

**Practice Problems Ib**  
**Math 250, Spring 2026 – Jacek Polewczak**

**Problem 1.**

Show that if an object moves subject only to a central force (that is  $\mathbf{r}''(t) = c\mathbf{r}(t)$ , where  $c$  is a constant), then the object moves in a plane.

*Hint:* Show that  $\mathbf{r}(t) \times \mathbf{r}'(t)$  is a constant vector for all  $t \geq 0$ .

**Solution**

$$\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times c\mathbf{r}(t) = c[\mathbf{r}(t) \times \mathbf{r}(t)] = 0.$$

Integrating both sides with respect to  $t$ , we obtain  $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{c}$ , a constant vector (independent of  $t$ ). Therefore,  $\mathbf{r}(t)$  is perpendicular to the vector  $\mathbf{c}$  for each value of  $t$ , so the path is in a plane (whose normal is vector  $\mathbf{c}$ ).

**Problem 2.**

Find the natural domain for the function

$$f(x, y) = \frac{\sqrt{\frac{x}{y}}}{\sqrt{1 - x^2 - y^2}}$$

**Solution** The natural domain is the subset of  $\mathbf{R}^2$

$$\begin{aligned} D &= \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, \frac{x}{y} \geq 0\} \\ &= \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, x \geq 0, y > 0\} \cup \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, x \leq 0, y < 0\} \end{aligned}$$

**Problem 3.**

Describe the largest set  $S$  on which the following functions are continuous.

$$(a) \quad f(x, y) = \ln(1 - x^2 - y^2) \quad (b) \quad f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & xy \neq 0 \\ 1 & xy = 0 \end{cases} \quad (c) \quad f(x, y) = \frac{x^3 - 2x^4y - y^5 - 4}{x^2 + y^2 - 9}$$

**Solution**

(a)  $f(x, y) = \ln(1 - x^2 - y^2) = g(h(x, y))$ , where  $g(t) = \ln t$  and  $h(x, y) = 1 - x^2 - y^2$ . Functions  $g(t)$  and  $h(x, y)$  are continuous for  $t > 0$  and  $(x, y) \in \mathbf{R}^2$ , respectively. Therefore,  $f(x, y)$  being a composition of  $g$  and  $h$  is continuous on the domain  $S = \{(x, y) \in \mathbf{R}^2 : 1 - x^2 - y^2 > 0\} = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$  (inside of the circle with radius 1 centered at the origin).

(b) When  $xy \neq 0$ ,  $\sin(xy)/xy$  is a quotient of two continuous functions. Indeed, both  $\sin t$  and  $xy$  are continuous functions on  $\mathbf{R}$  and  $\mathbf{R}^2$ , respectively. Therefore, the only suspicious point is  $(0, 0)$ . However, we know that

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1, \text{ thus}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy} = 1.$$

Since  $f(0, 0) = 1$ , the definition of continuity implies that  $f(x, y)$  is continuous also at  $(0, 0)$ . Thus, the function is continuous for  $(x, y) \in \mathbf{R}^2$ , i.e.,  $S = \mathbf{R}^2$ .

(c) The function is a quotient of two polynomials. It is continuous for all  $(x, y) \in \mathbf{R}^2$  except at the points where the denominator is zero. The denominator is zero on the set  $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 9\}$  (the circle with radius 3 centered at the origin). The set  $S = \mathbf{R}^2 \setminus \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 9\}$  (the whole plane except for the circle with radius 3 centered at the origin).

**Problem 4.**

If  $f(x, y) = \frac{xy^2}{x^2 + y^4}$ , analyze the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ .

**Solution**

We let  $(x, y) \rightarrow (0, 0)$  along any non-vertical line through the origin. Then  $y = mx$ , where  $m$  is the slope and  $f(x, y) = f(x, mx) = \frac{m^2x}{1 + m^2x^2}$ . So,  $f(x, y) \rightarrow 0$  as  $(x, y) \rightarrow 0$  along  $y = mx$ . On a the vertical line through the origin  $x = 0$  and  $f(0, y) = 0$ , for  $y \neq 0$ . Thus  $f(x, y) \rightarrow 0$  as  $(x, y) \rightarrow 0$  along a vertical line. In spite of the fact that  $f(x, y) \rightarrow 0$  along any line through the origin, it does **NOT** show that the given limit is 0. Indeed, if we let  $(x, y) \rightarrow (0, 0)$  along the parabola  $x = y^2$ , we have

$$f(x, y) = f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2},$$

so  $f(x, y) \rightarrow \frac{1}{2}$  as  $(x, y) \rightarrow 0$  along  $x = y^2$ . Since different paths lead to different limiting values, the given limit does not exist.

**Problem 5.**

Use  $\epsilon - \delta$  proof to show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0.$$

**Solution**

Let  $\epsilon > 0$ . We want to find  $\delta > 0$  such that if  $0 < \sqrt{x^2 + y^2} < \delta$  then  $\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \epsilon$ . First, we notice that  $\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| = \frac{3x^2|y|}{x^2 + y^2}$ . But since  $x^2 \leq x^2 + y^2$ , so  $x^2/(x^2 + y^2) \leq 1$ , and therefore

$$\frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}.$$

Thus if we choose  $\delta = \epsilon/3$  and let  $0 < \sqrt{x^2 + y^2} < \delta$ , then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = \epsilon.$$

Hence,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0.$$