Solutions to Various Practice Problems I

Problem 1.
(a) \[ x_{1,2} = \frac{-2 \pm \sqrt{4 + 32 \cdot 3}}{16} = \frac{-1 \pm 5}{8} = \frac{1}{2}, -\frac{3}{4} \]
(b) \[ x_{1,2} = \frac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6} \approx 3.44948, \approx -1.44948 \]

Problem 2.
(a) We have, \[ x^2 + 2x - 15 = (x + 5)(x - 3). \]
Using the sign pattern method, the solution set of the inequality \[ x^2 + 2x - 15 \geq 0 \] is the set \( (-\infty, -5] \cup [3, +\infty) \).
(b) Note that the quadratic equation \[ x^2 + x + 1 = 0 \] has no real solutions. Since the coefficient in front of \( x^2 \) is positive, the graph of \( y = x^2 + x + 1 \) lies above the \( x \)-axis (CHECK IT !).

Another way to look at this case is to notice that after completion of square, \[ x^2 + x + 1 = \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} > 0. \]
In other words, \( x^2 + x + 1 > 0 \) for all \( x \). Thus, there is no solution to the inequality \( x^2 + x + 1 \leq 0 \), or equivalently, the solution set is the empty set.
(c) The inequality \( \frac{2x + 1}{2 - x} \leq 1 \) is equivalent to \( \frac{3x - 1}{2 - x} \leq 0 \). Indeed,
\[ \frac{2x + 1}{2 - x} \leq 1 \iff \frac{2x + 1 - (2 - x)}{2 - x} \leq 0 \iff \frac{3x - 1}{2 - x} \leq 0 \iff (3x-1)(2-x) \leq 0 \]
The solution set of the last inequality is the set \( (-\infty, \frac{1}{3}] \cup (2, +\infty) \).

Note: Solving the inequality \( \frac{2x + 1}{2 - x} \leq 1 \) by reducing it to \( 2x + 1 \leq 2 - x \) is incorrect and leads to a wrong solution! Check it!

Problem 3.
\[ (f \circ g)(x) = f(g(x)) = f\left( \frac{1}{x - 1} \right) = \sqrt{\frac{1}{x - 1} + 1} = \sqrt{\frac{x}{x - 1}} \]
Note that \( \sqrt{\frac{x}{x - 1}} \neq \frac{\sqrt{x}}{\sqrt{x - 1}} \). Do you know why?
\[ (g \circ f)(x) = g(f(x)) = g\left( \sqrt{x + 1} \right) = \frac{1}{\sqrt{x + 1} - 1} \]

Problem 4.
\[ \frac{f(-1 + h) - f(-1)}{h} = \frac{1}{-1 + h} \cdot \frac{1}{h} = \frac{-1 - (-1) - h}{(-1)(-1 + h)} = \frac{1}{(-1 + h)} \]

Problem 5.
Problem 6.
The slope, $m$, of the line $4x + 5y + 16 = 0$ is $m = -\frac{4}{5}$. Note, the equation $4x + 5y + 16 = 0$ is equivalent to the equation $y = -\frac{4}{5}x - \frac{16}{5}$. The slope of the line perpendicular to $4x + 5y + 16 = 0$ is equal to $\frac{5}{4}$. Therefore, the equation of the line perpendicular to $5x + 4y + 16 = 0$ and passing through $(-1, 0)$ is given by $y - 0 = \frac{5}{4}(x + 1)$, or equivalently $y = \frac{5}{4}x + \frac{5}{4}$. Another equivalent form is $4y - 5x - 5 = 0$.

Problem 7.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{3x - 8}\right) = \frac{2x}{3x - 8} + 5 = \frac{2x}{3x - 8} = \frac{2x}{x + 15x - 40} = \frac{2x}{16x - 40} = \frac{x}{8x - 20}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x}{x + 5}\right) = \frac{2x}{3x - 8} = \frac{2x}{x + 5} = \frac{2x}{6x - 8x - 40} = \frac{2x}{-2x - 40} = -\frac{x}{x + 20}$$

Problem 8.

(a) $\lim_{x \to -1} \frac{3x^2 + 4x + 1}{x + 1} = \lim_{x \to -1} \frac{(3x + 1)(x + 1)}{x + 1} = \lim_{x \to -1} (3x + 1) = \lim_{x \to -1} 3x + \lim_{x \to -1} 1 = -3 + 1 = -2$

(b) $\lim_{x \to -\infty} \frac{-2x^4 + 3x^3 - 7x - 10}{3x^4 + 6x^2 - x + 100} = \lim_{x \to -\infty} \frac{-2 + \frac{3}{x} - \frac{7}{x^2} - \frac{10}{x^4}}{3 + \frac{6}{x^2} - \frac{1}{x^3} + \frac{100}{x^4}} = -2 + \lim_{x \to -\infty} \frac{3}{x} - \lim_{x \to -\infty} \frac{7}{x^2} - \lim_{x \to -\infty} \frac{10}{x^4} = \frac{-2}{3}$

Problem 9.

(a) $\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x \to -2} (x - 2) = \lim_{x \to -2} x - \lim_{x \to -2} 2 = -2 - 2 = -4$

(b) $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} - 2} = \lim_{x \to 4} (\sqrt{x} + 2) = \lim_{x \to 4} \sqrt{x} + \lim_{x \to 4} 2 = 2 + 2 = 4$

(c) $\lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1} = \lim_{x \to 1} \frac{[(\sqrt{x + 3} - 2)(\sqrt{x + 3} + 2)]}{(x - 1)(\sqrt{x + 3} + 2)} = \lim_{x \to 1} \frac{1}{\sqrt{x + 3} + 2} = \frac{1}{\lim_{x \to 1} x + 3 + 2} = \frac{1}{4}$

(d) $\lim_{x \to 0} \frac{1 - \cos(2x)}{3x^2} = \lim_{x \to 0} \frac{1 - (\cos^2 x - \sin^2 x)}{3x^2} = \lim_{x \to 0} \frac{2 \sin^2 x}{3x^2} = \frac{2}{3} \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = \frac{2}{3} \cdot 1^2 = \frac{2}{3}$

Problem 10.
The only point (Why?) where the function may be discontinuous is $x = -1$. Now $f(x)$ is defined at $x = -1$, $f(-1) = 1$. Next, we check whether the limit $\lim_{x \to -1} f(x)$ exists. We have,
Problem 15.

\[ \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x \to -1} (x-1) = \lim_{x \to -1} x = -1 - 1 = -2. \]

So the limit \( \lim_{x \to -1} f(x) \) indeed exists and is equal to \(-2\).

Finally, we observe that the third condition of continuity is not satisfied: the limit \( \lim_{x \to -1} f(x) \) is not equal to the value of the function at \( x = -1 \). Indeed, \( \lim_{x \to -1} f(x) = -2 \) but \( f(-1) = 1 \). Therefore, \( f(x) \) is not continuous at \( x = -1 \) because the third condition of continuity is not satisfied.

Problem 11.

We need to check which of the three conditions of continuity at \( x = -1 \) is not satisfied. Since \( f(-1) = -1 \), the function \( f(x) \) is defined at \( x = -1 \) and the condition (1) of continuity is satisfied. Next, we check if the condition (2) is satisfied, i.e., whether the limit \( \lim_{x \to -1} f(x) \) exists. The expression \( 2x - 1 \) approaches \(-5 \) whenever \( x \) approaches \(-1 \) and \( x < -1 \). On the other hand, the expression \( x^2 - 4 \) approaches \(-3 \) whenever \( x \) approaches \(-1 \) and \( x > -1 \). Therefore, \( \lim_{x \to -1} f(x) \) does not exist. The second condition of continuity is not satisfied.

Problem 12.

(a) The average rate of change from \( x = 2 \) to \( x = 3 \) is given by

\[ \frac{f(3) - f(2)}{3 - 2} = \frac{-9 - (-4)}{3 - 2} = -5. \]

(b) The instantaneous rate of change at \( x = 3 \) is \( f'(3) \):

\[ f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{-(3 + h)^2 - (-9)}{h} = \lim_{h \to 0} \frac{-h^2 - 6h}{h} = \lim_{h \to 0} (-h - 6) = -6. \]

Problem 13.

(a) \( g'(s) = 4s + \frac{4}{s^2} - \frac{1}{\sqrt{s^3}} \)  

(b) \( h'(x) = 5 \left( x + \frac{1}{x} + \frac{1}{x^2} \right)^4 \left[ 1 - \frac{1}{x^2} - \frac{2}{x^5} \right] \)

(c) \( F'(x) = \frac{1}{2} \left( \frac{x^2 + 1}{x^4 + 2} + 10 \right)^{-1/2} \left[ \frac{2x(x^4 + 2) - (x^2 + 1)4x^3}{(x^4 + 2)^2} \right] \)

Problem 14.

Since \( g'(t) = \left( \sqrt{2t^2 + 3} \right)' = \frac{2t}{\sqrt{2t^2 + 3}} \), we have

\[ g''(t) = \left( \frac{2t}{\sqrt{2t^2 + 3}} \right)' = \frac{6}{(2t^2 + 3)^{3/2}} \quad \text{and} \quad g'''(t) = \left[ \frac{6}{(2t^2 + 3)^{3/2}} \right]' = -\frac{36t}{(2t^2 + 3)^{5/2}}. \]

Problem 15.

(a) \( h'(t) = \frac{(2t - 3)(t + 1) - (t^2 - 3t + 1) \cdot 1}{(t + 1)^2} = \frac{t^2 + 2t - 4}{(t + 1)^2} \)

(b) \( f'(x) = \frac{2e^{-x}}{2\sqrt{c^2x^2 + 2}} = \frac{e^{-x}}{\sqrt{c^2x^2 + 2}} \)