Solutions to Practice Problems I
Math 250, Spring 2024 – Jacek Polewczak

Problem 1.
Let $\mathbf{v} =< a, b, c >$ be a vector perpendicular to $< 4, 3, 6 >$ and $< -2, -3, -2 >$. Then

$$< a, b, c > \cdot < 4, 3, 6 >= 0 \quad \text{and} \quad < a, b, c > \cdot < -2, -3, -2 >= 0.$$

Therefore, $4a + 3b + 6c = 0$ and $-2a - 3b - 2c = 0$. This pair of equations for a, b, and c has infinitely many solutions. For example, with $c = 3$, we can solve the equations for a and b. The result is $a = -6$ and $b = 2$. Thus, vector $< -6, 2, 3 >$, which has length 7, is perpendicular to $< 4, 3, 6 >$ and $< -2, -3, -2 >$. Then the required vectors are:

$$\left(\frac{10}{7}\right) < -6, 2, 3 > \quad \text{and} \quad \left(\frac{-10}{7}\right) < -6, 2, 3 > .$$

Note: *Any choice of $c \neq 0$ in $4a + 3b + 6c = 0$ and in $-2a - 3b - 2c = 0$ works. On the other hand, $c = 0$ leads to the equations $4a + 3b = 0$ and $2a + 3b = 0$ without solutions.*

Problem 2.
Normals to the planes are $< 3, -2, 5 >$ and $< 4, -2, -3 >$, so the cosine of the smaller angle is

$$\cos\theta = \frac{|12 + 4 - 15|}{(38)^{1/2}(29)^{1/2}} = \frac{1}{\sqrt{1102}}.$$

Thus, the angle $\theta = 1.540668$ radians or $\theta = 88.27°$. Note that with

$$\cos\theta = \frac{12 + 4 - 15}{(38)^{1/2}(29)^{1/2}} = -\frac{1}{\sqrt{1102}},$$

$\theta = 1.600925$ radians or $\theta = 91.73°$.

Problem 3.
Suppose $P(x, y, z)$ is a point in the plane $ax + by + cz + d = 0$. The distance from any point $Q(x_0, y_0, z_0)$ to the plane equals the length of the orthogonal projection of the vector $\overrightarrow{PQ}$ onto a vector $\mathbf{n} =< a, b, c >$ normal to the plane, which is

$$\frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|ax_0 + bx_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

The point $(0, 0, 9)$ belongs to the plane $-3x+2y+z = 9$. The distance between the parallel planes $-3x+2y+z = 9$ and $6x - 4y - 2z = 19$ is

$$\frac{|6(0) - 4(0) - 2(9) - 19|}{(36 + 16 + 4)^{1/2}} = \frac{37}{\sqrt{56}} \approx 4.9443.$$

Problem 4.
The line segment between the points is perpendicular to the plane. Its midpoint, $(2, 1, 1)$, belongs to the plane we are looking for. Then, vector $< 6 - (-2), 1 - 1, -2 - 4 >=< 8, 0, -6 >$ is perpendicular to the plane. Therefore, $8(x - 2) + 0(y - 1) - 6(z - 1) = 0$ or $4x - 3z = 5$ is the equation of the plane.

Problem 5.
Two vectors in the plane are

$$< 4 - (-1), -2 - (-2), 1 - (-3) > =< 5, 0, 4 >$$

and

$$< 5 - 4, 1 - (-2), 6 - 1 > =< 1, 3, 5 >$$

A vector normal to the plane is $< 5, 0, 4 > \times < 1, 3, 5 >=< -12, -21, 15 >= -3 < 4, 7, -5 >$. Therefore, an equation of the lane is $4(x+1) + 7(y+2) - 5(x+3) = 0$ or $4x + 7y - 5z = -3$.

Problem 6.

The cross product of vectors normal to those planes is parallel to the line of intersection of the planes. Thus, a normal vector to the plane we seek is $< 4, -3, 2 > \times < 3, 2, -1 >=< -1, 10, 17 >$. Equation of the plane is $-1(x-6) + 10(y-2) + 17(z+1) = 0$ or $x - 10y - 17z = 3$.

Problem 7.

Parametric equation: $x = -1 + 4t, y = 3 + 2t, z = 2 - t.$

Problem 8.

(a) $P(1, -1, 0)$ is on the line and so are vectors $\mathbf{PQ} =< 1, 0, 3 >$ and $\mathbf{a} =< 2, 3, -6 >$. Then

$$d = \frac{|\mathbf{PQ} \times \mathbf{a}|}{|\mathbf{a}|} = \frac{| < -9, 12, 3 > |}{\sqrt{4+9+36}} = \frac{3\sqrt{26}}{7} \approx 2.1853. \quad \text{(see page 847 of the textbook)}$$

(b) $P(1, -2, 0)$ and $Q(0, 1, 0)$ are on the lines $x = 1 + 2t$, $y = -2 + 3t$, $z = -4t$ and $x = 3t$, $y = 1 + t$, $z = -5t$, respectively. Now, the length of the projection vector of $\mathbf{PQ} =< -1, 3, 0 >$ on vector

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} =< 2, 3, -4 > \times < 3, 1, -5 >=< -11, -2, -7 >$$

is the distance between these skew lines. Indeed, vector $\mathbf{n}$ is perpendicular to both lines. Therefore,

$$d = \frac{|\mathbf{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|\mathbf{PQ} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|} = \frac{| < -1, 3, 0 > \cdot < -11, -2, -7 > |}{| < -11, -2, -7 > |} = \frac{5}{\sqrt{174}} \approx 0.3790.$$

Problem 9.

$$\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{r}'(t)\right] = \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times c\mathbf{r}(t) = c[\mathbf{r}(t) \times \mathbf{r}(t)] = 0.$$

Integrating both sides with respect to t, we obtain $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{c}$, a constant vector (independent of t). Therefore, $\mathbf{r}(t)$ is perpendicular to the vector $\mathbf{c}$ for each value of t, so the path is in a plane (whose normal is vector $\mathbf{c}$).

Problem 10.

The natural domain is the subset of $\mathbf{R}^2$

$$\begin{aligned} D =& \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, \frac{x}{y} \geq 0\} \\ =& \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, x \geq 0, y > 0\} \cup \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1, x \leq 0, y < 0\} \end{aligned}$$

Problem 11.

(a) $f(x, y) = \ln(1 - x^2 - y^2) = g(h(x, y))$, where $g(t) = \ln t$ and $h(x, y) = 1 - x^2 - y^2$. Functions $g(t)$ and $h(x, y)$ are continuous for $t > 0$ and $(x, y) \in \mathbf{R}^2$, respectively. Therefore, $f(x, y)$ being a composition of g and h is continuous on the domain $S = \{(x, y) \in \mathbf{R}^2 : 1 - x^2 - y^2 > 0\} = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$ (inside of the circle with radius 1 centered at the origin).

(b) When $xy \neq 0$, $\sin(xy)/xy$ is a quotient of two continuous functions. Indeed, both $\sin t$ and xy are continuous functions on $\mathbf{R}$ and $\mathbf{R}^2$, respectively. Therefore, the only suspicious point is $(0, 0)$. However, we know that $\lim_{z \to 0} \frac{\sin z}{z} = 1$, thus

$$\lim_{(x,y) \to (0,0)} \frac{\sin(xy)}{xy} = 1.$$

Since $f(0, 0) = 1$, the definition of continuity implies that $f(x, y)$ is continuous also at $(0, 0)$. Thus, the function is continuous for $(x, y) \in \mathbf{R}^2$, i.e., $S = \mathbf{R}^2$.

(c) The function is a quotient of two polynomials. It is continuous for all $(x, y) \in \mathbf{R}^2$ except at the points where the denominator is zero. The denominator is zero on the set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 9\}$ (the circle with radius 3 centered at the origin). The set $S = \mathbf{R}^2 \setminus \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 9\}$ (the whole plane except for the circle with radius 3 centered at the origin).

Problem 12.
We let $(x, y) \to (0, 0)$ along any non-vertical line through the origin. Then $y = mx$, where m is the slope and $f(x, y) = f(x, mx) = \dfrac{m^2x}{1 + m^2x^2}$. So, $f(x, y) \to 0$ as $(x, y) \to 0$ along $y = mx$. On a the vertical line through the origin $x = 0$ and $f(0, y) = 0$, for $y \neq 0$. Thus $f(x, y) \to 0$ as $(x, y) \to 0$ along a vertical line. In spite of the fact that $f(x, y) \to 0$ along any line through the origin, it does **NOT** show that the given limit is 0. Indeed, if we let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2},$$

so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to 0$ along $x = y^2$. Since different paths lead to different limiting values, the given limit does not exist.

Problem 13.

Let $\epsilon > 0$. We want to find $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$ then $\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon$. First, we notice that $\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \frac{3x^2|y|}{x^2+y^2}$. But since $x^2 \leq x^2 + y^2$, so $x^2/(x^2 + y^2) \leq 1$, and therefore

$$\frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}.$$

Thus if wew choose $\delta = \epsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2 + y^2} - 0\right| \leq 3\sqrt{x^2 + y^2} < 3\delta = \epsilon.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0.$$

Problem 14.
$f_x = 2x/y,\ f_y = -x^2/y^2;\quad f_x(2, -1) = -4$ and $f_y(2, -1) = -4$.
The tangent plane is $z = -4(x - 2) - 4(y + 1) + f(2, -1)$, or $z = -4x - 4y$.

Problem 15.
(a) The directional unit vector is $\mathbf{u} = \dfrac{1}{2} < -1, \sqrt{3} >$.

Thus, $D_{\mathbf{u}}f(x, y) =< -y\exp(-xy), -x\exp(-xy) > \cdot \dfrac{< -1, \sqrt{3} >}{2}$ and

$$D_{\mathbf{u}}f(-1, 1) =< -\mathrm{e}, \mathrm{e} > \cdot \frac{< -1, \sqrt{3} >}{2} = \frac{\mathrm{e} + e\sqrt{3}}{2} \approx 3.7132.$$

(b) The directional unit vector is $\mathbf{u} = \dfrac{1}{2} < \sqrt{2}, -1, -1 >$.

Thus, $D_{\mathbf{u}}f(x, y) =< 2x, 2y, 2z > \cdot \dfrac{1}{2} < \sqrt{2}, -1, -1 >$ and

$$D_{\mathbf{u}}f(1, -1, 2) = \sqrt{2} - 1 \approx 0.4142.$$

Problem 16.

$f(x,y)=x^2+4xy+y^2$, $g(x,y)=x-y-6$. $\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=x-y-6=0$ is equivalent to

$$<2x+4y,4x+2y>=\lambda<1,-1>,\quad x-y=6 \quad\Longrightarrow\quad 2x+4y=\lambda,\quad 4x+2y=-\lambda,\quad x-y=6.$$

Critical point is $(3,-3)$ (with the corresponding $\lambda=-6$) and the minimum is $f(3,-3)=-18$.

Problem 17.

Minimize the square of the distance to the plane, $f(x,y,z)=x^2+y^2+z^2$, subject to $g(x,y,z)=x+3y-2z-4=0$.

$\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=x+3y-2z-4=0$ is equivalent to

$<2x,2y,2z>=\lambda<1,3,-2>,\quad x+3y-2z-4=0 \quad\Longrightarrow\quad 2x=\lambda,\quad 2y=3\lambda,\quad 2z=-2\lambda,\quad x+3y-2z=4.$

Critical point is $(2/7,6/7,-4/7)$ (with the corresponding $\lambda=4/7$). The nature of the problem indicates that this will give a minimum rather than a maximum (**WHY ???**). The least distance to the plane is

$$\left[f\left(\frac{2}{7},\frac{6}{7},-\frac{4}{7}\right)\right]^{\frac{1}{2}}=\left(\frac{8}{7}\right)^{\frac{1}{2}}\approx 1.0690.$$

Problem 18.

$<8,-3,-1>$ is the normal to $8x-3y-z=0$. $\nabla F(x,y,z)=<4x,6y,-1>$ is normal to $z=2x^2+3y^2$ at (x,y,z). $4x=8$ and $6y=-3$, if $x=2$ and $y=-1/2$; then $z=8.75$. At $(2,-1/2,8.75)$.

Problem 19.

$\nabla f(x,y)=<2x-2a\cos y,2ax\sin y>=<0,0>$ at $(0,\pm\pi/2)$, $(a,0)$.

$D=f_{xx}f_{yy}-f_{xy}^2=(2)(2ax\cos y)-(2a\sin y)^2$, $f_{xx}=2$. At $(0,\pm\pi/2)$: $D=-4a^2<0$, so $(0,\pm\pi/2)$ are saddle points.

At $(a,0)$: $D=4a^2>0$ and $f_{xx}(a,0)>0$, so $(a,0)$ is a local minimum.

Problem 20.

Let s be the distance from the origin to (x,y,z) on the plane. Then $s^2=x^2+y^2+z^2$ and The equation of the plane is $x+2y+3z=12$.

Minimize $s^2=f(y,z)=(12-2y-3z)^2+y^2+z^2$.

$\nabla f(y,z)=<-48+12z+10y,-72+12y+20z>=<0,0>$ at $(12/7,18/7)$.

$D(12/7,18/7)=f_{yy}(12/7,18,7)f_{zz}(12/7,18/7)-[f_{yz}(12/7,18/7)]^2=56>0$ and $f_{yy}(12/7,18/7)=10>0$, so the point $(12/7,18/7)$ is a local minimum.

$s^2=504/49$, so the shortest distance is $s=(6\sqrt{14})/7\approx 3.2071$.

Problem 21.

Let L denote the sum of the edge lengths for a box of dimensions x, y, z.

Minimize $L=4x+4y+4z$ subject to $V_0=xyz$.

$L(x,y)=4x+4y+4V_0/(xy)$, $x>0$, $y>0$.

$$\nabla L(x,y)=4\left\langle\frac{x^2y-V_0}{x^2y},\frac{xy^2-V_0}{xy^2}\right\rangle=<0,0>\quad\Longrightarrow\quad x^2y=V_0\quad\text{and}\quad xy^2=V_0\quad\Longrightarrow\quad x=y.$$

Therefore (since $x^2y=V_0$), $x=y=V_0^{\frac{1}{3}}$. Finally, also $z=V_0^{\frac{1}{3}}$.

$L_{xx}=8V_0/(x^3y)$; $D=L_{xx}L_{yy}-L_{xy}^2=[8V_0/(x^3y)][8V_0/(xy^3)]-(4V_0/(x^2y^2)]$.

At $(V_0^{\frac{1}{3}},V_0^{\frac{1}{3}})$: $D>0$ and $L_{xx}>0$, so the point $(V_0^{\frac{1}{3}},V_0^{\frac{1}{3}})$ is a local minimum.

There are no other critical points, and as $(x,y)\to(0^+,0^+)$, $L(x,y)\to\infty$.

Conclusion: The optimal box is a cube of the dimensions $(V_0^{\frac{1}{3}},V_0^{\frac{1}{3}},V_0^{\frac{1}{3}})$.