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A KINETIC THEORY FOR SWARMING WITH BIRTH AND DEATH EVENTS

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Abstract. We consider a system of self-propelled agents interacting via pairwise attractive and repulsive Morse potentials and subject to gaussian noise in two dimensions. Earlier work showed that depending on interaction parameters, a catastrophic and an H–stable regime could arise, with diverse aggregation morphologies including mills, rings, clumps, flocks in the catastrophic regime and rigid body rotators and flocks in the H–stable one. Here, we consider both regimes and investigate the role of noise in promoting transitions between patterns, and in causing swarm disassembly. We find that within the catastrophic regime increasing noise intensity leads to a first order transition between translational flocks and compact stationary swarms and yet higher noise levels lead to swarm breakup. Within the H-stable regime instead we find .... Hysterisis.

The aggregation of self–propelled particles into coherent patterns is a ubiquitous process found in many chemical, physical, biological, and engineered systems [1]. Patterns may be static or dynamic and may arise over several spatio–temporal scales. For example, interacting droplets

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or ferromagnetic particles floating on fluid layers may form ordered structures [2, 3], organic molecules adsorbed on surfaces may self assemble into monolayers [4] while actin filaments, cells, myxobacteria and flagellated bacteria may form colonies, swarms or biofilms [5, 6, 7]. Interactions among members of these ensembles may be electrostatic, chemotactic, hydrophobic, electromagnetic, of Van der Waals type, and may also depend on the surrounding environment or cell culture.

More complex organisms such as insects, animals and humans also self-assemble, forming schools of fish, flocks of birds, locust swarms, or moving crowds that have inspired a new generation of mathematical modelers. In all these systems, agents organize following direct visual, tactile, auditory or other sensory couplings, giving rise to coherent bodies that may impart protection, enhanced mobility or other advantages to their members.

As our understanding of biological and biologically inspired self-assembly increases there is also great interest in applying this knowledge to design and create non–biological inanimate systems with novel properties and control possibilities [11]. Swarms of multiple, task-specific entities, such as unmanned land vehicles, search robots, underwater gliders, aerial drones have been tested. Expanded fabrication capabilities at the nano and mesoscale levels may also lead to the possibility of creating swarming nanostructures to monitor the presence of pathogens in seawater [12], or for biomedical purposes within the human body [13].

Beginning with the seminal work of Viczek and collaborators in the mid-ninties, and using the basic ingredients of direct interaction among agents, self–propulsion and the absence of central coordination, many discrete, rule-based, kinetic and hydrodynamic models have been presented. Some of these descriptions have considered idealized, portable swarming systems, while others have focused on specific organisms or vehicles, introducing ad–hoc behaviors, experimentally derived parameters and validated field testing.

One of the most studied discrete models within the swarming literature was initially introduced in Ref. [14] and is characterized by a set of individual self-propelled particles interacting via repulsive–attractive potentials. The model has been studied in detail [15, 16, 17] and several characterizations of morphologies and ensemble behavior as a function of parameter choices, noise and external fields have been presented, both in two and three dimensions [19, 20]. Ad–hoc features to study specific animal systems have also have also been considered [8, 9]. In other work the corresponding continuum descriptions have been developed using kinetic theory and presenting hydrodynamic equations that bridge the microscopic and macroscopic pictures. The vast body of work conducted on swarming systems however, has traditionally considered a fixed number of particles N, and events such as annihilation or creation of agents have not received much attention. On the other hand, the possibility of swarming agents increasing or decreasing while in motion, due for example, to birth or death events is certainly realistic and may occur in many natural systems.

This paper aims to develop a kinetic theory for collective agents where particle number is not kept constant but where basic mechanisms may exist that allow for particle creation and destruction. We will do this by first considering the case of a fixed number of particles s as a building block and later allowing s to vary. In Section 1 we thus describe the dynamics of a swarm of s particles and find the probability density function for the particles to occupy positions $\{\mathbf{x}_s\} = (\mathbf{x}_1, \dots \mathbf{x}_s)$ with velocities $\{\mathbf{v}_s\} = (\mathbf{v}_1, \dots \mathbf{v}_s)$ at time t. In Section 2 iwe consider the general case of variable particle numbers, while in Section 3 we ....

1. **A fixed particle number.** In this section we describe the dynamics for a fixed set of N discrete particles and review the kinetic equations they lead to, as shown in detail in Ref. [**?**, 10]. These will serve as a basis to derive the kinetic equations when the number of particles is changing, due to birth and death events, and N is no longer fixed. Within this context, each of our $1 \leq i \leq N$ particles obeys the following equations of motion

$$\dot{\mathbf{x}}_i = \mathbf{v}_i, \tag{1}$$

$$m_i \dot{\mathbf{v}}_i = (\alpha - \beta |\mathbf{v}_i|^2)\mathbf{v}_i - \nabla_{\mathbf{x}_i} \sum_{i \neq j} U(|\mathbf{x}_i - \mathbf{x}_j|), \tag{2}$$

where m_i is particle mass, which for simplicity we fix at $m_i = m = 1$. The first term on the right–hand–side of Eq. 1 is a non-conservative part where particle i exchanges energy with the environment via a self–propelling term $\alpha \mathbf{v}_i$ and via a frictional term $\beta |\mathbf{v}_i|^2 \mathbf{v}_i$. The total energy exchange is zero when $|\mathbf{v}_i|^2 = \alpha/\beta$, giving rise to a preferred particle speed. The other term is the contribution to the dynamics from a pairwise interaction potential U_i, given by the Morse potential

$$U(r) = \sum_{j \neq i} -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{3}$$

where C_a and C_r represent the amplitude of the attractive and repulsive contributions, respectively, and ℓ_a and ℓ_r their corresponding ranges. Equilibrium configurations have been thoroughly investigated in the parameter space defined by $C \equiv C_r/C_a$ and $\ell \equiv l_r/l_a$ [16].

We can now introduce $\rho_N(\{\mathbf{x}_N\}, \{\mathbf{v}_N\}, t)$ as the N–particle probability density function, so that the probability of finding each of the N distinguishable particles at positions $\{\mathbf{x}_N\} = (\mathbf{x}_1, \ldots \mathbf{x}_N)$ with velocities $\{\mathbf{v}_N\} = (\mathbf{v}_1, \ldots \mathbf{v}_N)$ at time t and within a volume $d\mathbf{x}_i d\mathbf{v}_i$ for each particle is given by $\rho_N(\{\mathbf{x}_N\}, \{\mathbf{v}_N\}, t) \prod_{i=1}^N d\mathbf{x}_i d\mathbf{v}_i$. Since ρ_N is a probability density function, its integral over phase space must be normalized to one

$$\int \rho_N(\{\mathbf{x}_N\}, \{\mathbf{v}_N\}, t) \prod_{i=1}^{N} d\mathbf{x}_i d\mathbf{v}_i = 1 \tag{4}$$

As shown in Ref. [10] we can write the Liouville equation for this system as

$$\frac{\partial \rho_N}{\partial t} + \sum_{i=1}^{N} \left[\nabla_{\mathbf{x}_i}(\mathbf{v}_i \rho_N) + \nabla_{\mathbf{v}_i}(\mathbf{v}_i \rho_N)\right] = 0. \tag{5}$$

We now consider the reduced s particle distribution function $f_s(\{\mathbf{x}_s\}, (\{\mathbf{v}_s\}, t)$ defined as

$$f_s(\{\mathbf{x}_s\}, (\{\mathbf{v}_s\}, t) = \langle \sum_{i=1}^{N} \delta(\mathbf{x_i} - \mathbf{x}) \delta(\mathbf{v_i} - \mathbf{v}) \rangle \tag{6}$$

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