longrightarrow \quad 2x + 4y = \lambda,\quad 4x + 2y = -\lambda,\quad x - y = 6.$$

Critical point is $(3,-3)$ (with the corresponding $\lambda = -6$) and the minimum is $f(3,-3) = -18$.

Problem 4.
Use the method of Lagrange's multipliers to find the least distance between the origin and the plane $x + 3y - 2z = 4$.

Solution

Minimize the square of the distance to the plane, $f(x,y,z) = x^2 + y^2 + z^2$, subject to $g(x,y,z) = x + 3y - 2z - 4 = 0$.
$\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = x + 3y - 2z - 4 = 0$ is equivalent to

$$< 2x, 2y, 2z >= \lambda < 1, 3, -2 >,\quad x + 3y - 2z - 4 = 0 \quad \Longrightarrow \quad 2x = \lambda,\quad 2y = 3\lambda,\quad 2z = -2\lambda,\quad x + 3y - 2z = 4.$$

Critical point is $(2/7, 6/7, -4/7)$ (with the corresponding $\lambda = 4/7$). The nature of the problem indicates that this will give a minimum rather than a maximum (**WHY ???**). The least distance to the plane is

$$\left[f\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7} \right) \right]^{\frac{1}{2}} = \left(\frac{8}{7} \right)^{\frac{1}{2}} \approx 1.0690.$$

Problem 5.
Find a point on the surface $z = 2x^2 + 3y^2$ where the the tangent plane is parallel to the plane $8x - 3y - z = 0$.

Solution

$< 8, -3, -1 >$ is the normal to $8x - 3y - z = 0$. $\nabla F(x, y, z) =< 4x, 6y, -1 >$ is normal to $z = 2x^2 + 3y^2$ at (x, y, z). $4x = 8$ and $6y = -3$, if $x = 2$ and $y = -1/2$; then $z = 8.75$. At $(2, -1/2, 8.75)$.

Problem 6.
Find all critical points; indicate whether each such point gives a local minimum, a local maximum, or whether it is a saddle point.

$$f(x, y) = x^2 + a^2 - 2ax \cos y; \quad S = \{(x, y) : -\infty < x < \infty, -\pi < y < \pi\}$$

Solution

$\nabla f(x, y) =< 2x - 2a \cos y, 2ax \sin y >=< 0, 0 >$ at $(0, \pm\pi/2)$, $(a, 0)$.
$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(2ax \cos y) - (2a \sin y)^2$, $f_{xx} = 2$. At $(0, \pm\pi/2)$: $D = -4a^2 < 0$, so $(0, \pm\pi/2)$ are saddle points.
At $(a, 0)$: $D = 4a^2 > 0$ and $f_{xx}(a, 0) > 0$, so $(a, 0)$ is a local minimum.

Problem 7.
Use the Second-Partials Test to find the shortest distance from the origin to the plane $x + 2y + 3z = 12$.

Solution

Let s be the distance from the origin to (x, y, z) on the plane. Then $s^2 = x^2 + y^2 + z^2$ and The equation of the plane is $x + 2y + 3z = 12$.
Minimize $s^2 = f(y, z) = (12 - 2y - 3z)^2 + y^2 + z^2$.
$\nabla f(y, z) =< -48 + 12z + 10y, -72 + 12y + 20z >=< 0, 0 >$ at $(12/7, 18/7)$.
$D(12/7, 18/7) = f_{yy}(12/7, 18, 7)f_{zz}(12/7, 18/7) - [f_{yz}(12/7, 18/7)]^2 = 56 > 0$ and $f_{yy}(12/7, 18/7) = 10 > 0$, so the point $(12/7, 18/7)$ is a local minimum.

Problem 8.
Use the Second-Partials Test to find the shape of the rectangular box of volume V_0 for which the sum of the edge lengths is least.

Solution

Let L denote the sum of the edge lengths for a box of dimensions x, y, z.
Minimize $L = 4x + 4y + 4z$ subject to $V_0 = xyz$.
$L(x, y) = 4x + 4y + 4V_0/(xy)$, $x > 0$, $y > 0$.

$$\nabla L(x, y) = 4\left\langle \frac{x^2y - V_0}{x^2y}, \frac{xy^2 - V_0}{xy^2} \right\rangle =< 0, 0 > \quad \Longrightarrow \quad x^2y = V_0 \quad \text{and} \quad xy^2 = V_0 \quad \Longrightarrow \quad x = y.$$

Therefore (since $x^2y = V_0$), $x = y = V_0^{\frac{1}{3}}$. Finally, also $z = V_0^{\frac{1}{3}}$.
$L_{xx} = 8V_0/(x^3y)$; $D = L_{xx}L_{yy} - L_{xy}^2 = [8V_0/(x^3y)][8V_0/(xy^3)] - (4V_0/(x^2y^2)]$.
At $(V_0^{\frac{1}{3}}, V_0^{\frac{1}{3}})$: $D > 0$ and $L_{xx} > 0$, so the point $(V_0^{\frac{1}{3}}, V_0^{\frac{1}{3}})$ is a local minimum.
There are no other critical points, and as $(x, y) \to (0^+, 0^+)$, $L(x, y) \to \infty$.
Conclusion: The optimal box is a cube of the dimensions $(V_0^{\frac{1}{3}}, V_0^{\frac{1}{3}}, V_0^{\frac{1}{3}})$.