Solved examples for line integrals and Green's Theorem
Math 250, Spring 2026 – Jacek Polewczak

Problem 1.

Evaluate $\int_C (1+xy)\,ds$, where C is the quarter-circle described by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \le t \le \pi/2$.

Solution

Here $x(t) = \cos t$ and $y(t) = \sin t$, so $x'(t) = -\sin t$ and $y'(t) = \cos t$. Thus

$$\int_C (1+xy)\,ds = \int_0^{\pi/2} (1+\cos(t)\sin(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^{\pi/2} (1+\cos(t)\sin(t))\sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= \int_0^{\pi/2} (1+\cos(t)\sin(t)) = \left[t + \frac{1}{2}\sin^2 t\right]_0^{\pi/2} = \frac{\pi}{2} + \frac{1}{2}$$

Problem 2.

Evaluate $\int_C (1+xy)\,dx$, where C is the quarter-circle described by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \le t \le \pi/2$.

Solution

Here $x(t) = \cos t$ and $y(t) = \sin t$, so $x'(t) = -\sin t$. We have

$$\int_C (1+xy)\,dx = \int_0^{\pi/2} [1+\cos(t)\sin(t)]x'(t)\,dt = -\int_0^{\pi/2} [1+\cos(t)\sin(t)]\sin t\,dt = -\int_0^{\pi/2} (\sin t + \cos(t)\sin^2 t)\,dt = -1 - \frac{1}{3}$$

Problem 3.

Evaluate $\int_C (1+xy)\,dy$, where C is the quarter-circle described by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \le t \le \pi/2$.

Solution

Here $x(t) = \cos t$ and $y(t) = \sin t$, so $y'(t) = \cos t$. We have

$$\int_C (1+xy)\,dy = \int_0^{\pi/2} [1+\cos(t)\sin(t)]y'(t)\,dt = \int_0^{\pi/2} [1+\cos(t)\sin(t)]\cos t\,dt = \int_0^{\pi/2} (\cos t + \sin(t)\cos^2 t)\,dt = 1 + \frac{1}{3}$$

Problem 4.

Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment C_2 from $(1,1)$ to $(0,0)$.

Solution

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

We have the following parametrizations for C_1 and C_2:

$$C_1: \quad x(t) = t, \quad y(t) = t^2, \quad 0 \le t \le 1$$
$$C_2: \quad x(t) = 1-t, \quad y(t) = 1-t, \quad 0 \le t \le 1$$

Thus

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{(x'(t))^2 + (y'(t))^2}\,dt == 2\int_0^1 t\sqrt{1+t^2}\,dt = \left[2\left(\frac{1}{8}\right)\left(\frac{2}{3}\right)\left(1+4t^2\right)^{3/2}\right]_0^1 = \frac{5\sqrt{5}-1}{6}$$

and

$$\int_{C_2} 2x\,ds = \int_0^1 2x\sqrt{(x'(t))^2+(y'(t))^2}\,dt = 2\int_0^1 (1-t)\sqrt{1+1}\,dt = \left[2\sqrt{2}\left(t-\frac{1}{2}t^2\right)\right]_0^1 = \sqrt{2} \tag{1}$$

Therefore

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5}-1}{6} + \sqrt{2}$$

Remark 1. *Here is another parametrization of C_2:*

$$C_2: \quad x(t) = 1-t^2, \quad y(t) = 1-t^2, \quad 0 \le t \le 1$$

Note that

$$\int_{C_2} 2x\,ds = \int_0^1 2x\sqrt{(x'(t))^2+(y'(t))^2}\,dt = 4\sqrt{2}\int_0^1 (1-t^2)t\,dt = \sqrt{2} \tag{2}$$

This shows that the line integral in (1) is the same as the line integral in (2). The line integral does not depend on a parametrization of the curve!!!

Problem 5.

Evaluate $\int_C y\,dx + x^2\,dy$, where (a) C is the line segment from $(1,-1)$ to $(4,2)$, (b) C is the arc C_2 of the parabola $x = y^2$ from $(1,-1)$ to $(4,2)$ and (c) C is the arc C_3 of the parabola $x = y^2$ from $(4,2)$ to $(1,-1)$.

Solution

(a) We have the following parametrizations of C_1:

$$C_1: \quad x(t) = 1+3t, \quad y(t) = -1+3t, \quad 0 \le t \le 1$$

We have $dx = 3dt$ and $dy = 3dt$ and

$$\int_{C_1} y\,dx + x^2\,dy = \int_0^1 (-1+3t)(3)\,dt + (1+3t)^2(3)\,dt = 27\int_0^1 (t^2+t)\,dt = 27\left[\frac{1}{3}t^3+\frac{1}{2}t^2\right]_0^1 = \frac{45}{2}$$

(b) We have the following parametrizations of C_2:

$$C_2: \quad x(t) = t^2, \quad y(t) = t, \quad -1 \le t \le 2$$

We have $dx = 2tdt$ and $dy = dt$ and

$$\int_{C_2} y\,dx + x^2\,dy = \int_{-1}^2 t(2t\,dt) + (t^2)^2\,dt = \int_{-1}^2 (2t^2+t^4)\,dt = \left[\frac{2}{3}t^3+\frac{1}{5}t^5\right]_{-1}^2 = \frac{63}{5}$$

(c) We have the following parametrizations of C_3:

$$C_3: \quad x(t) = t^2, \quad y(t) = -t, \quad -2 \le t \le 1$$

We have $dx = 2tdt$ and $dy = -dt$ and

$$\int_{C_3} y\,dx + x^2\,dy = \int_{-2}^1 (-t)(2t\,dt) + (t^2)^2\,(-dt) = -\int_{-2}^1 (2t^2+t^4)\,dt = -\left[\frac{2}{3}t^3+\frac{1}{5}t^5\right]_{-1}^2 = -\frac{63}{5}$$

Remark 2. *Problem 5 shows that line integral witch respect to x and y depend not only on the endpoints but also on the curve joining these points. Also the direction in which a curve is traced changes the sign of the value of the line integral.*

Problem 6.

Evaluate $\int\limits_C kz\,ds$, where k is constant and C is the circular helix with parametric equations $x(t) = \cos t$, $y(t) = \sin t$, and $z = t$, where $0 \le t \le 2\pi$.

Solution

We have $x'(t) = -\sin t$, $y'(t) = \cos t$, and $z'(t) = 1$, therefore

$$\int\limits_C kz\,ds = \int\limits_0^{2\pi} kt\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt = \int\limits_C kz\,ds = \int\limits_0^{2\pi} kt\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2}k\int\limits_0^{2\pi} t\,dt$$
$$= \sqrt{2}k\left[\frac{1}{2}t^2\right]_0^{2\pi} = 2\sqrt{2}k\pi^2$$

Problem 7.

Evaluate $\int\limits_C y\,dx + z\,dy + x\,dz$, where C consists of part of the twisted C_1 with parametric equations $x = t$, $y = t^2$, and $z = t^3$, where $0 \le t \le 1$, followed by the line segment C_2 from $(1,1,1)$ to $(0,1,0)$.

Solution

We have $dx = dt$, $dy = 2tdt$, and $dz = 3t^2$, therefore

$$\int\limits_{C_1} y\,dx + z\,dy + z\,dz = \int\limits_0^1 t^2\,dt + t^3(2t\,dt) + t(3t^2\,dt) = \int\limits_0^1 (t^2 + 3t^3 + 2t^4)\,dt = \left[\frac{1}{3}t^3 + \frac{3}{4}t^4 + \frac{2}{5}t^5\right]_0^1 = \frac{89}{60}$$

The parametric equations for the line segment from $(1,1,1)$ to $(0,1,0)$ are

$$x = 1 - t, \quad y = 1, \quad z = 1 - t, \quad 0 \le t \le 1$$

Then $dx = -dt$, $dy = 0$, and $dz = -dt$. Therefore,

$$\int\limits_{C_2} y\,dx + z\,dy + z\,dz = \int\limits_0^1 1(-dt) + (1-t)0 + (1-t)(-dt) = \int\limits_0^1 1(t-2)\,dt = \left[\frac{1}{2}t^2 - 2t\right]_0^1 == -\frac{3}{2}$$

Finally,

$$\int\limits_C y\,dx + z\,dy + z\,dz = \int\limits_{C_1} y\,dx + z\,dy + z\,dz = \int\limits_{C_2} y\,dx + z\,dy + z\,dz == \frac{89}{60} - \frac{3}{2} = -\frac{1}{60}$$

Problem 8.

Find the work done by the force $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{i} + z\mathbf{k}$ in moving a particle along the helix C described by the parametric equations $x = \cos t$, $y = \sin t$, and $z = t$ from $(1,0,0)$ to $(0,1,\frac{\pi}{2})$

Solution

We have $x(t) = \cos t$, $y(t) = \sin t$, and $z(t) = t$, thus

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t)) = -y\mathbf{i} + x\mathbf{i} + z\mathbf{k} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{i} + t\mathbf{k}$$

with $0 \le t \le \frac{\pi}{2}$. Next, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{i} + z(t)\mathbf{k} = -\cos(t)\mathbf{i} + \sin(t)\mathbf{i} + t\mathbf{k}$ and $\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{i} + \mathbf{k}$. Therefore,

$$\int\limits_C \mathbf{F}\cdot d\mathbf{r} = \int\limits_0^{\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int\limits_0^{\pi/2} [-\sin(t)\mathbf{i} + \cos(t)\mathbf{i} + t\mathbf{k}]\cdot[-\sin(t)\mathbf{i} + \cos(t)\mathbf{i} + \mathbf{k}]\,dt$$
$$= \int\limits_0^{\pi/2} [\sin^2 t + \cos^2 t + t]\,dt = \int\limits_0^{\pi/2} (1+t)\,dt = \left[t + \frac{1}{2}t^2\right]_0^{\pi/2} = \frac{\pi}{2}\left(1 + \frac{\pi}{4}\right)$$

Problem 9.

Evaluate $\oint\limits_C x^2\,dx + (xy+y^2)\,dy$, where C is the boundary of the region R bounded by the graphs of $y=x$ and $y=x^2$ and is oriented in a positive direction.

Solution

$R = \{(x,y) : 0 \le x \le 1, x^2 \le y \le x\}$ is a simple region. Using Green's Theorem with $P(x,y) = x^2$ and $Q(x,y) = xy + y^2$, we have

$$\oint\limits_C x^2\,dx + (xy+y^2)\,dy = \iint\limits_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \int_0^1\int_{x^2}^{x} (y-0)\,dydx = \int_0^1 \left[\frac{1}{2}y^2\right]_{y=x^2}^{y=x} dx = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx$$
$$= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{15}$$

Problem 10.

Evaluate $\oint\limits_C (y^2 + \tan x)\,dx + (x^3 + 2xy + \sqrt{y})\,dy$, where C is the circle $x^2+y^2=4$ oriented in a positive direction.

Solution

The simple region $R = \{(x,y) : x^2+y^2 = 4\}$ is the disk. Using Green's Theorem with $P(x,y) = y^2 + \tan x$ and $Q(x,y) = x^3 + 2xy + \sqrt{y}$, we have

$$\frac{\partial Q}{\partial x} = 3x^2 + 2y, \qquad \text{and} \qquad \frac{\partial P}{\partial y} = 2y$$

and so

$$\oint\limits_C (y^2+\tan x)\,dx + (x^3+2xy+\sqrt{y})\,dy = \iint\limits_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \iint\limits_R 3x^2\,dA \underset{\text{(using polar coordinates)}}{=} 3\int_0^{2\pi}\int_0^2 (r\cos\theta)^2 r\,dr\theta$$
$$= 3\int_0^{2\pi}\int_0^2 r^3\cos^2\theta\,drd\theta = 3\int_0^{2\pi}\left[\frac{1}{4}r^4\cos^2\theta\right]_{r=0}^{r=2} d\theta = 12\int_0^{2\pi}\cos^2\theta\,d\theta$$
$$= 6\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = 6\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi} = 12\pi$$

Problem 11.

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

The ellipse C can be represented by the parametric equations $z = a\cos t$ and $y = b\sin t$, where $0 \le t \le 2\pi$. We observe that the ellipse is traced in counterclockwise (positive) direction as t increases from 0 to 2π. We have

$$A = \frac{1}{2}\oint\limits_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

Problem 12.

Evaluate $\oint\limits_C (\exp(x) + y^2)\,dx + (x^2 - 3xy)\,dy$, where C is positively oriented closed curve lying on the boundary of the semia-annular region R bounded by the upper semicircles $x^2+y^2=1$ and $x^2+y^2=9$ and the x-axis.

Solution

The polar coordinates of region R

$$R = \{(r,\theta); 1 \le r \le 3, 0 \le \theta \le \pi\}$$

Using Green's Theorem with $P(x,y)=\exp(x)+y^2$ and $Q(x,y)=x^2-3xy$, we obtain

$$\frac{\partial Q}{\partial x}=2x+3y, \qquad \text{and} \qquad \frac{\partial P}{\partial y}=2y$$

and so

$$\oint_C (\exp(x)+y^2)\,dx+(x^2+3xy)\,dy=\iint_R \left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]\,dA=\iint_R (2x+y)\,dA \underset{\text{(using polar coordinates)}}{=}$$

$$\int_0^{\pi}\int_1^3 (2r\cos\theta+r\sin\theta)r\,dr d\theta=\int_0^{\pi}(2\cos\theta+\sin\theta)\left[\frac{1}{3}r^3\right]_1^3\,d\theta$$

$$=\frac{26}{3}\left[2\sin\theta-\cos\theta\right]_0^{\pi}=\frac{52}{3}$$

Problem 13.

Use Green Theorem to evaluate $\oint_C x^4\,dx+xy\,dy$, where C is the triangular curve consisting of the segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution

With $P(x,y)=x^4$ and $Q(x,y)=xy$ and $R=\{(x,y):0\le x\le 1, 0\le y\le 1-x\}$, we have

$$\oint_C x^4\,dx+xy\,dy=\iint_R \left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]\,dA=\int_0^1\int_0^{1-x}(y-0)\,dydx=\int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x}\,dx=\frac{1}{2}\int_0^1(1-x)^2\,dx$$

$$=-\frac{1}{6}\left[(1-x)^3\right]_0^1=\frac{1}{6}$$

Problem 14.

Use Green's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=(\exp(-x)+y^2,\exp(-y)+x^2)$ and C consists of the arc of the curve $y=\cos x$ from $(-\pi/2,0)$ to $(\pi/2,0)$ and the line segment from $(\pi/2,0)$ to $(-\pi/2,0)$.

Solution

The region R enclosed by the curve C is $R=\{(x,y),-\pi/2\le x\le \pi/2, 0\le y\le\cos x\}$ The above defined curve C is traversed clockwise, so $-C$ gives positive direction. We obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r}=-\int_{-C}(\exp(-x)+y^2)\,dx+(\exp(-y)+x^2)\,dy=\iint_R\left[\frac{\partial}{\partial x}(\exp(-y)+x^2)-\frac{\partial}{\partial y}(\exp(-x)+y^2)\right]\,dA$$

$$=-\int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x-2y)\,dydx=-\int_{-\pi/2}^{\pi/2}\left[2xy-y^2\right]_{y=0}^{y=\cos x}\,dx=-\int_{-\pi/2}^{\pi/2}(2x\cos x-\cos^2 x)\,dx \quad (\textit{integrate by parts } x\cos x)$$

$$=-\left[2x\sin x+2\cos x-\frac{1}{2}(x+\frac{1}{2}\sin(2x))\right]_{-\pi/2}^{\pi/2}=(\pi-\frac{1}{4}\pi-\pi-\frac{1}{4}\pi)=\frac{1}{2}\pi$$