A Kinetic theory for swarming with birth and death events

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December 13, 2011

1 Intro

Let us consider a particle system where the number of constituents may vary, due to birth or death events. In this case, the number of particles N is not a constant but may change in time, so that N is a variable itself. We thus denote by $f_N(\{\vec{x}_i\}, \{\vec{v}_i\}, t)$ the probability density of finding each of these N particles at position x_i, velocity v_i and within a volume $d\vec{x}_i, d\vec{v}_i$ in phase space. Conservation of probability requires that particles must be in any of the N-state ensembles so that

$$f_0(t) + \sum_{N=1}^{\infty} \int f_N(\{\mathbf{x}_i\}, \{\mathbf{v}_i\}, t) d\Omega_N = 1 \tag{1}$$

where $\Omega_N = d\mathbf{x}_1 \dots d\mathbf{x}_N d\mathbf{v}_1 d\mathbf{v}_N$. Each of the $f_N(\mathbf{x}_i, \mathbf{v}_i, t)$ functions obeys its related Liouville equation

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left[\nabla_{\mathbf{x}_i}(\dot{\mathbf{x}}_i f^{(N)}) + \nabla_{\mathbf{v}_i}(\dot{\mathbf{v}}_i f^{(N)}) \right] = \frac{\partial f^{(N)}}{\partial t}\Big|_{\mathrm{NC}}, \tag{2}$$

where the last term includes any non-conserved event, such as birth and death. Note that the total derivative of Eq. 1 must be zero, which implies that

$$\sum_{N=0}^{\infty} \int \frac{\partial f_N(\{\vec{x}_i\}, \{\vec{v}_i\}, t)}{\partial t} \Omega_N = 0. \tag{3}$$

Upon inserting the Liouville equation 2 in Eq. 3, we find that the non-conservative terms must satisfy

$$\sum_{N=0}^{\infty} \int \frac{\partial f^N}{\partial t}\Big|_{\mathrm{NC}} d\Omega_N = 0, \tag{4}$$

so that the non-conserved events combine to conserve the total probability. We now introduce a “death” term as follows:

$$\frac{\partial f^N}{\partial t}\Big|_{\mathrm{NC}} = -\sum_{i=1}^{N} \sigma(\mathbf{x}_i, \mathbf{v}_i) f^N + (N+1) \int f^{N+1}(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_N, \mathbf{v}) \sigma(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \tag{5}$$