Note that on the plots of the steady state values of s and g as a function of ρ_0, the s_0 curve has a global maximum. Because of the form of $f_{1,2}$, there is no closed form solution for the coordinates of the maximum. However, there is an easy closed form solution if $k_1 = k_2 = k$ and $\delta_1 = \delta_2 = \delta$, as is the case for our biologically estimated parameters. This solution is $(\rho_0, s) = (k, k/2)$.

Thus, it is natural to look for an approximate solution when $k_{1,2}$ and $\delta_{1,2}$ are each slightly detuned from equality. The basic idea is to expand everything in a power series in a small parameter ϵ.

Take the steady state formula for s from our manuscript, differentiate it, and set it equal to zero to look for the critical point. This is our governing equation. Then expand everything in a power series. Without loss of generality, we can do this as

$$k_1 = k + \epsilon K, \quad k_2 = k - \epsilon K$$

$$\delta_1 = \delta + \epsilon\Delta, \quad \delta_2 = \delta - \epsilon\Delta$$

$$\rho_0 = \rho_{00} + \epsilon\rho_{01}$$

.

Substituting the power series into the equation for s' and solving at $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ yields

$$\rho_{00} = k$$

and

$$\rho_{01} = K + \frac{\Delta k}{\delta}$$

Thus, the maximum solitarious density occurs at

$$\rho_0 \approx k + K + \frac{\Delta k}{\delta}$$

.

Substituting back into the original formula for s and keeping through $\mathcal{O}(\epsilon)$ gives us the maximum steady state solitarious density, which is

$$s_{max} \approx \frac{k}{2} + \frac{\Delta k}{2\delta}$$

I've checked how good this approximation is vis-a-vis the parameter sensitivities I discussed in yesterday's email. That is, I've set $k = 65$ and $\delta = 0.25$. I've let the deviation K be as large as $0.3k$ and the deviation Δ be as large as 0.3δ – that is, I've considered up to 30% deviation from the mean value.s Comparing the exact (numerical) values for the critical point's coordinates to the approximate values, you get up to 20% error for the ρ_0 coordinate and 15% error for the s coordinate.

Another interesting feature of the graph of homogeneous steady state s and g is that there is a point of equality between the two curves. By setting $s = g$ and throwing away the trivial case $\rho_0 = 0$, there is one positive solution, which is that $s = g$ when

$$\rho_0 = \frac{k_1}{2\delta_2}\left(\delta_1 - \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 + 4\delta_1\delta_2(k_2/k_1)^2}\right)$$

This solution is exact (no assumptions or approximations are necessary).