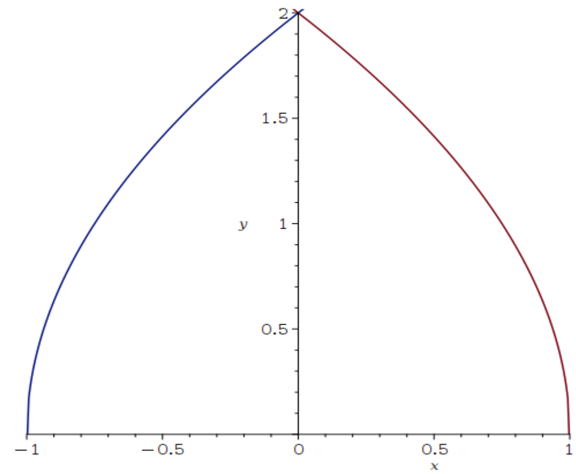
Change of variables examples
Math 250, Spring 2024 – Jacek Polewczak

Problem 1.

Use the change of variables $x = u^2 - v^2$ and $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, and $y \geq 0$ (see the figure to the right).



We will show that the transformation $T(u, v) = [u^2 - v^2, 2uv]$ maps the square $S = \{(u, v) : 0 \leq v \leq 1, 0 \leq v \leq 1\}$ onto region R.

The transformation maps the boundary of S into the boundary of R. We begin by finding the images of the sides of S into the boundary of R. The first side, S_1 of the square S is given by $v = 0$ and $0 \leq u \leq 1$ From the given equations we have $x = u^2$, $y = 0$, and so $0 \leq x \leq 1$ Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$, $0 \leq v \leq 1$ and, putting $u = 1$ in the given equations we get $x = 1 - v^2$ and $y = 2v$. Eliminating v we obtain $x = 1 - y^2/4$ with $0 \leq x \leq 1$, which is part of parabola. S_3 is given by $v = 1$, $0 \leq u \leq 1$, whose image is the parabolic arc $x = y^2/4 - 1$ with $-1 \leq x \leq 0$. Finally, S_4 is given by $u = 0$, $0 \leq v \leq 1$ whose image $x = -v^2$, $y = 0$, that is $-1 \leq x \leq 0$.

The Jacobian of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Thus,

$$\iint_R y\,dA = \iint_S 2uv\,|J(u, v)|\,dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\,dudv = 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\,dudv = 8 \int_0^1 \left[\frac{1}{4}u^4 v + \frac{1}{2}u^2 v^3\right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\,dv = \left[v^2 + v^4\right]_0^1 = 2.$$

Problem 2.

Evaluate the integral, $\iint_R \exp[(x + y)/(x - y)]\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$. Th integral of $\exp[(x + y)/(x - y)]$ is not easy to integrate. We make a change of variables suggested by the form of this function:

$$u = x + y, \qquad v = x - y.$$

These equation define the transformation T^{-1} from xy-plane to uv-plane. Solving the above equations for x and y we get

$$x = \frac{1}{2}(u + v), \qquad y = \frac{1}{2}(u - v),$$

In other words, $T(u, v) = [\frac{1}{2}(u + v), \frac{1}{2}(u - v)]$ is the transformation from uv-plane to xy-plane with the Jacobian of T

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Now we need to find the region S is uv-plane. The sides of R lie on the lines

$$y = 0, \quad x - y = 2, \quad x = 0, \quad x = y = 1,$$

thus the image lines in uv-plane are

$$u = v, \quad v = 2, \quad u = -v, \quad v = 1.$$

The region S is the trapezoidal region with vertices $(1,1)$, $(2,2)$, $(-2,2)$, and $(-1,1)$. Thus the region S is

$$S = \{(u,v) :\ 1 \le v \le 2,\ -v \le u \le v\}.$$

Therefore

$$\iint\limits_R \exp[(x+y)/(x-y)]\, dA = \iint\limits_S \exp(u/v)|J(u,v)|\, dA = \int_1^2 \int_{-v}^{v} \exp(u/v) du dv = \frac{1}{2}\int_1^2 [v\exp(u/v)]_{u=-v}^{u=v}\ dv$$

$$= \frac{1}{2}\int_1^2 (e - e^{-1})v\, dv = \frac{3}{4}(e - e^{-1}).$$

Problem 3.

Setup the integral $\int_3^4 \int_0^1 (x+2y)(2x+y)\, dxdy$ using a suitable change of variables.

We make a change of variables suggested by the form of this function:

$$u = x + 2y, \qquad v = 2x + y.$$

Solving the above system for (x,y), we get

$$x = -\frac{u}{3} + \frac{2v}{3}, \qquad y = -\frac{v}{3} + \frac{2u}{3}.$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

The corners of the rectangle in the xy-plane are $(0,3)$, $(1,3)$, $(1,4)$, and $(0,4)$, and they map into $(6,3)$, $(7,5)$, $(9,6)$, and $(8,4)$, respectively, in the uv-plane. This is parallelogram P. Note the transformations T and T^{-1} are linear transformations.

$$\int_3^4 \int_0^1 (x+2y)(2x+y)\, dxdy = \frac{1}{3}\int\limits_P uv\, dvdu.$$

Problem 4.

Evaluate the integral $\int\limits_R \exp[(x^3+y^3)/xy]\, dA$, where

$$R = \{(x,y) :\ y^2 - x \le 0,\ x^2 - y \le 0\},$$

using the change of variables $x = u^2 v$, $y = uv^2$.

The Jacobian is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2uv & u^2 \\ v^2 & 2uv \end{vmatrix} = 3u^2v^2.$$

The region R transforms into (*Check it!*)

$$S = \{(u,v) :\ 0 \le u \le 1,\ 0 \le v \le 1\},$$

and therefore,

$$\int\limits_R \exp[(x^3+y^3)/xy]\, dA = \int\limits_S \exp\left(\frac{u^6v^3 + u^3v^6}{u^3v^3}\right) |J(u,v)|\, dA = 3\int\limits_S u^2\exp(u^3)v^2\exp(v^3)\, dudv$$

$$= \frac{1}{3}\left(\int_0^1 3u^2\exp(u^3)\, du\right)^2 = \frac{1}{3}(e-1)^2.$$