Notes on the LSA

(Dated: August 16, 2011)

Abstract

Some notes on the semi-2D linear stability analysis

From the Linear Stability Analysis related to the $2D$ potentials we find that the possible unstable wavelengths k are given by the ones that satisfy the following

$$\lambda(k) = -2\pi k^2 \left[\frac{s_0 R_s r_s^2}{(1+k^2 r_s^2)^{3/2}} + \frac{g_0 R_g r_g^2}{(1+k^2 r_g^2)^{3/2}} - \frac{g_0 A_g a_g^2}{(1+k^2 a_g^2)^{3/2}} \right] \geq 0 \tag{1}$$

and where s_0 and g_0 are the steady state, equilibrium concentrations of solitary and gregarious locusts, repectively and given as

$$s_0 = \frac{\rho_0 \delta_1 k_1^2 (k_2^2 + \rho_0^2)}{\delta_1 k_1^2 k_2^2 + (\delta_1 k_1^2 + \delta_2 k_1^2)\rho_0^2 + \delta_2 \rho_0^4} \tag{2}$$

$$g_0 = \frac{\delta_2 \rho_0^3 (k_1^2 + \rho_0^2)}{\delta_1 k_1^2 k_2^2 + (\delta_1 k_1^2 + \delta_2 k_1^2)\rho_0^2 + \delta_2 \rho_0^4} \tag{3}$$

In order to make analytical progress we set $k_1 = k_2 = h$ and $\delta_1 = \delta_2 = \delta$. The use of h is to avoid confusion with the wavelength k. We also use the fact that $r_s = a_g$. Later it will be useful to recall that within our model $r_g < r_s$. Inserting these simplifications into Eq. 1 we find that $\lambda(k)$ can be written as

$$\lambda(k) = \frac{2\pi R_g \rho_0^3 r_g^2 k^2}{(h^2 + \rho_0^2)(1 + k^2 r_s^2)^{3/2}} \left[M - m(k) \right] \tag{4}$$

where

$$M = \frac{\rho_0^2 A_g r_s^2 - h^2 R_s r_s^2}{R_g \rho_0^2 r_g^2} \tag{5}$$

and

$$m(k) = \frac{(1 + k^2 r_s^2)^{3/2}}{(1 + k^2 r_g^2)^{3/2}}. \tag{6}$$

Note that eq. 4 holds for all parameter choices, provided that $k_1 = k_2$, $r_s = a_g$ and $\delta_1 = \delta_2$. Also note that since $m(k) > 0$ for instability to occur we must require $M > 0$, otherwise $\lambda(k)$ will always be negative. This is not a sufficient condition, but it is a necesary one and translates to

$$\rho_0 > \rho_c = \sqrt{\frac{R_s}{A_g}} h, \tag{7}$$

which in our case for $R_s = 41.5, A_g = 13.3$ and $h = 65$ implies that if $\rho_0 < \rho_c = 115$ the system will always be stable. Let us now look at the terms inside the square brackets of Eq. 4. We know that $r_g < r_s$. This implies that the function $m(k)$ is monotonically increasing. Hence, for $\lambda(k)$ to be positive, it is also necessary that M be greater that the minimum value of $m(k)$ which is attained at $k = 0$. Thus, a more stringent condition is that

$$M > m(k = 0) = 1 \tag{8}$$

Using the fact that $m(k \to \infty) = r_s^3/r_g^3 > 1$ we can now distinguish three cases

- if $M < 1$ then $\lambda(k) < 0$ and the homogeneous steady state is stable to perturbations of all wave numbers.
- if $1 \leq M \leq \frac{r_s^3}{r_g^3}$ then instabilities will arise for perturbations with wave numbers in a band extending from $k = 0$ to some finite k.
- if $M > \frac{r_s^3}{r_g^3}$ then the system is unstable to perturbations of *any* wave number.

The condition $M > 1$, to guarantee instability can be rewritten as

$$\rho_0 > \rho_{c,2} = \sqrt{\frac{R_s}{A_g - R_g \frac{r_g^2}{r_s^2}}} h \tag{9}$$

so that, we are now guaranteed instability as long as $\rho_0 > \rho_{c,2} = 116.7$ which is exactly what we see from numerical plots. This condition is now sufficient for instability to set in. Note that for the parameters at hand, the third case cannot be ever verified, since it can be shown that $M < r_s^3/r_g^3$ for the numbers we use in the paper. It may however, be a viable occurrence for other parameter choices.

I. MAXIMUM INSTABILITY

Finally, if we are interested in the most unstable wavelength, in the case where $M > 1$, we can calculate $\lambda'(k)$ to find

$$\lambda'(k) = \frac{2\pi R_g \rho_0^3 r_g^2 k}{h^2 + \rho_0^2} \left[M \frac{2 - r_s^2 k^2}{(1 + r_s^2 k^2)^{5/2}} - \frac{2 - r_g^2 k^2}{(1 + r_g^2 k^2)^{5/2}} \right]. \tag{10}$$

Note that the two terms in the parenthesis contain the function

$$g(y) = \frac{2 - y^2}{(1 + y^2)^{5/2}} \tag{11}$$

so that finding the maximum corresponds to setting the term in parenthesis to zero and finding values of $k = k^*$ so that

$$Mg(r_s k^*) = g(r_g k^*) \tag{12}$$

The $g(y)$ function is such that $g(0) = 2$, $g(y \to \infty) = 0$, it intersects the y axis at $y = \sqrt{2}$ initially decreases and attains a minimum at $y = 2$. We note that this means, specifically, that at $k = 0$ the function $Mg(r_s k = 0) > g(r_g k = 0)$ since $M > 1$ in this case. Hence, $omega'(k)$ starts out positive for small k. On the other hand, the function $Mg(r_s k)$ will be zero at the location $k_s = \sqrt{2}/r_s$, whereas the function $g(r_g k)$ will be zero at the location $k_g = \sqrt{2}/r_g$. We also know that $k_g > k_s$, due to our choice $r_s < r_g$. In particular, this means that for $k = k_s$ $Mg(r_s k = r_s k_s) = 0$ but $g(r_g k = r_g k_s)$ is still positive, because it will become zero for larger $k = k_g$.

These facts lead us to conclude that $Mg(r_s k = r_s k_s) = 0 < g(r_g k = k_s)$, and that there must exist a point $0 < k^* < k_s$ where the two terms are equal. In particular, for the choice of $r_s = 0.14$ this leads to $k^* < 10$. The value of k^* is the value of k that leads to the fastest growing perturbation. There is another solution to Eq. 12,but that will represent a minimum.

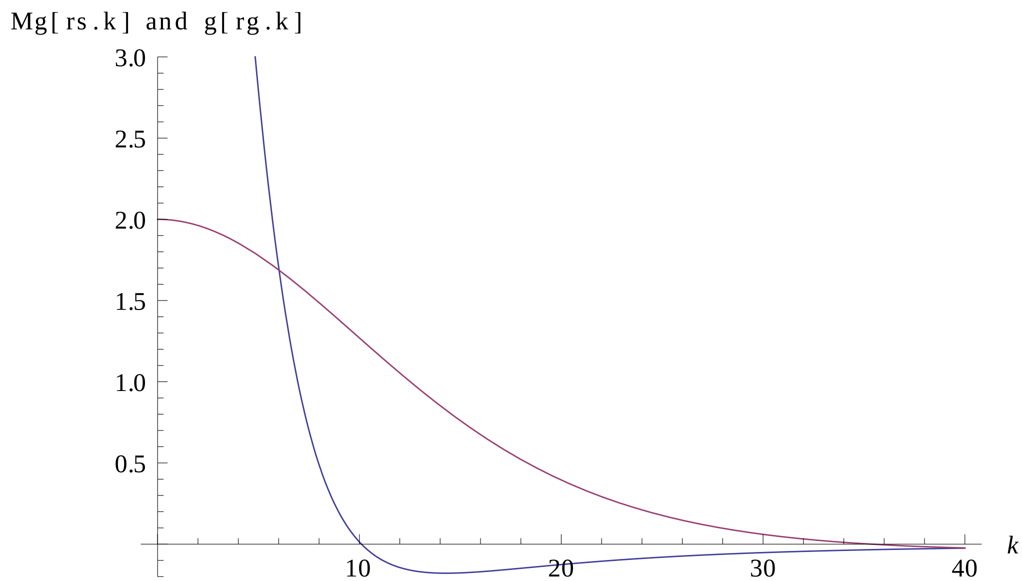

FIG. 1: The magenta curve is the function $g(r_g k)$ while the blue curve is the $Mg(r_s k)$ curve, where $M > 1$. Here, we set $M = 5$, $r_g = 0.04$ and $r_s = 0.14$. Note that as predicted, the two meet at a value $k^* < \sqrt{2}/r_s$ which is the point that makes the curve $Mg(r_s k) = 0$. In this case, $k^* \simeq 6.02$