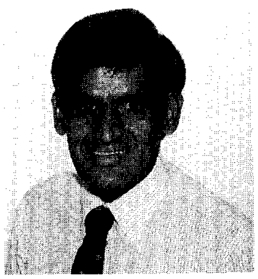

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The Field Equivalence Principle: Illustration of the Establishment of the Non-Intuitive Null Fields

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1. Introduction

The field Equivalence Principle, one of the fundamental concepts in electromagnetics, has numerous applications. However, for a beginning student, it is not easy to understand this concept thoroughly and to appreciate it. The dilemma faced by beginning students is illustrated by the example problem shown in Figure 1a. Here, we have sources in a finite Region I, and an arbitrary mathematical surface separating Regions I and II. The equivalent problems for the exterior and interior regions are specified in Figures 1b and 1c, respectively, with the use of electric and magnetic equivalent currents impressed on the boundary surface. The acceptance of the establishment by the equivalent sources of the non-intuitive null field for the exterior problem (by the equivalent sources and the original source for the interior problem) is commonly bothersome and not comfortably realized. In order to clarify this, we revisit Love's and Schelkunoff's forms of the Equivalence Principle. Subsequently, we discuss two simple, analytically tractable illustrative examples, consisting of plane-wave fields in two half-space regions, separated by an infinite planar surface. In particular, the emphasis will be on the establishment of the non-intuitive null fields developed by these equivalent sources. Various forms of equivalence will be illustrated by simple analytical field expressions. We believe that this tutorial presentation will be helpful to students and faculty.

2. Love's and Schelkunoff's Field Equivalence Principle

Let us consider two regions, denoted by I and II, and separated by a boundary surface, S, which may be an arbitrary mathematical surface, as shown in Figure 2. For simplicity, it is assumed that Region I contains sources J_1 and M_1, and that Region II is

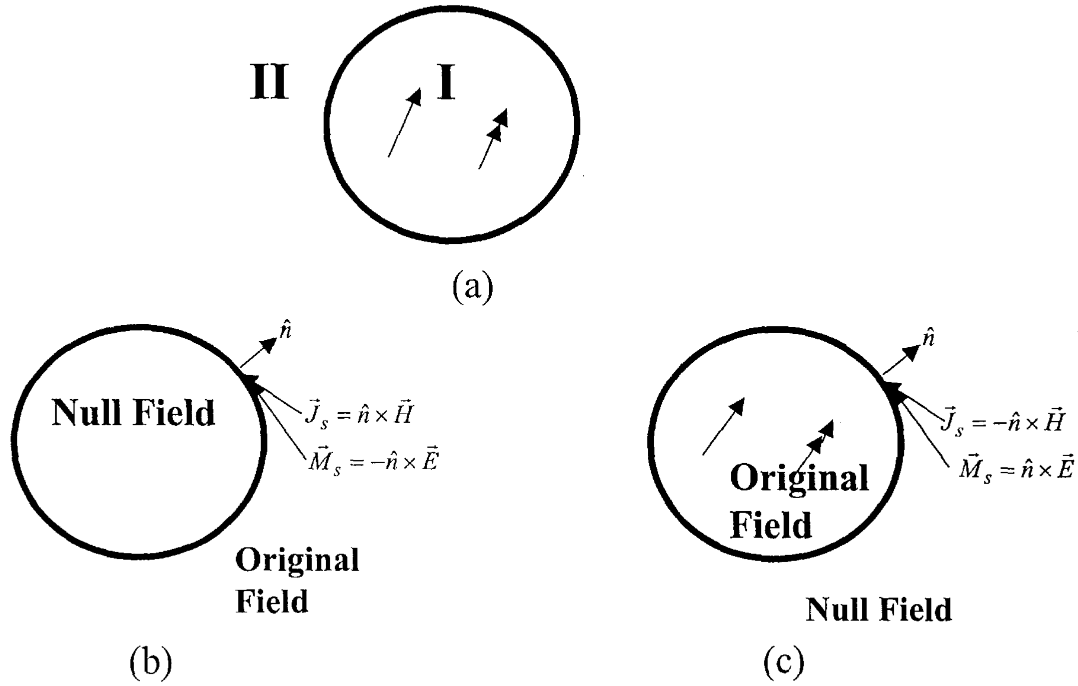

Figure 1. An illustration of the field Equivalence Principle: (a) The original problem containing sources in Region I and an arbitrary mathematical surface dividing Regions I and II; (b) Equivalence for the external problem; (c) Equivalence for the internal problem. Note the establishment of the null fields in these cases.

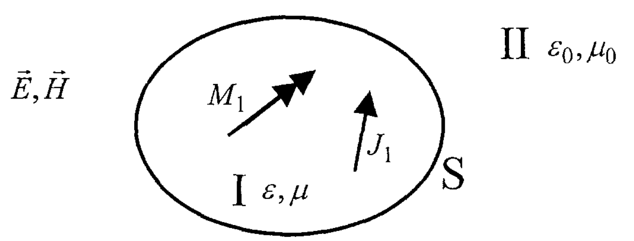

Figure 2. The geometry of the problem.

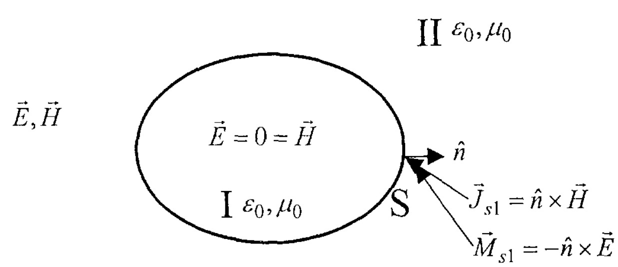

Figure 3. Love's equivalence for Region II.

source-free. Let us assume that Region II is free space, and that Region I contains some material of constitutive parameters ε and μ.

By invoking Love's Equivalence Principle [1], we can consider a problem equivalent to one of these regions, e.g., Region II, as shown in Figure 3. In this case, the original material in Region I is replaced by free space, and the sources are removed. Equivalent electric and magnetic currents, given below, are suspended in space on the surface S. These are related to the tangential components of the fields on the surface:

$$\vec{J}_{s1} = \hat{n} \times \vec{H},$$

$$\vec{M}_{s1} = -\hat{n} \times \vec{E},$$

where $\hat{n}$ is a unit normal vector, directed from S towards Region II.

The equivalent currents, radiating into free space, produce the correct original fields in Region II, and they produce null fields in Region I. The null fields produced by these sources are commonly bothersome and not comfortably accepted.

An equivalent problem may be set up for Region I, as shown in Figure 4 [1]. In this case, Region II is filled up with the same material as in Region I of the original problem, in Figure 2. We place equivalent currents on S, which are the negative of what we had in Figure 3. These equivalent currents will satisfy the boundary conditions for this problem, shown in Figure 4. Thus, the equivalent currents and the original sources, radiating in a homogeneous

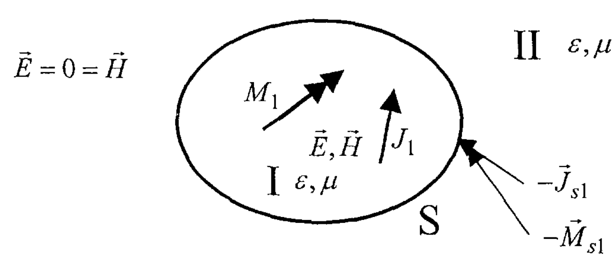

Figure 4. Love's equivalence for Region I.

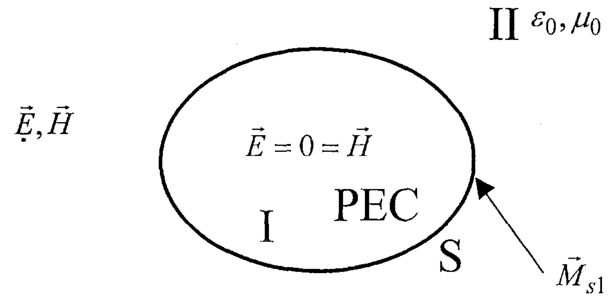

Figure 5. Schelkunoff's equivalence for Region II, with a PEC in Region I.

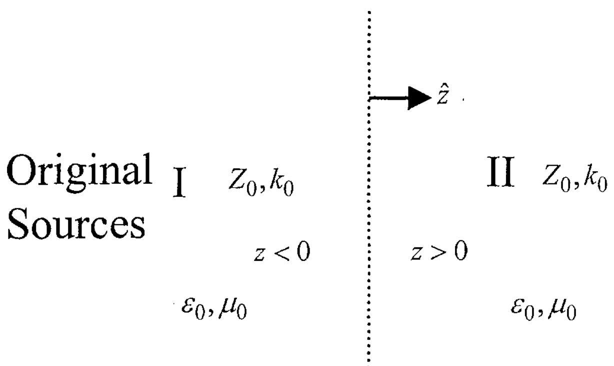

Figure 6. Plane-wave propagation in a homogeneous region.

medium of constitutive parameters ε and μ, will radiate the correct fields in Region I; whereas in Region II, they produce null fields. Again, the null fields produced by these sources are commonly bothersome and not comfortably accepted.

The validity of both forms of equivalence follows from the fact that the fields in each region satisfy Maxwell's equations, and that the boundary conditions for the tangential components of the fields are satisfied on the surface S. Thus, the uniqueness conditions for electromagnetic fields are satisfied [2, 3]. Stratton and Chu have also rigorously derived the Equivalence Theorem [4]. For a beginning student, it is hard to visualize the fact that the equivalent currents in Figure 3 produce the correct fields in Region II, whereas they produce null fields in Region I. Similarly, it is hard to visualize the fact that the equivalent fields produce the scattered fields in Region I in Figure 4, whereas in Region II they produce the negative of the incident fields.

We can construct variations of the Equivalence Principle by placing different media, sources, and/or fields in the unwanted region, e.g., in Region I in Figure 3, provided they satisfy Maxwell's equations. The equivalent currents must satisfy the boundary conditions. However, in many applications, we consider placing a perfect electric conductor (PEC) or perfect magnetic conductor (PMC) in the region containing null fields. Let us consider Figure 3. We can place a PEC medium in Region I. This will not affect the fields in Region I, since there is no interaction of the null fields and any material medium. The fields in Region II will also be unaffected, since the boundary conditions at S are still satisfied. We can use the Superposition Principle to find the fields produced by the two equivalent currents. It is noted that these are impressed currents, and not induced currents. The tangential electric currents impressed on a PEC surface will not radiate. This can be proven rigorously from the Lorentz Reciprocity Theorem [2], and is discussed in Appendix 1. We are left with the problem of finding the fields produced by the magnetic currents on the PEC surface S, as illustrated in Figure 5. This form of equivalence is referred to as Schelkunoff's Equivalence Principle [2, 5, 6]. While this problem may be formulated as an integral equation and solved numerically for the general case, a simple analytical problem is discussed in Section 3.2. We can obtain an equivalent problem, similar to Figure 5 but with an electric equivalent current on a PMC surface, S. It is possible to have two more equivalent problems for Figure 4, with the use of PECs or PMCs in Region II.

3. An Example of a Homogeneous Medium

We now discuss analytically tractable, illustrative examples of plane-wave propagation in simple geometries. The first example uses a homogeneous medium, and the second uses two different media, with a planar boundary normal to the propagation direction.

Let us consider two half-space regions, denoted by I ($z<0$) and II ($z>0$), with a source at $z=-\infty$ producing plane-wave fields, as shown in Figure 6. The infinite plane at $z=0$ separates the two regions. The medium in both regions is free space. There is no scattering, and, hence, the total fields everywhere are the incident fields:

$$\vec{E}^{inc}=\hat{x}E_0e^{-jk_0z}$$
$$\vec{H}^{inc}=\hat{y}\frac{E_0}{Z_0}e^{-jk_0z}. \tag{1}$$

In Equation (1), $k_0=2\pi/\lambda_0$ (where λ_0 is the free-space wavelength) is the wavenumber, and Z_0 ($=\sqrt{\mu_0/\varepsilon_0}$) is the intrinsic impedance of free space.

3.1 Love's Equivalence Principle

3.1.1 Equivalence to Region II

The equivalent currents on the boundary surface, S ($z=0$) in Figure 7 are given by

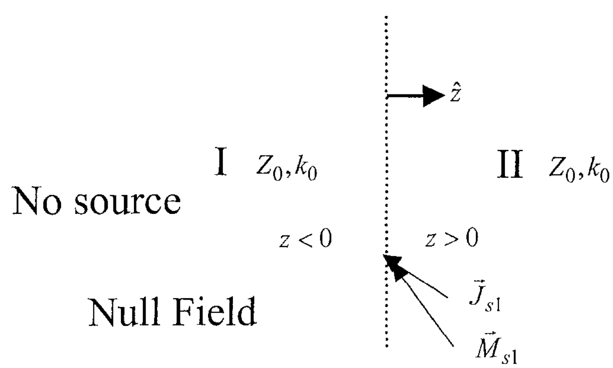

Figure 7. Love's Equivalence Principle applied to Region II.

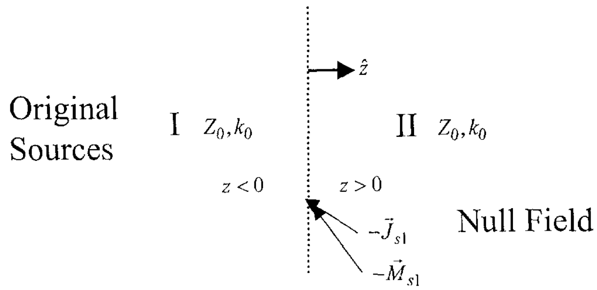

Figure 8. Love's Equivalence Principle applied to Region I.

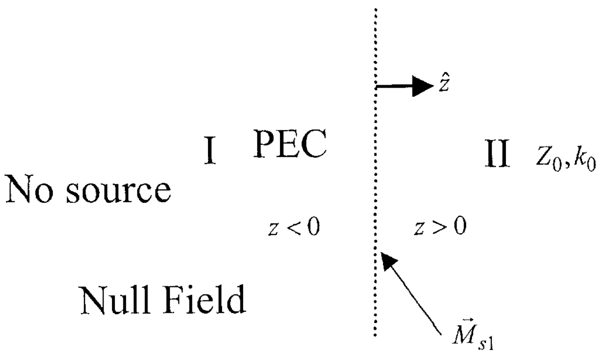

Figure 9. Schelkunoff's Equivalence Principle applied to Region II.

$$\vec{J}_{s1} = \hat{n}\times\vec{H} = \hat{z}\times\hat{y}\frac{E_0}{Z_0} = -\hat{x}\frac{E_0}{Z_0},$$
$$\vec{M}_{s1} = -\hat{n}\times\vec{E} = -\hat{z}\times\hat{x}E_0 = -\hat{y}E_0. \tag{2}$$

The fields produced by the electric current [2] may be obtained easily in closed form from the magnetic vector potential (see Appendix 2), and are given below:

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = \hat{x}\dfrac{E_0}{2}e^{jk_0 z} & \vec{E} = \hat{x}\dfrac{E_0}{2}e^{-jk_0 z} \\ \vec{H} = -\hat{y}\dfrac{E_0}{2Z_0}e^{jk_0 z} & \vec{H} = \hat{y}\dfrac{E_0}{2Z_0}e^{-jk_0 z} \end{array} \tag{3}$$

These fields satisfy Maxwell's equations and the boundary conditions on S.

Fields produced by the magnetic equivalent current [2] may be obtained by duality:

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = -\hat{x}\dfrac{E_0}{2}e^{jk_0 z} & \vec{E} = \hat{x}\dfrac{E_0}{2}e^{-jk_0 z} \\ \vec{H} = \hat{y}\dfrac{E_0}{2Z_0}e^{jk_0 z} & \vec{H} = \hat{y}\dfrac{E_0}{2Z_0}e^{-jk_0 z} \end{array} \tag{4}$$

From Equations (3) and (4) we obtain the total fields produced by the two equivalent currents given in Equation (5). These are null fields in Region I, and the incident fields in Region II.

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = 0 & \vec{E} = \vec{E}^{inc} \\ \vec{H} = 0 & \vec{H} = \vec{H}^{inc} \end{array} \tag{5}$$

Note that one cannot simply use the argument of symmetry, and state that the surface-current sheet should radiate symmetrically on both sides! The electric- and magnetic-current sheets produce different (odd and even) types of symmetry for the fields.

3.1.2 Equivalence to Region I

We now have the original sources producing the incident fields, and the equivalent currents, given below, on S. The equivalence to Region I is shown in Figure 8.

$$\vec{J}_{s2} = -\hat{z}\times\hat{y}\frac{E_0}{Z_0} = \hat{x}\frac{E_0}{Z_0} = -\vec{J}_{s1},$$
$$\vec{M}_{s2} = \hat{z}\times\hat{x}E_0 = \hat{y}E_0 = -\vec{M}_{s1}. \tag{6}$$

Clearly, the fields produced by the equivalent currents are the negative of the fields given by Equations (3) through (5). Thus, the contributions of the equivalent currents are found to be null fields in Region I, and the negative of the incident fields in Region II. Therefore, the total field in Region I is the incident field, and that in Region II is zero.

3.2 Schelkunoff's Equivalence Principle

3.2.1 Equivalence to Region II

Consider Love's Equivalence Principle applied to Region II above, in Section 3.1.1, and shown in Figure 9. Since Region I has null fields, PEC material may fill up Region I without affecting the fields in Regions I and II. The equivalent electric and magnetic currents are on the PEC surface, S. Clearly, the electric currents will not radiate (in this situation only, one may also apply the image theory to justify this observation). The fields produced by

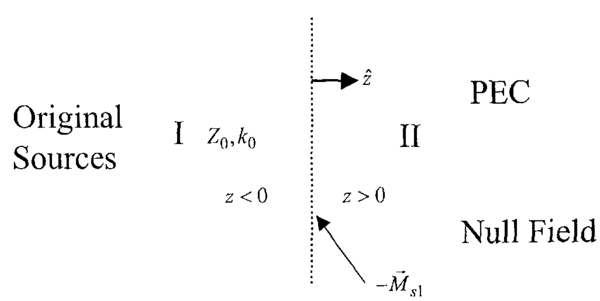

Figure 10. Schelkunoff's Equivalence Principle applied to Region I.

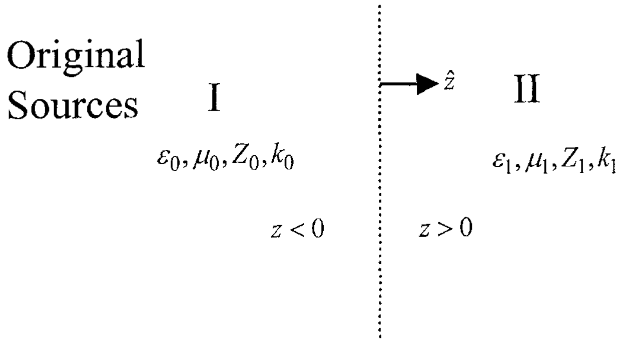

Figure 11. Plane-wave propagation normal to the interface between two dielectric media.

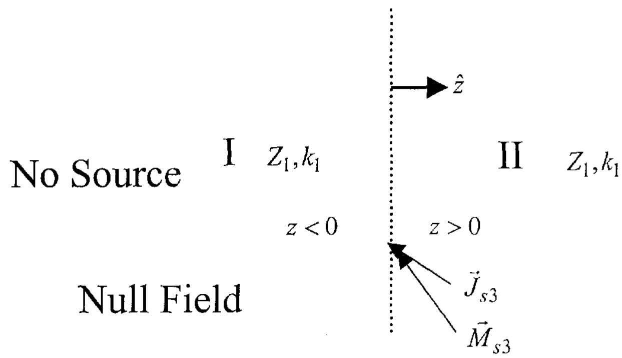

Figure 12. The equivalence for Region II.

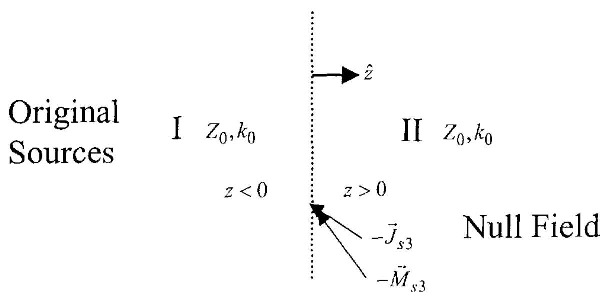

Figure 13. The equivalence for Region I.

the magnetic equivalent currents on the PEC surface may be obtained by using the Image Principle. We have twice as much magnetic equivalent current as on the $z=0$ boundary in Figure 7. These magnetic currents radiate in free space. The fields in Region II are twice those given by Equation (4). These are the original incident fields. The fields in Region I are fictitious while using the Image Principle.

3.2.2 Equivalence to Region I

As illustrated in Figure 10, we have PEC material in Region II. Region I contains the original sources at $z=-\infty$, and the equivalent currents on S. The electric currents on the PEC will not radiate. The total field consists of three parts: the original incident fields, the reflected fields produced by the reflection of the incident fields on the PEC surface S, and the fields radiated by the doubled equivalent magnetic currents in a homogeneous medium. The last component is due to the application of the Image Principle, and it is twice the negative of the amount given by Equation (4). The reflected fields and the fields produced by the magnetic currents, given below, for Region I are found to be the negative of each other:

$$\begin{aligned}
\vec{E}^{ref} &= -\hat{x}E_0 e^{jk_0 z} \\
\vec{H}^{ref} &= \hat{y}\frac{E_0}{Z_0} e^{jk_0 z} \\
\vec{E}\left(-2\vec{M}_{s1}\right) &= \hat{x}E_0 e^{jk_0 z} \\
\vec{H}\left(-2\vec{M}_{s1}\right) &= -\hat{y}\frac{E_0}{Z_0} e^{jk_0 z}
\end{aligned} \tag{7}$$

Thus, the total fields in Region I are the original incident fields given by Equation (1).

The Duality Principle may be invoked to obtain the fields for the problem of Schelkunoff's equivalence, using PMC media in the null regions.

4. Examples of Plane Wave Propagation in Two Different Media

4.1. Plane Wave Propagation Normal to a PEC Boundary

In the original problem discussed in Section 3, let Region II consist of PEC: now, the tangential electric field is zero on S. We only have the electric equivalent current, equal to twice the negative of the value given by Equation (2), and located on S. Clearly this current will produce twice the negative value of the field given by Equation (3) in Region I ($z<0$). This is indeed the reflected field given by Equation (7), so that the total field is the incident field plus the reflected field. The field produced by the electric equivalent currents in Region II is the negative of the incident field, and thus the total field in Region II will be zero. It is noted that the equivalent problem yields the actual null fields in the conductor.

4.2. Plane-Wave Propagation Normal to a Boundary Between Two Different Dielectric Media

Let Region II consist of a material of intrinsic impedance Z_1 and wavenumber k_1, and let Region I consist of free space, as shown in Figure 11. The actual fields in the two regions are given by [2]

$$z<0 \qquad \vec{E} = \hat{x}E_0\left(e^{-jk_0z} + \Gamma e^{jk_0z}\right) \qquad z>0 \qquad \vec{E} = \hat{x}E_0\left(1+\Gamma\right)e^{-jk_1z} \tag{8}$$

$$\vec{H} = \hat{y}\frac{E_0}{Z_0}\left(e^{-jk_0z} - \Gamma e^{jk_0z}\right) \qquad \vec{H} = \hat{y}\frac{E_0}{Z_1}\left(1+\Gamma\right)e^{-jk_1z} \tag{9}$$

where $\Gamma = (Z_1 - Z_0)/(Z_1 + Z_0)$.

4.2.1 Equivalence to Region II

Figure 12 shows the equivalent problem valid for Region II. The equivalent currents on S are given by

$$\begin{aligned} \vec{J}_{s3} &= \hat{n}\times\vec{H} = \hat{z}\times\hat{y}\frac{E_0}{Z_1}\left(1+\Gamma\right) = -\hat{x}\frac{E_0}{Z_1}\left(1+\Gamma\right), \\ \vec{M}_{s3} &= -\hat{n}\times\vec{E} = -\hat{z}\times\hat{x}E_0\left(1+\Gamma\right) = -\hat{y}E_0\left(1+\Gamma\right). \end{aligned} \tag{10}$$

These equivalent currents radiate in a homogeneous region containing a medium of intrinsic impedance Z_1 and wavenumber k_1.

The fields produced by the electric currents in the two regions are obtained by inspection of Equation (3):

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = \hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{jk_1z} & \vec{E} = \hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{-jk_1z} \\ \vec{H} = -\hat{y}\dfrac{E_0}{2Z_1}\left(1+\Gamma\right)e^{jk_1z} & \vec{H} = \hat{y}\dfrac{E_0}{2Z_1}\left(1+\Gamma\right)e^{-jk_1z} \end{array} \tag{11}$$

Similarly, the fields produced by the magnetic equivalent currents are obtained from an inspection of Equation (4), and are given in Equation (12):

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = -\hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{jk_1z} & \vec{E} = \hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{-jk_1z} \\ \vec{H} = \hat{y}\dfrac{E_0}{2Z_1}\left(1+\Gamma\right)e^{jk_1z} & \vec{H} = \hat{y}\dfrac{E_0}{2Z_1}\left(1+\Gamma\right)e^{-jk_1z} \end{array} \tag{12}$$

Thus, the total fields in Region II are the same as those given by Equation (9), and the fields in Region I are null fields. It is noted that Region I has the same material as that of Region II, in this equivalence.

4.2.2 Equivalence to Region I

We now have equivalent currents on S that are the negative of those in Equation (10). The medium in Region II is replaced by free space, the same as that in Region I, since we have null fields in Region II. Equations (13) and (14) give the fields produced by the electric and magnetic equivalent currents, respectively, as shown in Figure 13. In Equation (13), we have used the expression for the electric current in the form

$$-\vec{J}_{s3} = \hat{x}\frac{E_0}{Z_0}\left(1-\Gamma\right),$$

since $\dfrac{\left(1+\Gamma\right)}{Z_1} = \dfrac{\left(1-\Gamma\right)}{Z_0}$, obtained by matching H_y at $z=0$ in Equations (8) and (9).

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = -\hat{x}\dfrac{E_0}{2}\left(1-\Gamma\right)e^{jk_0z} & \vec{E} = -\hat{x}\dfrac{E_0}{2}\left(1-\Gamma\right)e^{-jk_0z} \\ \vec{H} = \hat{y}\dfrac{E_0}{2Z_0}\left(1-\Gamma\right)e^{jk_0z} & \vec{H} = -\hat{y}\dfrac{E_0}{2Z_0}\left(1-\Gamma\right)e^{-jk_0z} \end{array} \tag{13}$$

$$\begin{array}{ll} z<0 & z>0 \\ \vec{E} = \hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{jk_0z} & \vec{E} = -\hat{x}\dfrac{E_0}{2}\left(1+\Gamma\right)e^{-jk_0z} \\ \vec{H} = -\hat{y}\dfrac{E_0}{2Z_0}\left(1+\Gamma\right)e^{jk_0z} & \vec{H} = -\hat{y}\dfrac{E_0}{2Z_0}\left(1+\Gamma\right)e^{-jk_0z} \end{array} \tag{14}$$

The sum of the fields produced by the two equivalent currents in Region I are the reflected fields. With the addition of the incident fields, we obtain the correct total fields in this region. In Region II, the fields of Equations (13) and (14) add to yield the negative of the incident fields, so that the total fields are zero in this region.

Schelkunoff's equivalence may be obtained such that only one of the two equivalent currents is non-zero. When employed for equivalence in Region I, care must be exercised to include the appropriate reflected fields.

5. Acknowledgments

We wish to acknowledge our discussions with Prof. Aluizio Prata of the University of Southern California. This tutorial was originally motivated by a contribution from A. J. Booysen of Grintek Antennas, Centurion, South Africa. We are very grateful to Dr. Ross Stone for a thorough review of this contribution and for his many valuable suggestions.

6. Appendix 1
Proof that the Impressed Currents on a PEC Surface Do Not Produce Any Fields

The Lorentz Reciprocity Theorem takes the following form, when the domain of the problem is infinite space [2]:

$$\iiint\left(\vec{E}^a\cdot\vec{J}^b - \vec{H}^a\cdot\vec{M}^b\right)dV = \iiint\left(\vec{E}^b\cdot\vec{J}^a - \vec{H}^b\cdot\vec{M}^a\right)dV. \tag{15}$$

In this equation, the integration extends over the volume where the sources are present. The superscript a refers to one set of sources and the corresponding fields, and the superscript b refers to a second set. Let the a source be the tangential impressed electric current placed on a PEC, and let the b source be a Hertzian electric dipole at any point in space. The tangential $\vec{E}$ field produced by the Hertzian dipole on the PEC is zero (the boundary condition on the PEC). Therefore, the right-hand side of Equation (15) is zero, since the integration is carried out over the region of a sources, i.e., the PEC.

The integral of the left-hand-side term, which is proportional to the electric field produced by the impressed electric current on the PEC at the location of the Hertzian dipole and along the dipole, is also zero. By choosing the Hertzian dipole at different locations and with different orientations, we can show that the electric field produced by the impressed electric current on the PEC is zero everywhere. Similarly, by choosing a magnetic dipole for the b source, we can show that the magnetic field produced by the impressed electric current on a PEC is zero, although this result is obvious when the E field is zero everywhere.

6. Appendix 2 The Fields of a Planar Current

The planar current is given by

$$\vec{J} = -\hat{x}\frac{E_0}{Z_0}.$$

The magnetic vector potential is [3]

$$\vec{A} = \frac{\mu_0}{4\pi}\iint \vec{J}\frac{e^{-jk_0R}}{4\pi R}ds'.$$

Then,

$$\vec{A} = -\hat{x}\frac{E_0}{Z_0}\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{e^{-jk_0\sqrt{\rho'^2+z^2}}}{\sqrt{\rho'^2+z^2}}\rho' d\rho' d\phi'$$

$$= -\hat{x}\frac{E_0}{Z_0}\frac{\mu_0}{2}\int_0^{\infty}\frac{e^{-jk_0\sqrt{\rho'^2+z^2}}}{\sqrt{\rho'^2+z^2}}\frac{d\left(\rho'^2+z^2\right)}{2}$$

$$= -\hat{x}\frac{E_0}{2Z_0}\frac{\mu_0}{2}\int_0^{\infty} -\frac{2}{jk_0}d\left(e^{-jk_0\sqrt{\rho'^2+z^2}}\right)$$

$$= \hat{x}\frac{E_0}{2Z_0}\frac{\mu_0}{jk_0}\left(e^{-jk_0z}\right).$$

The integral vanishes at the upper limit with the assumption of a small loss in the medium. In the limiting case of vanishing losses, the result is still valid. Then,

$$\vec{E} = -j\omega\vec{A}$$

since

$$\nabla\bullet\vec{A} = 0,$$

and

$$\vec{E} = \hat{x}\frac{E_0}{2}e^{-jk_0z}.$$

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