DOUBLE MILLING IN SELF-PROPELLED SWARMS FROM KINETIC THEORY

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Abstract. We present a kinetic theory for swarming systems of interacting, self-propelled discrete particles. Starting from the Liouville equation for the many-body problem we derive a kinetic equation for the single particle probability distribution function and the related macroscopic hydrodynamic equations. General solutions include flocks of constant density and fixed velocity and other non-trivial morphologies such as compactly supported rotating mills. The kinetic theory approach leads us to the identification of macroscopic structures otherwise not recognized as solutions of the hydrodynamic equations, such as double mills of two superimposed flows. We find the conditions allowing for the existence of such solutions and compare to the case of single mills.

Key words. Interacting particle systems, swarming, kinetic theory, milling patterns.

AMS subject classifications. 92D50, 82C40, 82C22, 92C15

1. Introduction. Viewed as a collective, interacting individuals often flow into spectacular coherent patterns [5]. Systems that self-organize can be observed in nature across a wide variety of spatio-temporal scales: schools of fish, flocks of birds and swarms of insects among animal species; morphogenetic and bacterial growth at the cellular and subcellular levels. While each of these groups follow specific physical laws, all are able to organize in the absence of a leader, allowing order to arise even when starting from disordered configurations [23].

The ubiquity of the self-organizing phenomenon has lead to the development of several minimal models to describe a collection of interacting agents, both as discrete particles [31, 17, 9] or as a continuous density [28, 29, 30]. In particular, models of individuals driven by self-propelling forces and pairwise attractive and repulsive interactions have been shown to self-organize in various morphologies. Translationally invariant flocks, rotating mills, rings and clumps have all been observed and classified, so that specific interaction and propulsion values can be associated to specific collective configurations [20, 21, 15]. Aggregation patterns have also been identified in discrete models where averaging in direction or velocity is performed [10, 11] or where different zones of interaction and averaging are considered [1, 2].

However, as the number of particles grows, it becomes increasingly difficult to follow the dynamics of each individual agent. Indeed, to find the time evolution of a configuration of N individuals there are about N^2 interactions to compute, which in the limit of large N may yield to prohibitive calculations. A more compact, continuum approach where particles are represented by a density field, becomes thus desirable. While several continuum models based on heuristic derivations have been presented in the literature, few attempts have been made at trying to derive the hydrodynamic equations starting from discrete systems [4, 8, 18, 12, 13].

Furthermore, even when parallels between microscopic and macroscopic descriptions exist, there might be cases in which the fullness of a microscopic solution is

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not immediately captured by the corresponding macroscopic one. For example, in simulations of rotating mills, discrete particle systems show the possibility of *two* compactly supported structures of roughly the same number of particles circulating in opposite directions. The corresponding macroscopic solution would be a "boring" stationary density since the two mills average out their velocities to zero. Due to the non-linearity of the problem, the trivial superposition of two rotating mills which solve the macroscopic problem, is not necessarily a solution itself.

This paper aims to bridge general microscopic descriptions of self-propelled interacting swarming systems to their macroscopic counterparts, using kinetic theory [6, 24] as middle ground. Here, the exact location and velocity of particles are considered irrelevant, but not to the extent that average velocities can be computed *tout court* at every position and every time step. Rather, several velocities may be possible, so that the focus is on determining the probability density function $f(\mathbf{x}, \mathbf{v}, t)$ that at time t a particle is at position $\mathbf{x}$ with velocity $\mathbf{v}$. Starting from a set of discrete swarming equations we shall thus derive the kinetic equation for $f(\mathbf{x}, \mathbf{v}, t)$ and hence present the corresponding hydrodynamic description. Solutions will be matched to the discrete case and most importantly, our kinetic model will allow us to identify the presence of a new class of solutions, those of double mills, which elude a strictly macroscopic derivation.

2. Discrete Model. We consider N interacting, self-propelled particles with Rayleigh friction in $\mathbb{R}^d$, governed by the following equations of motion [20, 15, 7, 8]

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \nabla_{\mathbf{x}_i} \displaystyle\sum_{j \neq i} U(|\mathbf{x}_i - \mathbf{x}_j|). \end{cases}$$

Here U is a pairwise interaction potential and $\alpha, \beta > 0$ are effective values for propulsion and friction forces. A common choice for U is the Morse potential composed of attractive and repulsive components

$$U(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{2.1}$$

with C_a, C_r attractive and repulsive strengths and ℓ_a, ℓ_r their respective length scales. While the Morse potential is a common choice for interacting swarming systems, in this formulation we keep U general. In order to deal with the large number particle limit, we scale the amplitude of the potential through an effective mass normalization. For simplicity, we assume identical particles of mass m with total mass fixed at $M = Nm$. The "weak coupling scaling" assumption for the mean-field limit, see [27] and [16, Part I], assumes that the potential is modulated by a factor M/N, i.e.,

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \dfrac{M}{N}\nabla_{\mathbf{x}_i} \displaystyle\sum_{j \neq i} U(|\mathbf{x}_i - \mathbf{x}_j|), \end{cases} \tag{2.2}$$

The above scaling of the potential allows the total kinetic and potential energy to bear the same N dependence since, in the unnormalized case, the total kinetic energy is a sum of N terms and the total pairwise potential energy scales as N^2. The weak coupling limit assumption can be justified as representing a scenario where a particle

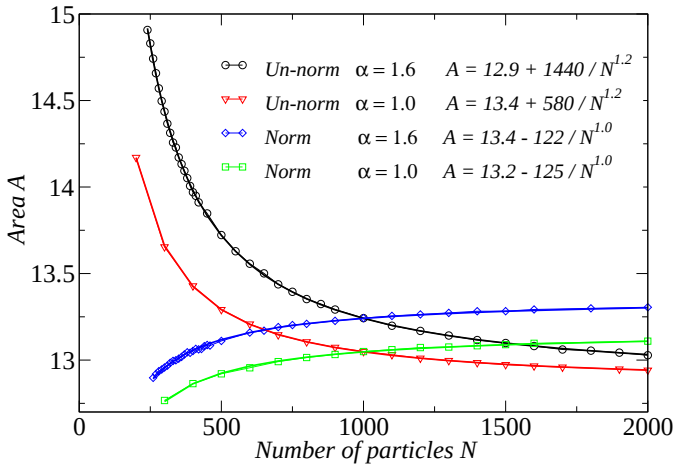

FIG. 2.1. *Area of unidirectional milling structures as a function of particle number N. The upper curve represents the unnormalized potentials, the lower one the potential normalized so that the total mass is kept at $M = Nm = 1000$. Milling parameters are chosen as $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 3$, $\ell_r = 0.5$ while the self-propulsion terms are set at $\alpha = 1.6$, $\beta = 0.5$ and at $\alpha = 1.0$, $\beta = 0.5$. The normalized and unnormalized curves match at $N = 1000$, which is the reference point for the unnormalized potential. Curves can be fitted as $A \sim A_0 + B/N^{\xi}$, where $\xi = 1.2$ or $\xi = 1.0$ for the unnormalized and normalized potentials, respectively.*

located at x_j imparts an interaction potential on the position x, proportional to its mass $m = M/N$, as in Coulomb or gravitational interactions.

As a result, the mass normalization in (2.2) allows for the kinetic and potential energy to be the sum of terms that should both scale as N. While interaction amplitudes are now N dependent, the fundamental character of the resulting morphologies do not change since we simply introduce a multiplicative factor for U. As an example, in the case of the Morse potential, patterns of aggregation depend on the relative amplitudes $C = C_r/C_a$ and $\ell = \ell_r/\ell_a$, so that N dependencies in the Morse potential do not affect the qualitative features of the observed patterns. The latter were classified in two dimensions in terms of C and ℓ in [15] using the concept of H-stability of potentials in the unnormalized case, where $M = N$ in (2.2).

The most relevant set of parameters for biological applications concerns long-range attraction and short-range repulsion leading to $C > 1$ and $\ell < 1$. For these potentials, there exists a unique minimum of the pairwise potential and a typical distance minimizing the potential energy. However, the curve $C\ell^d = 1$ divides this parameter region into the catastrophic regime ($C\ell^d < 1$) and the H-stable regime ($C\ell^d > 1$). In the H-stable regime and for unnormalized potentials, particle simulations in two dimensions lead to crystalline-like patterns. Here, for sufficiently large N, agents find an optimal spacing and maintain a fixed relative distance from each other even as N is further increased. Conversely, in the catastrophic regime, particles settle at a typical rotational speed of magnitude $|v| = \sqrt{\alpha/\beta}$, and single and double mills are observed. The area of these rotating structures decreases as a function of N. We refer to [15] for full details.

Due to the fact that qualitative features depend solely of the ratios C and ℓ, both normalized and unnormalized potentials yield the same type of patterns, whether they be mills or flocks, however the way these structures scale with N is very different in the normalized and unnormalized case. In Figure 2.1 we show single mill areas

of discrete particle systems in two dimensions as a function of N in the so called catastrophic regime, where, in the unnormalized case, rotating mills are expected to decrease in size, even to lose their inner radius, as the number of agents increases. As it can be seen, mill areas scale very differently in the unnormalized regime compared to the normalized case. We observe the tendency of mills to equilibrate to a density in a fixed annular region as the number of agents increases, as is typical of catastrophic potentials. The asymptotic value of the area as $N \to \infty$ depends on the intrinsic particle speed with higher values of $\sqrt{\alpha/\beta}$ yielding higher asymptotic limits. This is simply due to higher fluctuations in particle position in the collapsed state.

While coherent flocks and single mills states are the most common patterns observed in biological swarms [23, 26], double-mill patterns are also reported in the biological literature, for instance *M. xanthus* cells show distinct cell subpopulations swarming in two opposite directions during part of their life cycle [19].

3. Collisionless Kinetic Model. In this section, we briefly present the derivation of a mean-field kinetic model from the interaction particle dynamics (2.2). Let us denote by $f^{(N)}(\{\mathbf{x}_i\},\{\mathbf{v}_i\},t)$ the N-particle probability density function, so that the probability of finding each of the i particles at position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ within a volume $d\mathbf{x}_i\, d\mathbf{v}_i$ in phase space is $f^{(N)}(\{\mathbf{x}_i\},\{\mathbf{v}_i\},t)\prod_i d\mathbf{x}_i\, d\mathbf{v}_i$. Conservation of mass allows to write the time evolution of $f^{(N)}$ according to the following Liouville equation

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N}\Big[\mathrm{div}_{\mathbf{x_i}}(\dot{\mathbf{x}}_i f^{(N)}) + \mathrm{div}_{\mathbf{v}_i}(\dot{\mathbf{v}}_i f^{(N)})\Big] = 0, \tag{3.1}$$

where we now set $M = 1$ in (2.2). The one-particle distribution function $f^{(1)}(\mathbf{x}_1,\mathbf{v}_1,t)$ is defined as

$$f^{(1)}(\mathbf{x}_1,\mathbf{v}_1,t) = \int f^{(N)}\, d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N.$$

As standard in kinetic theory [27, 16, 25] we can now take the mean-field limit of Eq. 2.2 assuming particles are uncorrelated. We integrate the Liouville equation (3.1) to find the corresponding BBGKY hierarchy; restricting our analysis to the case of $f^{(1)}$ we find

$$\frac{\partial f^{(1)}}{\partial t} + \int \mathrm{div}_{\mathbf{x}_1}(\mathbf{v}_1 f^{(N)})\, d\mathbf{\Omega}_1 + \int \mathrm{div}_{\mathbf{v}_1}(\dot{\mathbf{v}}_1 f^{(N)})\, d\mathbf{\Omega}_1 = 0,$$

where $d\mathbf{\Omega}_1 = d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N$ is the volume element. The spatial divergence term reduces to $\mathbf{v}_1 \nabla_{\mathbf{x}_1} f^{(1)}$ while for the momentum term we obtain the following two contributions

$$\mathrm{div}_{\mathbf{v}_1}[(\alpha - \beta|\mathbf{v}_1|^2)\mathbf{v}_1 f^{(1)}] \qquad \text{and} \qquad \mathrm{div}_{\mathbf{v}_1}\int (\nabla_{\mathbf{x}_1}\sum_{j\neq 1} U_{1,j})\, f^{(N)}\, d\mathbf{\Omega}_1,$$

where $U_{1,j}$ is a shorthand notation for the fact that the two body potential is evaluated at $|\mathbf{x}_1 - \mathbf{x}_j|$. Since particles are indistinguishable the last term can be recast as

$$(N-1)\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(N)}\, d\mathbf{\Omega}_1 = (N-1)\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2,$$

with the notation $d\mathbf{\Omega_1} = d\mathbf{x}_2 d\mathbf{v}_2\, d\mathbf{\Omega}_2$ and where $f^{(2)}$ is the pair correlation function

$$f^{(2)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2) = \int f^{(N)} d\mathbf{\Omega}_2.$$

We furthermore assume correlations are negligible and factorize $f^{(2)}$ as

$$f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) = f^{(1)}(\mathbf{x}_1, \mathbf{v}_1)\, f^{(1)}(\mathbf{x}_2, \mathbf{v}_2).$$

We simplify the notation so that $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1) \equiv f(\mathbf{x}, \mathbf{v})$. The integral of f in the velocity coordinate is the macroscopic density of the system

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}.$$

Since the interaction term is independent of $\mathbf{v}$ we can also write

$$\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2 = (\nabla_{\mathbf{x}} U \star \rho)\, f,$$

where the $\star$ notation implies convolution between its arguments in the $\mathbf{x}$ spatial coordinate. The Liouville equation (3.1) now reduces to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \frac{N-1}{N}\, \mathrm{div}_{\mathbf{v}}\, [(\nabla_{\mathbf{x}} U \star \rho) f] = 0,$$

which in the limit of very large N yields

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \mathrm{div}_{\mathbf{v}}\, [(\nabla_{\mathbf{x}} U \star \rho) f] = 0. \tag{3.2}$$

The procedure used to find the equation for f is well-known in kinetic theory [27, 16] and the mean-field limit can be rigorously proved in the case of $\alpha = \beta = 0$ and smooth potentials [22, 3, 14]. A review of these results can be found in [16, Part I]. More precisely, let us consider the empirical measure associated to the system of N particles (2.2) given by

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\mathbf{x}_i(t), \mathbf{v}_i(t))},$$

where $\delta_{(\mathbf{x}_0, \mathbf{v}_0)}$ is the Dirac delta at $(\mathbf{x}_0, \mathbf{v}_0)$ in phase space.

Given an interaction potential $U \in C^2_r(\mathbb{R}^d)$, such that U is a rapidly decaying C^2 function as $|x| \to \infty$, then the empirical measure is always a solution in the sense of distributions of the kinetic equation (3.2), as a simple adaptation of [16, Lemma I.2] shows. Moreover, a stability result for empirical measures as distributional solutions of the kinetic equation should be expected although the proofs in the above references do not apply directly. More precisely, assume the initial data of the particle system (2.2) is chosen in such a way that $\mu_N(0) \to f_0$ weakly-* as measures, then all limit points of the empirical measure in the weak-* topology as measures should be distributional solutions of the kinetic equation (3.2) with initial data f_0. Such a result is in fact a proof of convergence of the particle method based on (2.2) for the kinetic equation (3.2) and will be treated elsewhere. Let us remark that the typical Morse potential (2.1) is only Lipschitz for which the previous theory does not apply.

Finally, let us point out that we can find the evolution of the total energy E associated to the kinetic equation defined as the sum of the potential and kinetic energies, E_U and E_K, respectively

$$E(t) = \frac{1}{2} \int U(|x-y|)\rho(\mathbf{x}, t)\rho(\mathbf{y}, t) d\mathbf{x} d\mathbf{y} + \frac{1}{2} \int |\mathbf{v}|^2 f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}. \tag{3.3}$$

The total energy verifies

$$\frac{dE}{dt} = \int |\mathbf{v}|^2(\alpha - \beta|\mathbf{v}|^2) f(\mathbf{x},\mathbf{v},t) d\mathbf{x} d\mathbf{v}.$$

The above relation implies energy conservation if our distribution is concentrated on the velocity sphere $\beta|\mathbf{v}|^2 = \alpha$. In this case, since there is no energy dissipation, we cannot expect any equilibration of the solution as $t \to \infty$ but we might expect stability results for certain particular initial conditions.

One of the objectives of the next sections is to show that all proposed continuum models for our swarming system 2.2 can be derived from the kinetic equation (3.2) and its variants. Moreover, all patterns observed in particle simulations can be considered as particular solutions of the kinetic equation (3.2) as will be shown below.

3.1. Macroscopic-Hydrodynamic equations. As usual in kinetic theory, we obtain continuum-like equations by computing the evolution of macroscopic quantities starting from (3.2). These macroscopic quantities are the velocity moments of $f(\mathbf{x},\mathbf{v},t)$. Let us consider the coarse grained velocity field $\mathbf{u}(\mathbf{x},t)$ defined through

$$\rho\mathbf{u} = \int \mathbf{v} f(\mathbf{x},\mathbf{v})\, d\mathbf{v}$$

and the energy-flux $\mathbf{q}_K$, the pressure tensor $\hat{\sigma}_K$ and the temperature θ defined through fluctuation terms as

$$\mathbf{q}_K = \frac{1}{2}\int |\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v}, \qquad \hat{\sigma}_K = \int (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v},$$

and

$$\delta_K = \int |\mathbf{v}-\mathbf{u}|^2 f\, d\mathbf{v} = d\,\rho\,\theta.$$

Integrating (3.2) in $d\mathbf{v}$ we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0. \tag{3.4}$$

Integrating the same equation over $\mathbf{v}d\mathbf{v}$ and using integration by parts, we find the momentum equation

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u}) = (\alpha - \beta|\mathbf{u}|^2)\rho\,\mathbf{u} - (\nabla_{\mathbf{x}} U \star \rho)\,\rho - \mathrm{div}_{\mathbf{x}}\,\hat{\sigma}_K - 2\,\beta\,\mathbf{q}_K - 2\,\beta\,\mathbf{u}\,\hat{\sigma}_K - \beta\,\delta_K\,\mathbf{u}.$$

We should now close the moment system: if we assume that fluctuations are negligible and that the distribution is monokinetic, i.e., $f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x},t))$, where δ stands for the Dirac delta, the macroscopic system reduces to

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0, \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = (\alpha - \beta|\mathbf{u}|^2)\mathbf{u} - \nabla_{\mathbf{x}} U \star \rho. \end{cases} \tag{3.5}$$

The system of equations (3.5) was already proposed in Ref. [8] based on computations of the empirical measure associated to N particles. Here, the same description is

recovered and understood as a first-order closure hydrodynamic system to the kinetic Eq. (3.2). In Ref. [8] the authors discussed the validity of this approximation based on numerical comparisons of the N-particle system and the hydrodynamic system (3.5). They concluded that the hydrodynamic system is a good approximation close to equilibrium patterns situations and performed a linear stability analysis for the simple flocking solution around $\rho = \rho_0$ and $|u| = \sqrt{\alpha/\beta}$. Double mills however cannot be simply explained with this hydrodynamic approach due to the averaging in velocity.

Based on the mesoscopic description given by the kinetic equation (3.2), we can also write the equations for energy transport. The kinetic energy density $\mathcal{E}_K$ and potential energy density $\mathcal{E}_U$ are defined, respectively, as

$$\mathcal{E}_K = \frac{1}{2}\int |\mathbf{v}|^2 f d\mathbf{v} = \rho\frac{|\mathbf{u}|^2}{2} + \frac{\delta_K}{2} = \rho\frac{|\mathbf{u}|^2}{2} + \frac{d}{2}\rho\theta,$$

and $\mathcal{E}_U = \frac{1}{2}(U \star \rho)\,\rho$, so that the total energy is given by $\mathcal{E} = \mathcal{E}_K + \mathcal{E}_U$. In order to determine the evolution of the total energy density we multiply (3.2) by $|\mathbf{v}|^2/2$, integrate over $d\mathbf{v}$ and integrate by parts to obtain

$$\frac{\partial \mathcal{E}_K}{\partial t} + \int \frac{|\mathbf{v}|^2}{2}\mathbf{v}\nabla_{\mathbf{x}} f d\mathbf{v} - \int |\mathbf{v}|^2(\alpha - \beta|\mathbf{v}|^2)\, f\, d\mathbf{v} + \int \mathbf{v}(\nabla_{\mathbf{x}} U \star \rho) f\, d\mathbf{v} = 0.$$

As before, we can rewrite all integral terms as

$$\int \frac{|\mathbf{v}|^2}{2}\mathbf{v}\nabla_{\mathbf{x}} f d\mathbf{v} = \mathrm{div}_{\mathbf{x}}\,\mathcal{E}_K\mathbf{u} + \mathrm{div}_{\mathbf{x}}\,\mathbf{q}_K + \mathrm{div}_{\mathbf{x}}\,\mathbf{u}\,\hat{\sigma}_K, \quad \int (\nabla_{\mathbf{x}} U \star \rho)\, f\, \mathbf{v}\, d\mathbf{v} = (\nabla_{\mathbf{x}} U \star \rho)\,\rho\mathbf{u},$$

and

$$\int (\alpha - \beta|\mathbf{v}|^2)|\mathbf{v}|^2\, f d\mathbf{v} = 2\alpha\mathcal{E}_K + \beta\int |\mathbf{v}|^4\, f d\mathbf{v}.$$

Using the chain rule and the continuity equation for ρ we find

$$\frac{\partial \mathcal{E}_U}{\partial t} = -\mathrm{div}_{\mathbf{x}}\,\mathcal{E}_U\mathbf{u} + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) + (\nabla_{\mathbf{x}} U \star \rho)\,\mathbf{u}\right]\rho,$$

so that

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}_{\mathbf{x}}\mathcal{E}\,\mathbf{u} + \mathrm{div}_{\mathbf{x}}(\mathbf{q}_K + \mathbf{u}\,\hat{\sigma}_K) = \\ -\beta\int |\mathbf{v}|^4 f d\mathbf{v} + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) - (\nabla_{\mathbf{x}} U \star \rho)\mathbf{u}\right]\rho + 2\alpha\mathcal{E}_K. \qquad (3.6)\end{aligned}$$

Finally, the latter term can be expressed as

$$\int |\mathbf{v}|^4 f d\mathbf{v} = |\mathbf{u}|^2(2\mathcal{E}_K + \delta_K) + 8\mathbf{u}\mathbf{q}_K + \tau_K + 4g_K,$$

where τ_K and g_K are defined as

$$\tau_K = \int |\mathbf{v} - \mathbf{u}|^4 f d\mathbf{v} \qquad \text{and} \qquad g_K = \int \left[\mathbf{u}\cdot(\mathbf{v} - \mathbf{u})\right]^2 d\mathbf{v}.$$

Again, in the limit of small fluctuations $\theta \simeq 0$, the above reduces to $2\mathcal{E}_K = |\mathbf{u}|^2 + \delta_K \sim |\mathbf{u}|^2$, and the energy density transport equation can be written as

$$\frac{\partial \mathcal{E}}{\partial t} + \operatorname{div}_{\mathbf{x}} \mathcal{E}\,\mathbf{u} = 2\alpha|\mathbf{u}|^2 - \beta|\mathbf{u}|^4 + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) - (\nabla_{\mathbf{x}} U \star \rho)\mathbf{u}\right]\rho.$$

The system of equations (3.5) complemented by the above energy transport equation constitute the hydrodynamic description in the small fluctuation closure, i.e., $f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x},t))$.

3.2. Single-Milling and Flocking Patterns: Monokinetic solutions. We now try to find weak solutions to the kinetic equation (3.2) of the monokinetic form

$$f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x},t)), \tag{3.7}$$

where the constraints on ρ and $\mathbf{u}$ will be imposed by the weak formulation analysis. In this ansatz all fluctuating terms are strictly zero, since all microscopic velocities are identically set to $\mathbf{u}$ and we can find explicit weak solutions.

LEMMA 3.1. *Given $\rho(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$ as smooth as needed. Then, the ansatz* (3.7) *is a distributional solution of the kinetic equation* (3.2) *if and only if $(\rho,\mathbf{u})$ is a solution of the system* (3.5).

Proof. Imposing that (3.7) is a distributional solution of (3.2), we multiply by a test function $\psi(\mathbf{x},\mathbf{v},t)$. Integrating over $d\mathbf{v}$ and evaluating the Dirac delta, we find

$$\int \frac{\partial f}{\partial t}\,\psi\,dtd\mathbf{x}\,d\mathbf{v} = -\int \left[\frac{\partial \psi}{\partial t}\right](\mathbf{x},\mathbf{u}(\mathbf{x},t),t)\,\rho\,dt\,d\mathbf{x}. \tag{3.8}$$

This quantity can be written through the chain rule as follows

$$\frac{\partial}{\partial t}\left[\psi(\mathbf{x},\mathbf{u}(\mathbf{x},t),t)\right] = \left[\frac{\partial \psi}{\partial t}\right](\mathbf{x},\mathbf{u}(\mathbf{x},t),t) + \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t)\cdot[\nabla_{\mathbf{v}}\psi](\mathbf{x},\mathbf{u}(\mathbf{x},t),t).$$

In order to simplify the notation, we use the bracket notation where $[\psi]$ denotes the function $\psi(\mathbf{x},\mathbf{u}(\mathbf{x},t),t)$. We can now write

$$\int \frac{\partial f}{\partial t}\,\psi\,dtd\mathbf{x}\,d\mathbf{v} = -\int \frac{\partial[\psi]}{\partial t}\,\rho\,dt\,d\mathbf{x} + \int \frac{\partial \mathbf{u}}{\partial t}\cdot[\nabla_{\mathbf{v}}\psi]\,\rho\,dt\,d\mathbf{x},$$

$$\int \operatorname{div}_{\mathbf{x}}(\mathbf{v}f)\psi\,dt\,d\mathbf{x}\,d\mathbf{v} = -\int \rho\,\mathbf{u}\cdot\nabla_{\mathbf{x}}[\psi]\,dt\,d\mathbf{x} + \int \rho\,(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u}\cdot[\nabla_{\mathbf{v}}\psi]\,dt\,d\mathbf{x},$$

$$\int \operatorname{div}_{\mathbf{v}}[(\alpha-\beta|\mathbf{v}|^2)\mathbf{v}\,f]\,\psi\,dt\,d\mathbf{x}\,d\mathbf{v} = -\int (\alpha-\beta|\mathbf{u}|^2)(\mathbf{u}\cdot[\nabla_{\mathbf{v}}\psi])\,\rho\,dt\,d\mathbf{x},$$

and

$$\int \operatorname{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho)f\right]\psi\,dt\,d\mathbf{x}\,d\mathbf{v} = -\int (\nabla_{\mathbf{x}} U \star \rho)\cdot[\nabla_{\mathbf{v}}\psi]\,\rho\,dt\,d\mathbf{x}.$$

Now, it is a simple matter of collecting terms in the above expressions to deduce the sufficiency part of the statement. The necessity part is deduced by choosing ψ to be

velocity independent, $\psi = \phi(\mathbf{x}, t)$ or $\psi = \varphi(\mathbf{x}, t)\mathbf{v}_i$, $i = 1, \dots 3$ with φ a test function in $(\mathbf{x}, t)$. □

As a particular case of the previous lemma, we find particular distributional solutions of the kinetic equation (3.2). Let us look for solutions in which we impose that $\beta|\mathbf{u}(\mathbf{x}, t)|^2 = \alpha$, i.e., that the frictional terms balance each other. Moreover, if we seek steady solutions, the monokinetic distribution (3.7) is a distributional solution of (3.2) if and only if its arguments $(\rho, \mathbf{u})$ satisfy

$$\begin{cases} \operatorname{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0, \\ (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = -(\nabla_{\mathbf{x}}U \star \rho), \\ \beta|\mathbf{u}(\mathbf{x})|^2 = \alpha. \end{cases} \tag{3.9}$$

One family of solutions for these equations is given by ρ and $\mathbf{u}$ both uniform and constant in time. This is of course the translationally invariant, flocking solution. We can obtain another family of monokinetic solutions by assuming that particles undergo a circular motion with constant speed $\sqrt{\alpha/\beta}$. These single milling solutions were proposed in [20] and shown numerically. We will come back to this particular set of solutions below.

3.3. Double Milling Patterns: Hydrodynamic Superpositions at Kinetic Level. In the previous section, we showed that particular solutions to the kinetic equation (3.2) are found by imposing the monokinetic ansatz. Here, we look for the conditions that must be met in order to ensure that a linear superposition of such monokinetic distributions is a distributional solution of Eq. (3.2). For concreteness, we consider the case of two populations with density ρ_1 and ρ_2 and with velocity $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, so that

$$f = \rho_1\delta(\mathbf{v} - \mathbf{u}_1(\mathbf{x}, t)) + \rho_2\delta(\mathbf{v} - \mathbf{u}_2(\mathbf{x}, t)). \tag{3.10}$$

With this definition we find

$$\rho = \rho_1 + \rho_2, \tag{3.11}$$

$$\rho\mathbf{u} = \rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2. \tag{3.12}$$

We thus insert (3.10) in the kinetic equation (3.2), multiply by the test function ψ and, as done before, complete the temporal and spatial integrals so that

$$\begin{aligned} &\int \rho_1 \left[\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1\cdot\nabla_{\mathbf{x}})\mathbf{u}_1 + (\nabla_{\mathbf{x}}U \star \rho) - (\alpha - \beta|\mathbf{u}_1|^2)\mathbf{u}_1\right]\cdot[\nabla_{\mathbf{v}}\psi_1]\,dt\,d\mathbf{x} \\ &+ \int \rho_2 \left[\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2\cdot\nabla_{\mathbf{x}})\mathbf{u}_2 + (\nabla_{\mathbf{x}}U \star \rho) - (\alpha - \beta|\mathbf{u}_2|^2)\mathbf{u}_2\right]\cdot[\nabla_{\mathbf{v}}\psi_2]\,dt\,d\mathbf{x} \\ &+ \int \left(\frac{\partial \rho_1}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho_1\mathbf{u}_1)\right)[\psi_1]\,dt\,d\mathbf{x} + \int \left(\frac{\partial \rho_2}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho_2\mathbf{u}_2)\right)[\psi_2]\,dt\,d\mathbf{x} = 0. \end{aligned} \tag{3.13}$$

Here the subscripts $\psi_{1,2}$ signify that the quantities of interest are evaluated at $\mathbf{v} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{u}_2$, respectively. As before, choosing ψ to be only a function of t and $\mathbf{x}$ leads to the following conservation equation

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2) = 0,$$

which in light of (3.11) and (3.12) is the usual continuity equation (3.4). We can also choose $\psi = \mathbf{v}$ so that (3.13) gives

$$\sum_{i=1}^{2} \rho_i \left[\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla_{\mathbf{x}})\mathbf{u_i} - (\alpha - \beta|\mathbf{u}_i|^2)\mathbf{u}_i \right] + \sum_{i=1}^{2} \mathbf{u}_i \left[\frac{\partial \rho_i}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho_i \mathbf{u}_i) \right] + (\nabla_{\mathbf{x}} U \star \rho)\rho = 0. \tag{3.14}$$

This is the general condition for the two sub-populations to satisfy so that their superposition is a solution to the kinetic equation. We present this result in the following

LEMMA 3.2. *Given $\rho_i(\mathbf{x}, t)$ and $\mathbf{u}_i(\mathbf{x}, t)$, $i = 1, 2$, as smooth as needed satisfying* (3.11) *and* (3.12). *Then, the ansatz* (3.10) *is a distributional solution of the kinetic equation* (3.2) *if and only if $(\rho_i, \mathbf{u}_i)$ satisfy* (3.4) *and* (3.14).

Based on this Lemma, we can again look for steady state solutions satisfying $\beta|\mathbf{u}_i(\mathbf{x})|^2 = \alpha$, $i = 1, 2$ and leading to

$$\rho(\nabla_{\mathbf{x}} U \star \rho) + \rho_1(\mathbf{u}_1 \cdot \nabla_{\mathbf{x}})\mathbf{u}_1 + \rho_2(\mathbf{u}_2 \cdot \nabla_{\mathbf{x}})\mathbf{u}_2 + \mathbf{u}_1 \mathrm{div}_{\mathbf{x}}(\rho_1 \mathbf{u}_1) + \mathbf{u}_2 \mathrm{div}_{\mathbf{x}}(\rho_2 \mathbf{u}_2) = 0.$$

The above formulation differs from the monokinetic case of (3.7) in a non-trivial way. We can nonetheless recast the above result by assuming $2\rho_1 = 2\rho_2 = \rho$ and by imposing $\mathbf{u}_1 = -\mathbf{u}_2 = \mathbf{u}$ so that

$$\rho(\nabla_{\mathbf{x}} U \star \rho) + \rho \mathbf{u} \nabla_{\mathbf{x}} \mathbf{u} + \mathbf{u} \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0.$$

Simplifying ρ we find, that similarly to the monokinetic case, the superposition

$$f = \frac{1}{2}\rho\delta(\mathbf{v} - \mathbf{u}(\mathbf{x})) + \frac{1}{2}\rho\delta(\mathbf{v} + \mathbf{u}(\mathbf{x})).$$

is a stationary solution to the kinetic equation if and only if

$$\begin{cases} (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + (\nabla_{\mathbf{x}} U \star \rho) = 0, & \text{whenever } \rho \neq 0 \\ \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, \\ \beta|\mathbf{u}(\mathbf{x})|^2 = \alpha. \end{cases} \tag{3.15}$$

Summarizing, we conclude that double milling solutions, where half the particles travel at the same speed $\mathbf{u}$ and the rest at its exact opposite $-\mathbf{u}$, exist provided (3.15) is satisfied. Furthermore, this result allows us to conclude that steady solutions $\rho, \mathbf{u}$ for the single mill case, are also double milling solutions with dual velocities since (3.4) and (3.15) are the same. Note that in the dual-velocity case the average macroscopic velocity is zero and therefore, double milling solutions cannot be explained by the hydrodynamic model with a single macroscopic velocity. These richer solutions arise solely by imposing two subpopulations moving at opposite speeds and which naturally emerge from the kinetic approach.

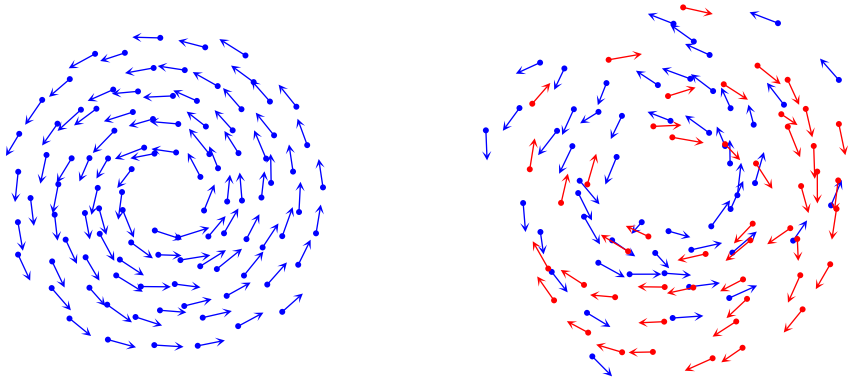

FIG. 3.1. *Single and double milling solutions from numerical simulations of* (2.2) *in the unnormalized case where* $M = N$ *and parameters are chosen as* $N = 100$, $C_a = 0.5$ $C_r = 1.0$, $\ell_a = 3$, $\ell_r = 0.5$, $\alpha = 1.0$, $\beta = 0.5$. *Random initial conditions were chosen for particles in the left panel, whereas* $N/2$ *particles were initiated rotating clockwise and* $N/2$ *counterclockwise in the right panel. General random initial conditions typically yield single rotational structures.*

3.4. Density profiles for Single and Double Milling Patterns. Steady milling solutions to (3.9) can be found by setting $\mathbf{u}$ in a rotatory state:

$$\mathbf{u} = \sqrt{\frac{\alpha}{\beta}}\,\frac{\mathbf{x}^{\perp}}{|\mathbf{x}|}.$$

In this case the transport term can be rewritten as

$$(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \frac{\alpha}{\beta}\,\frac{\mathbf{x}}{|\mathbf{x}|^{\mathbf{2}}} \qquad \text{for } |x| \neq 0$$

so that, using (3.9) as shown in [20]

$$U \star \rho = D - \frac{\alpha}{\beta}\ln|\mathbf{x}| \qquad \text{whenever } \rho \neq 0, \tag{3.16}$$

where D is a constant with $\rho = 0$ near the origin and $\mathbf{u}\cdot\nabla_{\mathbf{x}}\rho = 0$ whenever $\rho \neq 0$. Let us remark that a radial density will satisfy the constraint $\mathbf{u}\cdot\nabla_{\mathbf{x}}\rho = 0$. Solutions to Eq. (3.16) were numerically found in Ref. [20] and matched to single mill patterns in Ref. [8]. From our previous discussion, this solution also applies to superimposed mills of velocities $\mathbf{u}, -\mathbf{u}$. While the total velocity is zero, the two populations coexist rotating in opposite directions such that the total density satisfies Eq. (3.16). Let us remark that a uniform distribution on the circle of radius $R_0 > 0$, i.e. ρ being a Delta dirac on the circle of radius R_0, is a measure-valued solution to Eq. (3.16). This can be trivially checked since D can be chosen in such a way that (3.16) is verified. A rigorous proof of existence of radial integrable solutions with support in an interval $[R_0, R_1]$ with $0 < R_0 < R_1$ to Eq. (3.16) is still lacking.

4. Kinetic Model for interacting particles with random noise. Finally, let us study the case of the interacting particle system with random noise, i.e.,

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \dfrac{M}{N}\nabla_{\mathbf{x}_i}\displaystyle\sum_{j\neq i} U(|\mathbf{x}_i - \mathbf{x}_j|) + \sqrt{2\sigma}\,\Gamma(t), \end{cases} \tag{4.1}$$

where $\Gamma(t)$ is the Wiener process of unit variance and $\sigma > 0$ the noise strength. Using Ito's formula and the same procedure as in section 2, it is easy to derive the following kinetic Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \sigma\Delta_{\mathbf{v}} f. \tag{4.2}$$

In order to derive some asymptotic limit equations starting from the kinetic equation (4.2), we need to have a dimensionless formulation of the equation in a suitable form. We introduce the effective self-propulsion strength $\alpha = \alpha_1 - \alpha_0$ with $\alpha_0, \alpha_1 > 0$, and where α_0 is the Stokes frictional component and α_1 the self-propulsion generated by the organisms. With this choice the kinetic equation is rewritten as

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha_1 - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \mathrm{div}_{\mathbf{v}}[\alpha_0 \mathbf{v} f + \sigma\nabla_{\mathbf{v}} f]. \tag{4.3}$$

Let us first remark that $\mathcal{T}_{FP} = \frac{1}{\alpha_0}$ is the natural relaxation time for the Fokker-Planck operator $\mathrm{div}_{\mathbf{v}}[\alpha_0\mathbf{v}f + \sigma\nabla_{\mathbf{v}} f]$. As usual, let us consider

$$\mathcal{V}_{th} = \sqrt{\frac{\sigma}{\alpha_0}},$$

the typical value of fluctuations in particle velocity, called thermal speed. Let us introduce time and length units T and $\mathrm{L} = \ell_a$ respectively, which define the time and length units of the observation and the typical speed $\mathrm{U} = \mathrm{L}/\mathrm{T}$. Then, we can define dimensionless variables, denoted by primed quantities, as

$$t = \mathrm{T}\, t', \qquad \mathbf{x} = \mathrm{L}\,\mathbf{x}', \qquad \mathbf{v} = \mathcal{V}_{th}\mathbf{v}',$$

$$f'(t', \mathbf{x}', \mathbf{v}') = \mathrm{L}^d\ \mathcal{V}_{th}^d\ f(\mathrm{T}t', \mathrm{L}\mathbf{x}', \mathcal{V}_{th}\mathbf{v}'), \qquad \text{and} \qquad U'(\mathbf{x}') = C_a\ U(\mathrm{L}\mathbf{x}').$$

With this change of scales, we finally find the following dimensionless kinetic equation

$$\frac{\partial f}{\partial t} + \eta\mathbf{v}\cdot\nabla_{\mathbf{x}} f + \gamma_1 \mathrm{div}_{\mathbf{v}}[\mathbf{v}\, f] - \gamma_2 \mathrm{div}_{\mathbf{v}}[|\mathbf{v}|^2\mathbf{v}\, f] - \chi\mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \frac{1}{\epsilon} L f. \tag{4.4}$$

where primes have been eliminated for notational simplicity and the operator Lf defined as $Lf \equiv \mathrm{div}_{\mathbf{v}}[\mathbf{v}f + \nabla_{\mathbf{v}} f]$. Here, η, γ_1, γ_2 and χ are dimensionless parameters given by

$$\eta = \frac{\mathcal{V}_{th}}{U}, \qquad \gamma_1 = T\alpha_1, \qquad \gamma_2 = T\beta\mathcal{V}_{th}^2,$$

$$\chi = \frac{C_a}{U\mathcal{V}_{th}} \qquad \text{and} \qquad \epsilon = \frac{\mathcal{T}_{FP}}{T} = \frac{1}{T\alpha_0}.$$

With this dimensionless formulation, the potential becomes $U(r) = -e^{-r} + Ce^{-r/\ell}$ and we can find two different regimes in which the kinetic equation may be approximated by distinct macroscopic equations.

4.1. Weak-interaction/Strong-noise regime. We choose the following relation between the dimensionless parameters: $\eta \simeq \gamma_1 \simeq \gamma_2 \simeq \chi \simeq \epsilon^{-1/2}$. In this regime, the dominant mechanisms are the noise and the linear Stokes friction. In order to obtain macroscopic equations, we use the standard Hilbert expansion method. Inserting the following Hilbert expansion

$$f_\epsilon = f^{(0)} + \sqrt{\epsilon} f^{(1)} + \epsilon f^{(2)} + \dots \qquad \text{and} \qquad \rho_\epsilon = \rho + \sqrt{\epsilon}\rho^{(1)} + \dots \tag{4.5}$$

into (4.4) and identifying terms with equal power of $\sqrt{\epsilon}$, we get:

- ϵ^{-1} terms: $Lf^{(0)} = 0$ which implies that $f^{(0)}(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\ M(\mathbf{v})$, where $M(\mathbf{v})$ is the Maxwellian distribution with unit temperature.
- $\epsilon^{-1/2}$ terms:

$$\begin{aligned} Lf^{(1)} &= \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(0)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(0)}] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f^{(0)}\right] \\ &= \mathbf{v}\cdot[\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho]\ M(\mathbf{v}) + \rho\left[d-(d+3)|\mathbf{v}|^2+|\mathbf{v}|^4\right]\ M(\mathbf{v}). \end{aligned}$$

  This equation can be readily inverted remarking that $L(\mathbf{v}M(\mathbf{v})) = -\mathbf{v}M(\mathbf{v})$ and

$$L\left[(\frac{1}{2}|\mathbf{v}|^2 - \frac{1}{4}|\mathbf{v}|^4)M(\mathbf{v})\right] = \left[d-(d+3)|\mathbf{v}|^2+|\mathbf{v}|^4\right]\ M(\mathbf{v})$$

  to obtain

$$f^{(1)}(t,x,\mathbf{v}) = -\mathbf{v}\cdot[\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho]\ M(\mathbf{v}) + \rho\left[\frac{1}{2}|\mathbf{v}|^2 - \frac{1}{4}|\mathbf{v}|^4\right]\ M(\mathbf{v})$$

- ϵ^0 terms:

$$\begin{aligned} Lf^{(2)} = {}& \partial_t f^{(1)} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(1)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(1)}] \\ & - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho) f^{(1)}\right] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho^{(1)}) f^{(0)}\right], \end{aligned}$$

  with $\rho^{(1)} = \int f^{(1)}\, d\mathbf{v}$. However, since $\int h\, d\xi = 0$ is a necessary condition for the equation $L(f) = h$ to admit a solution, we conclude

$$\partial_t\Big(\int_{\mathbb{R}^d} f^{(0)}\, d\mathbf{v}\Big) + \mathrm{div}_{\mathbf{x}}\Big(\int_{\mathbb{R}^d} \mathbf{v} f^{(1)}\, d\mathbf{v}\Big) = \partial_t\rho - \nabla_{\mathbf{x}}\cdot\big(\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho\big) = 0.$$

Therefore, in the $\epsilon \to 0$ limit regime we expect the macroscopic density to be well approximated by the solution to the equation

$$\partial_t\rho = \nabla_{\mathbf{x}}\cdot\big((\nabla_{\mathbf{x}} U\star\rho)\rho\big) + \Delta_{\mathbf{x}}\rho. \tag{4.6}$$

4.2. Strong-interaction/Strong-noise regime. We choose the following relation between the dimensionless parameters: $\eta \simeq \gamma_1 \simeq \gamma_2 \simeq 1$ and $\chi \simeq \epsilon^{-1}$. In this regime, the dominant mechanisms are the noise, the linear Stokes friction and the interaction term. We use again the Hilbert expansion

$$f_\epsilon = f^{(0)} + \epsilon f^{(1)} + \dots \qquad \text{and} \qquad \rho_\epsilon = \rho + \epsilon\rho^{(1)} + \dots$$

into (4.4) and identifying terms with equal power of ϵ, we get:

- ϵ^{-1} terms:

$$Lf^{(0)} = -\mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f^{(0)}\right]$$

which implies that $f^{(0)}(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\ M(\mathbf{v} - (\nabla_{\mathbf{x}} U \star \rho))$.

- ϵ^{0} terms:

$$L(f^{(1)}) + \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f^{(1)}\right] = \partial_t f^{(0)} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(0)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(0)}]$$

and integration with respect to $\mathbf{v}$ yields the mass conservation

$$\partial_t \rho = \nabla_{\mathbf{x}} \cdot \big((\nabla_{\mathbf{x}} U \star \rho)\rho\big). \tag{4.7}$$

Let us point out again that both equations (4.6) and (4.7) were proposed in Refs. [29, 30] as continuum models for swarming and are here recovered through our kinetic theory.

5. Conclusions. We have developed a full kinetic theory aimed at describing self-propelling swarming systems driven by general pairwise interactions. Our analysis leads to a new class of macroscopic solutions to swarming systems consisting of double, superimposed rotating mills. These are indeed observed in discrete simulations but they cannot be identified from the corresponding hydrodynamic equations since the inherent dual velocity distributions yield a macroscopic average of zero. We find the conditions under which double mills can coexist. Due to the non linearity of the problem these solutions are not trivial since in general the superposition of two existing solutions does not necessarily satisfy the governing equations of motion. We have also extended the kinetic theory to the interacting particle system with random noise effects. The resulting kinetic Fokker-Planck equation lead, under suitable scalings, to macroscopic equations for the density of organisms already proposed in the literature. We have demonstrated that the kinetic theory approach lead to an unified hierarchy of swarming models bridging the particle description to all the hydrodynamic and continuum descriptions available in the literature. Future development includes a full numerical solution of the kinetic equations, both in one and two dimensions.

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REFERENCES

[1] A. Barbaro, K. Taylor, P. F. Trethewey, L. Youseff and B. Birnir, *Discrete and continuous models of the dynamics of pelagic fish: application to the capelin*, preprint.
[2] B. Birnir, *An ODE model of the motion of pelagic fish*, J. Stat. Phys., 128 (2007), pp. 535–568.
[3] W. Braun and K. Hepp, *The Vlasov Dynamics and Its Fluctuations in the 1/N Limit of Interacting Classical Particles*, Commun. Math. Phys., 56 (1977), pp. 101113.
[4] M. Burger, V. Capasso and D. Morale, *On an aggregation model with long and short range interactions*, Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal, 8 (2007), pp. 939–958.

[5] S. Camazine, J.-L. Deneubourg, N.R. Franks, J. Sneyd, G. Theraulaz and E. Bonabeau, *Self-Organization in Biological Systems*, Princeton University Press, 2003.
[6] C. Cercignani, R. Illner, M. Pulvirenti, *The mathematical theory of dilute gases*, Springer series in Applied Mathematical Sciences 106, Springer-Verlag, 1994.
[7] Y.-L. Chuang, Y. R. Huang, M. R. D'Orsogna and A. L. Bertozzi, *Multi-vehicle flocking: scalability of cooperative control algorithms using pairwise potentials*, IEEE International Conference on Robotics and Automation, 2007, pp. 2292-2299.
[8] Y.-L. Chuang, M. R. D'Orsogna, D. Marthaler, A. L. Bertozzi and L. Chayes, *State transitions and the continuum limit for a 2D interacting, self-propelled particle system*, Physica D, 232 (2007), pp. 33-47.
[9] I.D. Couzin, J. Krause, N.R. Franks and S.A. Levin, *Effective leadership and decision making in animal groups on the move*, Nature, 433 (2005), pp. 513-516.
[10] F. Cucker and S. Smale, *On the mathematics of emergence*, Japan. J. Math., 2 (2007), pp. 197-227.
[11] F. Cucker and S. Smale, *Emergent behavior in flocks* IEEE Trans. Automat. Control, 52 (2007), pp. 852-862.
[12] P. Degond and S. Motsch, *Macroscopic limit of self-driven particles with orientation interaction*, preprint.
[13] P. Degond and S. Motsch, *Large-scale dynamics of the Persistent Turing Walker model of fish behavior*, preprint.
[14] R. Dobrushin, *Vlasov equations*, Funct. Anal. Appl., 13 (1979), pp. 115123.
[15] M.R.D'Orsogna, Y.-L. Chuang, A. L. Bertozzi, L. Chayes, *Self-propelled particles with soft-core interactions: patterns, stability, and collapse*, Phys. Rev. Lett., 96 (2006), pp. 104302-1/4.
[16] F. Golse, *The Mean-Field Limit for the Dynamics of Large Particle Systems*, Journées équations aux dérivées partielles, **9** (2003), pp. 1-47.
[17] G. Gregoire, H. Chate, *Onset of collective and cohesive motion*, Phy. Rev. Lett., 92 (2004), pp. 025702-1/4.
[18] S.-Y. Ha and E. Tadmor, *From particle to kinetic and hydrodynamic descriptions of flocking*, 1 (2008), pp. 415-435.
[19] A.L. Koch, D. White, *The social lifestyle of myxobacteria*, Bioessays 20 (1998), pp. 10301038.
[20] H. Levine and W.-J. Rappel, *Self-organization in systems of self-propelled particles*, Phys. Rev. E, 63 (2000), pp. 017101-1/4.
[21] A. Mogilner, L. Edelstein-Keshet, L. Bent and A. Spiros, *Mutual interactions, potentials, and individual distance in a social aggregation*, J. Math. Biol., 47 (2003), pp. 353-389.
[22] H. Neunzert, *The Vlasov equation as a limit of Hamiltonian classical mechanical systems of interacting particles*, Trans. Fluid Dynamics, 18 (1977), pp. 663678.
[23] J. Parrish, L. Edelstein-Keshet, *Complexity, pattern, and evolutionary trade-offs in animal aggregation*, Science, 294 (1999), pp. 99101.
[24] *Topics in kinetic theory*, T. Passot, C. Sulem and P.L. Sulem editors, Fields Institute Communications 46, American Mathematical Society, 2005.
[25] G. Russo and P. Smereka, *Kinetic theory for bubbly flow. I. Collisionless case*, SIAM J. Appl. Math., 56 (1996), pp. 327-357.
[26] T.C. Schneirla, *Army Ants: A Study in Social Organization*, W.H. Freeman, 1971.
[27] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Modern Phys., 52 (1980), pp. 569–615.
[28] J. Toner, Y. Tu, *Long-range order in a two-dimensional dynamical xy model: How birds fly together*, Phys. Rev. Lett., 75 (1995), pp. 43264329.
[29] C.M. Topaz and A.L. Bertozzi, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math., 65 (2004), pp. 152–174.
[30] C.M. Topaz, A.L. Bertozzi, and M.A. Lewis, *A nonlocal continuum model for biological aggregation*, Bulletin of Mathematical Biology, 68 (2006), pp. 1601-1623.
[31] T. Vicsek, A. Czirk, E. Ben-Jacob, I. Cohen, O. Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett., 75 (1995), pp. 12261229.