Pursuit on an Organized Crime Network

Charles Z. Marshak
UCLA Department of Mathematics

M. Puck Rombach, Andrea L. Bertozzi
UCLA Department of Mathematics and Institute of Pure and Applied Mathematics

Maria R. D'Orsogna*
UCLA Department of Biomathematics and CSUN Department of Mathematics
(Dated: April 28, 2015)

We model the hierarchal evolution of an organized criminal network via antagonistic *recruitment* and *pursuit* processes. Within the recruitment phase, a criminal kingpin enlists new members into the network, who in turn seek out other affiliates. New recruits are linked to established criminals according to a probability distribution that depends on the current network structure. At the same time, law enforcement agents attempt to dismantle the growing organization using ad-hoc pursuit strategies that initiate on the lower level nodes and that unfold as self-avoiding random walks. The global details of the organization are unknown to law enforcement, who must explore the hierarchy node by node. We halt the pursuit at a pre-set stopping time, encoding if and when an arrest should be made; the criminal network is assumed to be eradicated if the kingpin is arrested. We first analyze recruitment and study the large scale properties of the growing network; later we add pursuit and use numerical simulations to study the eradication probability in the case of three strategic stopping times, the time to first eradication and related costs. We find that eradication becomes increasingly costly as the network increases in size so the optimal way of arresting the kingpin is to intervene at the early stages of network formation. We discuss our results in the context of dark network disruption and their implications on possible law enforcement strategies.

PACS numbers: 89.65.Ef, 89.75.Hc, 87.23.Ge

I. BACKGROUND

In recent years, scientists from many different fields have begun tackling criminal socio-economic systems using tools from statistical mechanics, complex networks, partial differential equations and game theory [1–3]. This interdisciplinary effort has helped shed light on the formation of crime hotspots [4–6], the dynamics of criminal behavior [7, 8], the mechanics of gang rivalries [9] and recidivism trends [10]. We bring some of these tools to the problem of organized crime. Underworld syndicates profit from exploitation, theft, intimidation, and murder, with estimates of 5% of the global GDP stemming from money laundering associated to criminal enterprises [11]. Be it the terrorist attacks led by radical Islamic groups, the human trafficking overseen by the Russian mafia, or the drug smuggling directed by South American cartels, criminal networks benefit at the expense of law-abiding people. Governments must find effective intervention strategies to protect their citizens, maintain internal order, and safeguard the channels of the legal economy. In this work, we focus on the dynamics of organized crime networks, modeling their formation and analyzing the effects of possible network disruption strategies used by law enforcement.

Since organized criminal associations mostly operate in sophisticated and secretive ways, the most pertinent way of studying them is through the well-known framework of dark networks [12, 13]. Here, actors and links are predominantly hidden to possible disruptors. Criminals must decide who to interact with and how to balance the threat of possibly being arrested with the profits afforded by criminal collaboration [12]. On the other hand, police agents must eradicate the criminal enterprise without full knowledge of the global network. McBride *et al.* modeled the formation and disruption of dark networks where the limited information available to law enforcement agents is embodied by a probability distribution that governs dismantling attempts [12, 14, 15]. In other cases, law enforcement may have access to the full layout of the criminal organization, in contrast to the above dark network scenario. Several models have been presented to study network eradication when the entire structure is known [16, 17]. Typically, each criminal is assigned a utility value based on his or her prominence in the organization, which is then used by law enforcement to orchestrate optimal intervention tactics. The work of Duijn *et al.* tests several disruption strategies on a well-documented drug trafficking network in the Netherlands [13]. Here, disruption is modeled as a process of node removal from a dynamic network, where some of the disruption strategies assume full knowledge of the network. New players are not introduced, but once a node has been eliminated, the

* dorsogna@csun.edu

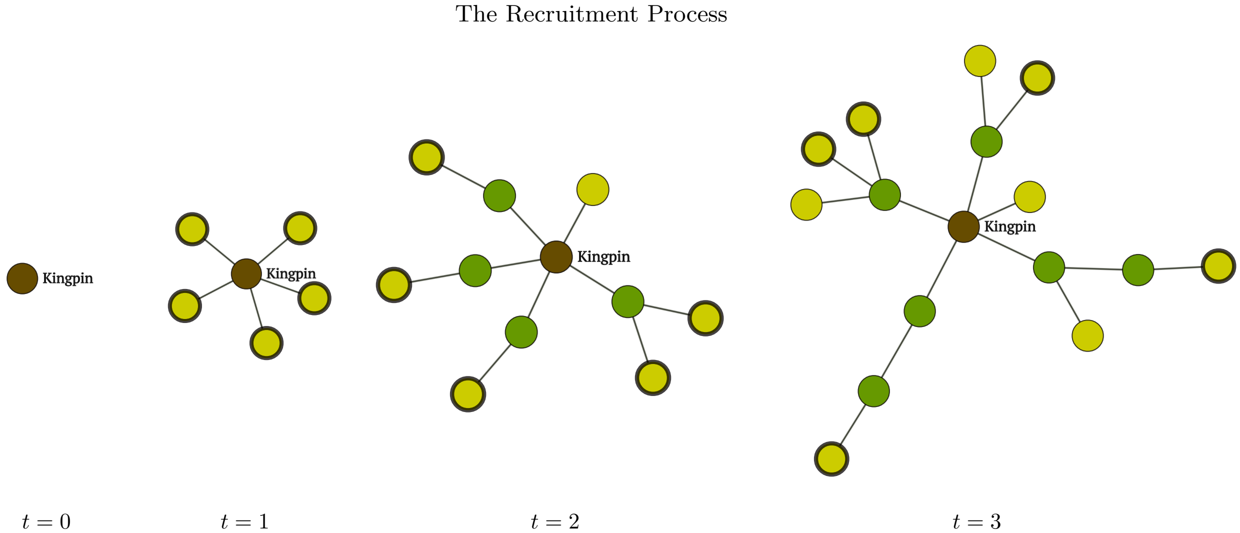

Figure 1. A simulated recruitment process for $t = 0, 1, 2, 3$. The network starts with a single criminal, the kingpin, and then evolves according to the preferential attachment mechanism described in the text. Here the number of new criminals introduced into the network is given by the recruitment index $k = 5$. The lighter color nodes represent criminals without underlings, and will be referred to as street criminals. Of these, the nodes with the darker boundary are those freshly recruited at a given time step. For example, the number of street criminals when $t = 3$ is 9, of which five are new recruits.

network is allowed to "recover" and links among nodes may readjust. The authors found that interventions are most effective at the very early stages of the disruptive process, since in the long run perturbations and reorganizations lead to a more robust and resilient network.

We study disruption strategies on a hierarchical, growing, dark criminal network. Within the context of this paper, a hierarchal criminal network is defined to be one where every criminal has precisely one link to a more senior member, except for the organization's head, whom we refer to as the kingpin. The hierarchical structure will inform the disruption procedure significantly and is not a feature of the above described work. Concurrent to law enforcement eradication attempts, we also include a mechanism for criminal recruitment which is also not included in the work of McBride *et al.* and Dujin *et al.* As we shall see, the hierarchal structure and the interplay between the two antagonistic trends – recruitment and disruption – lead to interesting dynamics and implications on the optimal strategies to be used in attempting to eradicate the criminal organization.

We model the professional connections between criminals, not their social ties, leading to the choice of a hierarchical network. Criminal investigations frequently record social links among suspects, making it difficult to distill professional connections from data. Hence, it is not surprising that there is an ongoing debate as to what extent criminal networks do in fact possess underlying hierarchal structures [18]. However, while not always obvious from the available data, a well-defined professional hierarchy is important for large, coordinated criminal operations [18–20]. For example, the Cosa Nostra mafia network and the Hells Angels biker gang network, manifest clear hierarchies beneath their stratified social structure [21–23]. On the other hand, the September 11th terrorist network, the Dutch drug trafficking network and various American gang networks are examples of networks whose structure benefit from decentralization [13, 19, 20, 24]. Ultimately, a professional network structure is more transparent for mathematical analysis.

We provide a mechanism for criminal recruitment using a variation of preferential attachment models that are now implemented in a variety of contexts [25–32]. Originally these mechanisms were proposed to study the topology of the internet [25, 26]. The central assumption, roughly speaking, is that the "rich get richer", meaning that webpages with many existing links are more likely to be connected to newly introduced ones. Though the motivation of our model stems from organized crime, we used these preferential attachment models to guide the design and analysis of our recruitment mechanism. In particular, we use the notion of "social distance" attachment proposed by Boguñá *et al.* [27, 33] where each node of a static network is associated to a set of socially relevant features such as profession, religion or location that lead to an ad-hoc metric quantifying the "social" distance between nodes. The probability that a link is created is inversely proportional to this distance. We adapt the notion of social distance to a dynamically evolving criminal network.

Finally, our arrest mechanism draws on probabilistic node and link removal processes that have been developed in recent years. In particular, Bollobás and Roirden addressed robustness and vulnerability issues in graphs upon the uniform removal of random links. The authors determined under which conditions node removal could lead to the largest connected component in the network being dismantled with high probability [34]. On the other hand, node removal cascades arise when the elimination of a single node triggers the removal of others. Since this type of behavior arises in communication networks when the failure of a tower forces the network to compensate, node removal cascades are often used to model wireless networks and power grids [35, 36]. They have also been widely adapted to other disciplines and are used to model contagion [37], neuronal [38–40], and terrorist networks [41]. While all the applications described above are not necessarily related to organized crime, we will draw upon these many different perspectives to best model criminal recruitment and police disruption on the dark criminal network we describe below.

II. OVERVIEW

In this section, we present an overview of our dynamical, hierarchal criminal network stemming from a so-called kingpin. Our model alternates between two key processes: the recruitment of criminals to the network and the concurrent, antagonistic pursuit and disruption by law enforcement.

To recruit new criminals into the network, we use a preferential attachment mechanism, a schematic of which is shown in Fig. 1. Criminal nodes are added to the network at a constant rate, each forming a link to a more senior member. We assume the entire network structure to be initially hidden to law enforcement except for the visible "street-level" criminals at the end of the network that don't have any further underlings. These street criminals are the ones that are directly involved in thefts, burglaries or drug dealings while more nested members of the hierarchy are assumed to act more like masterminds: the higher up a criminal is in the network, the less likely he or she is to overtly engage in criminal enterprises. As a consequence, any police intervention must begin from street criminals and later progress to other linked nodes, higher up in the hierarchy, so that the network structure becomes gradually visible to law enforcement only in a node by node fashion.

The distance between a given criminal node on the network and visible street-level activity is defined as the smallest number of connections separating the given node from any street criminal. The closer a criminal is to visible street-level activity, the more vulnerable he or she is to detection and arrest. As such, senior criminals will seek to maintain a buffer between themselves and street criminals. Criminals without any underlings will instead aggressively recruit new members in order to avoid exposure to law enforcement. At the same time they have more access to potential recruits due to the criminal activities they are engaged in. Combining these two heuristics, we assume that within the recruitment process prospective criminals are most likely to establish a link with street criminals, rather than with more nested members of the criminal hierarchy. This description of recruitment is echoed in Ref. [42], and a precise mathematical formulation will be given in the next section.

This model	Network theory
Kingpin	Root
Underlings of criminal j	Children of node j
Criminal network	Rooted directed tree
$\mathcal{C}(t)$: Criminals (including kingpin) at time t	Vertices or nodes (including root) at time t
$\mathcal{S}(t)$: Set of criminals without underlings	Set of leaves at time t, i.e. nodes of out degree 0

Table I. A table comparing the terminology used in this paper and that of standard network theory.

The goal of law enforcement is to capture the kingpin and eradicate the network. As mentioned above, law enforcement can begin pursuing the kingpin only from end nodes with no underlings. Given a street criminal, an officer will begin an investigation by performing a self-avoiding random walk on the network, "interrogating" criminals as they go. At any point during the pursuit, the officer can decide to "arrest" the criminal under investigation and all associated underlings. By removing criminals from the network, the kingpin becomes more vulnerable to future capture. However, since the overall structure of the network is unknown to law enforcement, the random walk may lead to a dead-end: in moving from node to node, the officer may reach a new street criminal without underlings, with no further investigation possible. In this case, the pursuit is terminated and deemed unsuccessful. If instead the kingpin is reached and arrested, the criminal network is assumed to be eradicated. Before illustrating our recruitment and arrest model more in detail, in Table I we list standard network terminology and the corresponding nomenclature used here, for context. For example, in standard network terminology, our network is a tree, the street criminals are the leaves and the kingpin is the root.

III. RECRUITMENT

We now focus on the mathematical aspects of the recruitment process. We start with an initial criminal network formed solely by the kingpin. The network evolves recursively so that at time t it contains a set $\mathcal{C}(t)$ of criminals, including the kingpin. Of these, the subset $\mathcal{S}(t)$ denotes street criminals, those without any underlings. We also introduce the metric $\text{dist}(j;t)$ to denote the distance separating criminal j from street activity and defined as the minimum number of links between criminal j and any other street criminal in the hierarchy beneath it. At every time step increment, from t to $t+1$, we add k new recruits to the network according to a preferential attachment mechanism. Every node j is assigned a weight

Parameter	Description
t	Time
k	Recruitment index – number of new criminals added to the network at each time step
a	Node weight parameter in $w(j;t)$.
n	Number of criminals on network at time t, $n = kt + 1$

Table II. The parameters of the recruitment mechanism.

$w(j;t)$ to quantify the relative likelihood that it will link a new criminal underling. Since, as discussed above, a plausible assumption is that street criminals are the most likely to recruit new criminals into the organization, we let $w(j;t)$ be inversely proportional to the distance between j and $\mathcal{S}(t)$ so that

$$w(j;t) = \frac{1}{\text{dist}(j;t) + a}, \tag{1}$$

where a is a parameter, which we set to $a = 1$ for simplicity. With these choices, $w(j;t)$ embodies the proximity of criminal j to visible street-level activity on a scale from zero to one, with one being the closest possible. If criminal j is a street criminal, then $\text{dist}(j;t) = 0$ and $w(j;t) = 1/a = 1$, the maximum possible value. On the other hand, as the network keeps growing, higher level nodes can become progressively detached from street activity, so that in principle $\text{dist}(j;t) \to \infty$ and $w(j;t) \to 0$. Note that the choice $a \to \infty$ leads to a uniform weight $w(j;t)$ for all criminals. In this case, the recruitment process can be described as the growth of a recursive tree [30, 31]. On the other extreme, the choice $a \to 0$ would restrict the recruitment process exclusively to street criminals, barring higher ranking criminals from adding subordinates. Our decision to use a finite, non zero value $a = 1$ ensures that recruitment is not exclusive to $\mathcal{S}(t)$.

After evaluating $w(j;t)$ for all existing nodes, we iteratively introduce k new criminals to the network. We add them one by one to nodes that are selected according to the relative weights $w(j;t)$. Note that each existing criminal can add multiple underlings within a single time step, since the recruitment of one new member does not exclude the possibility of a different new member being recruited by same criminal. We call k the recruitment index. In Fig. 2 we depict a particular network configuration, including the explicit weights $w(j;t)$ assigned to each criminal j.

A. Out Degree Distribution

We can now investigate the statistics related to our recruitment model. First, we explore how the total number of underlings a criminal has varies throughout the network, *i.e.* we analyze the *out degree probability distribution.* By out degree we indicate the number of nodes

The Criminal Network and $w(j;t)$

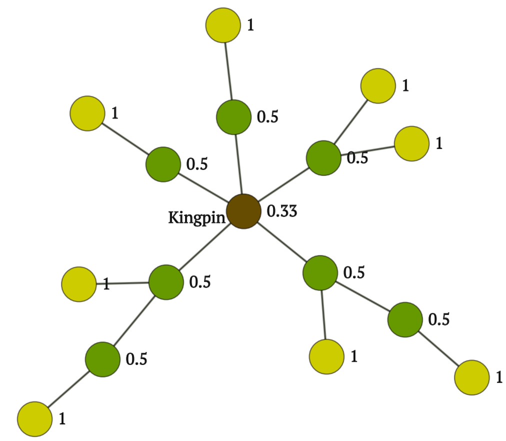

Figure 2. The criminal network at $t = 3$ with the values of $w(j;t)$ explicitly shown. Here, the recruitment index $k = 5$ and the initial configuration was that of a single kingpin. All criminals j within $\mathcal{S}(t)$ have weight $w(j;t) = 1$ since on these nodes $\text{dist}(j;t) = 0$.

in the hierarchy directly beneath a criminal, excluding higher nodes from the enumeration. For example, in Fig. 2 the out degree of the kingpin is 5 criminals. Of these five, the upper left two have out degree one and the other three have out degree two. We do not impose any limitations on the number of underlings connected to any given node, either directly or indirectly. In principle thus, a node can have an infinitely large number of subordinates. However, due to the choices made in modeling the attachment weights $w(j;t)$ we expect that as the organization grows and more senior criminals become more entrenched within the network, their likelihood of adding new recruits decreases in favor of criminals that are closer to street activity. As a result, we expect our organized crime network to grow several hierarchal levels and to have a few key players, such as the kingpin, who directly oversee a relatively large number of criminals while the rest have at most one or two underlings that "report" to them. We introduce the time dependent out degree probability distribution $P(d;t)$ for a randomly selected node to have d direct underlings at time t. At the onset of network growth, when the only criminal present is the kingpin, $P(0;0) = 1$. As the number of added nodes $n = kt + 1$ increases, we conjecture that for large enough d, $P(d; t \to \infty)$ can be approximated via an exponential form

$$P_\infty(d) \equiv P(d; t \to \infty) = c_1 e^{-c_2 d}, \tag{2}$$

for constants c_1, c_2. Following the above discussion on the nature of $w(j;t)$, we expect that as $t \to \infty$ most of the new underlings will be connected to existing street criminals, resulting in $P_\infty(d = 0) \simeq P_\infty(d = 1)$. Using this approximation and Eq. 2 for $d \geq 1$, we expect $c_1, c_2, P_\infty(d = 0)$ to be related by

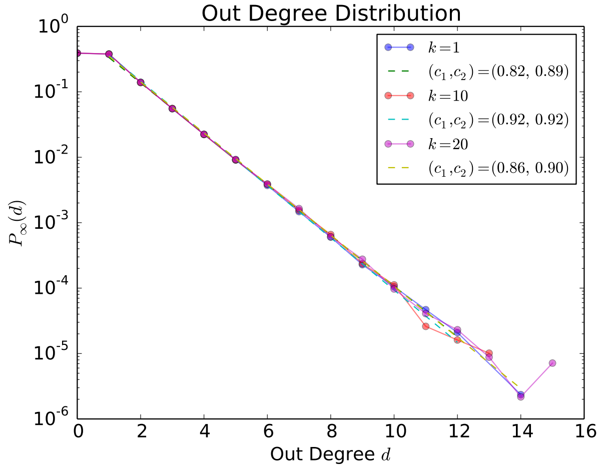

Figure 3. The out degree distribution $P_\infty(d)$ of nodes on a criminal network determined from numerical simulations for $t \to \infty$. The three curves correspond to recruitment indices $k = 1, 10, 20$. Simulations were terminated when the total number of criminals exceeded 5×10^3. All curves for $P_\infty(d)$ are averaged over 100 runs. The tail of the degree distribution is noisy due high degree nodes occurring sporadically. Running the simulations for longer times will extend the range of the domain in d, but not the form of $P_\infty(d)$. We conjecture the degree distribution to follow an exponential law independent of k, and fit it to a decaying exponential distribution as discussed in the text. The value for $P_\infty(d = 0) = 0.39$ for all values of k.

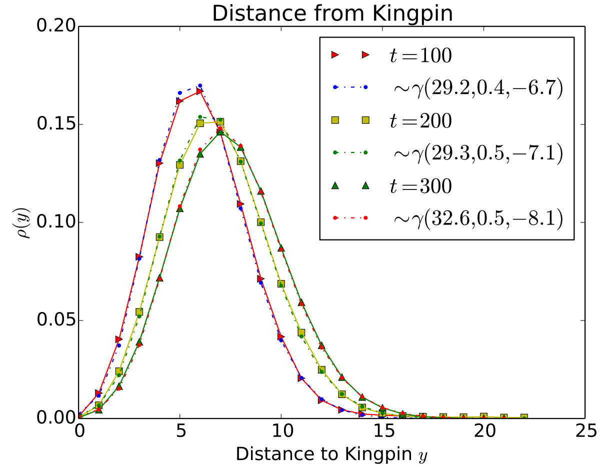

Figure 4. The distribution of criminal position relative to the kingpin after 100 runs. The three curves represent the different times t at which the recruitment was stopped. Here, $t = 100, 200, 300$. The recruitment index was set at $k = 5$ and the initial configuration of the network was that of the single kingpin. The probabilities were fit using the shifted gamma density $\rho_{\alpha,\beta,s}(y)$ found in Eq. 5 and with parameters $\gamma(\alpha, \beta, s)$ specified in the legend. Similar shaped curves arise for larger values of k.

$$c_1 \simeq [1 - P_\infty(d = 0)](e^{c_2} - 1). \quad (3)$$

In Fig. 3, we grew the network to 5×10^3 criminals and found Eq. 2 to accurately describe the out degree distribution for $d \geq 1$, with Eq. 3 being accurate to first approximation. We also varied k between 1 and 20 and did not notice substantial variations in fitted parameter values. Moreover, we considered smaller sized networks (not shown here) with 500 criminals and found that Eq. 2 still described the data well. These results suggest that at large times, network structure is independent of its rate of growth. In particular, as $t \to \infty$ our results show that the probability that any given node has no underlings is given by the k-independent, universal value $P_\infty(d = 0) = 0.39$. The average out-degree d is given by

$$\langle d \rangle = c_1 \sum_{d=1}^{\infty} d e^{-c_2 d} = \frac{1 - P_\infty(d = 0)}{1 - e^{-c_2}}. \quad (4)$$

B. Criminal Density and Position

We now investigate how criminals are positioned relative to the kingpin, at the core of the network. We expect the distribution of criminals relative to the kingpin to become more uniform as the network grows in scale. In Fig. 4 the recruitment process is stopped at a fixed time t, when we measure $\rho(y)$, the ratio of criminals at a distance y from the kingpin with respect to the total number of nodes. Our measured $\rho(y)$ is then fitted to a shifted gamma probability density, the continuous analog of negative binomial distribution [43] and given by

$$\rho_{\alpha,\beta,s}(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (y - s)^{\alpha - 1} e^{-\beta(y-s)} \quad (5)$$

for $y > s$ and $\alpha > 1$. Our choice was motivated by the fact that this distribution is supported only on a portion of the horizontal axis. Using the fitted values of $\rho_{\alpha,\beta,s}(y)$ the probability that a criminal is a distance y from the kingpin can be estimated as

$$\int_s^y \rho_{\alpha,\beta,s}(y')\, dy'. \quad (6)$$

From Fig. 4 we note that as t increases, the average distance from the kingpin increases and that the distribution of criminals becomes broader, as can be expected. We also used different initial conditions, starting the recruitment process on given, already established networks and found that, at long times, the shifted gamma probability density remained a valid approximation for $\rho(y)$.

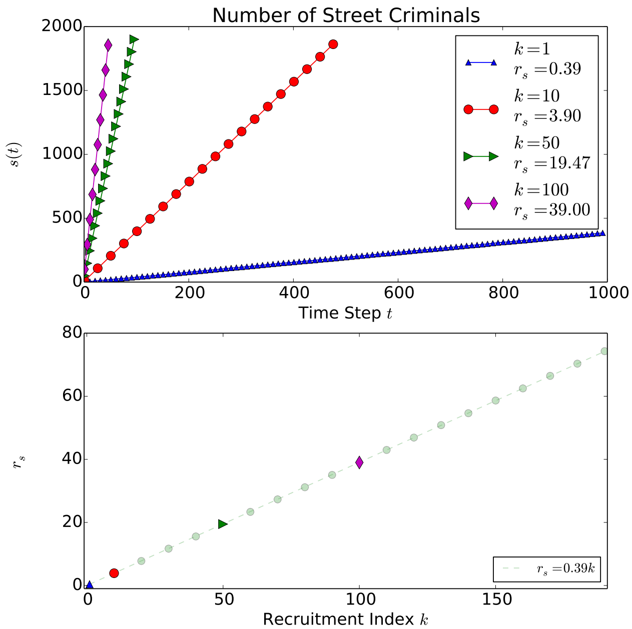

Figure 5. (Top) Number of street criminals $s(t)$ as a function of time for recruitment rates $k = 10, 50, 100$. In each case, the recruitment process was terminated when the total number of criminals exceeded 5×10^3 and averaged over 100 runs. We fit the data to $s(t) = r_s t + 1$ and expect $r_s \simeq P_\infty(d = 0)k$ with the universal factor $P_\infty(d = 0) = 0.39$. This scaling is confirmed by the fitted values of r_s as can be seen from the values shown in the legend. (Bottom) The slope values r_s as a function of k with the r_s values in the top display shown in dark symbols. This data is then linearly fit as shown in the bottom legend, confirming our conjecture $r_s \simeq P_\infty(d = 0)k$.

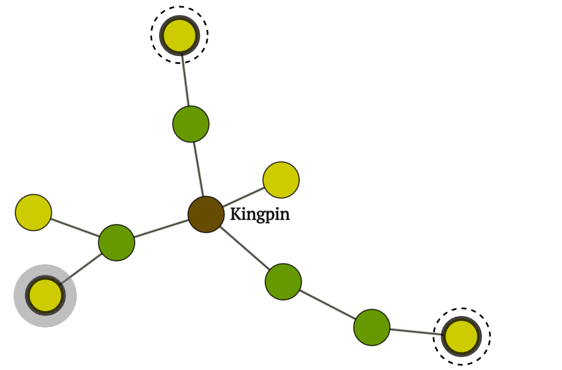

Figure 6. Schematic of the addition of new nodes from time t to time $t + 1$. Here, $k = 5$ and the new nodes are depicted in light color. The addition of a street criminal to a more senior one, represented by the node surrounded by a solid ring on the lower left hand side, will increase $s(t+1)$ by one unit with respect to $s(t)$. Vice-versa, the addition of a new node to an existing street criminal, as shown by the nodes surrounded by the dashed circles, will not change the number of street criminals, since for every new member of $\mathcal{S}(t + 1)$, one from $\mathcal{S}(t)$ will be removed.

C. Street Criminals

In our model, street criminals are the nodes in $\mathcal{S}(t)$ without any underlings. We assume theirs is the only activity to be visible to law enforcement making street criminals the most vulnerable to arrest. At the same time, due to the choices made for $w(j; t)$, street criminals also have the highest probability of recruiting new members into the organization. Since the network grows linearly in time, we expect the total number of street criminals $s(t)$ to increase accordingly, and that the proportion of street criminals with respect to the total number of nodes will remain fixed at the universal value $P_\infty(d = 0) = 0.39$ as shown in Fig. 3. In Fig. 5 we plot $s(t)$ and fit the data to a linear form $s(t) = r_s t + 1$, where the unitary intercept is chosen since at $t = 0$ the only criminal present is the kingpin, who does not have any underlings yet. We expect $r_s \simeq P_\infty(d = 0)k$ as verified in the lower panel of Fig. 5.

We can also write an iterative equation for $s(t + 1)$ [44, 45]. At a given time t the likelihood of adding a new street criminal to the network is given by the probability of adding a new node to a senior criminal, one that already has underlings. Conversely, the net number of street criminals will not change upon adding a new node to existing street criminals, since for every new street criminal added to the enumeration, there will be one that will be removed, having added a new underling. This is illustrated in Fig. 6. The weight associated to adding a new node to a senior criminal is given by

$$\sum_{j \in [\mathcal{C}(t) - \mathcal{S}(t)]} w(j; t) = \sum_{j \in \mathcal{C}(t)} w(j; t) - \frac{s(t)}{a} \quad (7)$$

since the number of street criminals is $s(t)$ and their weight is given by $1/a$. The total probability of adding a criminal to a senior node is thus given by Eq. 7, normalized with respect to the total weight $\sum_{j \in \mathcal{C}(t)} w(j; t)$. We can now write our iterative equation for $s(t + 1)$. The number of added new street criminals is given by the probability of adding a street criminal to a senior one as described above, multiplied by the total number of available new criminals at each time step, given by k. Using $a = 1$ we find

$$s(t + 1) \simeq s(t) + \frac{\sum_{j \in \mathcal{C}(t)} w(j; t) - s(t)}{\sum_{j \in \mathcal{C}(t)} w(j; t)} k. \quad (8)$$

We can use this relationship to heuristically determine the weight of the entire tree $\sum_{j \in \mathcal{C}(t)} w(j; t)$ at long times. The latter can be assumed to scale linearly, since the total number of members of the network grows at the same rate, and at least a fraction of nodes $P_\infty(d = 0)$ will be associated to a finite, unitary weight. We thus posit

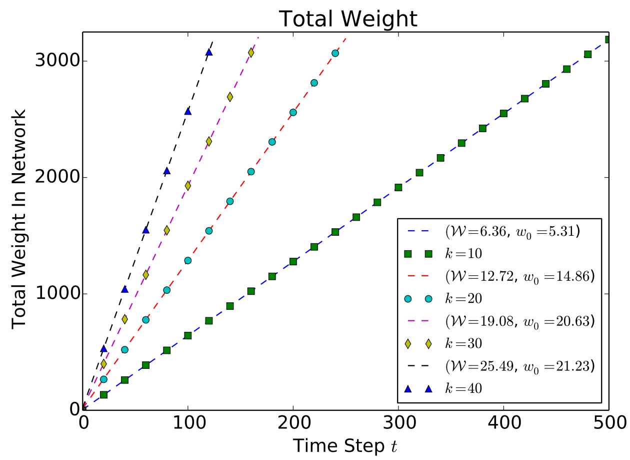

Figure 7. The total weight of the network $\sum_{j\in\mathcal{C}(t)} w(j;t)$ as a function of time averaged over 100 runs for $k = 10, 20, 30, 40$. At the onset the network consisted of the kingpin alone; simulations were stopped when the total number of criminals exceeded 5×10^3. We fit the data to a linear form $\mathcal{W}t + w_0$ and find that the extrapolated values of $\mathcal{W}$ shown in the figure legend are in excellent agreement with the ones predicted from Eq. 10, given by $\mathcal{W} = 6.39, 12.79, 19.18, 25.57$ for $k = 10, 20, 30, 40$, respectively.

$\sum_{j\in\mathcal{C}(t)} w(j;t) \simeq \mathcal{W}t$ as $t \to \infty$. We also use $s(t) \simeq r_s t$ as $t \to \infty$ so the recursion relation Eq. 8 at long times becomes

$$r_s \simeq \frac{\mathcal{W} - r_s}{\mathcal{W}} k, \tag{9}$$

yielding

$$\mathcal{W} \simeq \frac{k r_s}{k - r_s} \simeq \frac{k P_\infty(d=0)}{1 - P_\infty(d=0)}, \tag{10}$$

and where the last term is obtained via $r_s \simeq k P_\infty(d = 0)$. In Fig. 7 we plot the total weight of the network as a function of time for $k = 10, 20, 30, 40$ and find that it scales linearly, as we had assumed. We also find that the corresponding numerical fits yield good agreement with the estimates from Eq. 10.

IV. POLICE PURSUIT

In this section we introduce police agents to our model and describe the pursuit mechanisms they are engaged in on a network that is concurrently growing in time. The ultimate goal of law enforcement is to reach the kingpin, capture him or her, and dismantle the expanding criminal organization. The ultimate goal of the criminal enterprise is to expand as much as possible. As discussed earlier, it is reasonable to assume that the global structure of the network is unknown to law enforcement agents who can begin their "investigative" activities only at the bottom of the network, populated by street criminals. Once a criminal network is formed, at every time step t we choose a random street criminal in $\mathcal{S}(t)$ as the initial search node for the officer. This street criminal is now considered to be under investigation. The officer can decide whether to "arrest" the criminal in question or move to one of its associates, chosen among the nodes that are linked to the current suspect. In the next subsection we will illustrate three different ways of making this strategic decision. For now, we note that if the choice to arrest the current suspect is made, the latter is removed and this particular investigative round at time t is complete. If the choice to migrate to a linked criminal is made, the pursuit continues: the officer is now faced with the same decision on whether to arrest or continue investigating. We impose that a node that has already been visited by law enforcement cannot be visited again. A sequence of investigative choices thus lead to a self-avoiding random walk on the criminal network at time t. The pursuit ends if an arrest is made or if the officer reaches either the kingpin or another street criminal.

If the kingpin is reached, we consider the attempts of law enforcement to be successful: the criminal organization has been dismantled at time t and the process is terminated. Vice-versa, if the investigative unit ends on another criminal in $\mathcal{S}(t)$, we consider law enforcement intervention at this time to have failed: no arrests will be made, and the network stays unchanged. Finally, if law enforcement decides to perform an arrest on a given node, all of its underlings in the hierarchal structure will be removed as well. Note that since the node at which the arrest takes place can be a few links removed from the original starting point, an arrest does not necessarily imply that the first street criminal to be investigated will be removed as well. The case of an arrest may be considered a partial success, since eliminating a few nodes on the hierarchical structure may make the kingpin more vulnerable in future time steps. After the pursuit phase at time t is completed and assuming the kingpin has not been reached, the criminal network is grown according to the recruitment methods described in the previous section. Pursuit and recruitment are then iterated at time $t + 1$ and until the network reaches a given size n^*, with the network evolving dynamically in time. Note that since at the onset of our simulation the only criminal present is the kingpin, we do not introduce the police pursuit at time $t = 0$, since the kingpin would be arrested immediately and the criminal organization would not grow. Rather we consider an existing network, as shown in Fig. 8 for a full ternary tree of height three with forty criminals, as the initial configuration on which the pursuit is started. A full ternary network is one where all nodes, except street criminals, have exactly three underlings. A complete ternary network is one where all nodes, except street criminals and the nodes immediately above them, have exactly three underlings. We use these full or complete networks as initial conditions for an "unthreatened" criminal organization, prior to police intervention, and assume that once law enforcement investigations have begun the network grows according to the recruitment rules described in the previous section.

A. Pursuit Strategies

We now discuss possible strategies that law enforcement may employ in deciding whether to arrest a criminal or continue investigating other, neighboring nodes. As already discussed, the police is assumed not to have full knowledge of the entire organization but only of the criminals under investigation and their linked nodes. Because this is a dark network, the pursuit process may be unsuccessful with law enforcement reaching street criminals that are low on the criminal hierarchy, representing a dead-end.

The first possible pursuit method we consider is the fixed investigation number strategy, where starting from a given street criminal, law enforcement officials will investigate p successive nodes before making an arrest, assuming they have not reached a new street criminal along their self-avoiding random walk. This strategy will be denoted by $Q_A(p)$. The second method is the minimum out degree strategy, where law enforcement officials will keep investigating until a node of at least out degree q is reached, similarly assuming the self-avoiding random walk does not lead to other street criminals. In this case, it is assumed that police agents are seeking to maximize the influence of the criminal to be arrested, since the higher q is, the more direct underlings the suspect will be affiliated to. We denote this strategy by $Q_D(q)$. Finally, within the infinite investigation strategy, the pursuit is stopped only upon reaching the kingpin or a dead-end street-criminal. This last strategy will be denoted by Q_I. While there are many more possible disruption strategies, we view these as the building blocks of possible pursuit methods. Note that none of the above strategies require full knowledge of the network layout, since the out degree is a local measurement. In Fig. 8 we show law enforcement officials pursuing strategy $Q_A(q = 3)$ on an initial full ternary tree of height three.

Each of the above methods has its own advantages. We can view $Q_A(p)$ as a cost-avoidance strategy that assumes that at most p investigations will be needed to reach the kingpin. Under $Q_D(q)$ only criminals of a certain "influence" will be arrested, with at least q direct underlings. Q_I is the most covert and expensive, as it disrupts the network by kingpin capture alone. Since no intermediate arrests will be performed, the network can effectively grow as if law enforcement were not present. We can relate the strategies as follows

$$\lim_{p\to\infty} Q_A(p) = \lim_{q\to\infty} Q_D(q) = Q_I, \tag{11}$$

$$Q_A(1) = Q_D(1). \tag{12}$$

Note that as the parameters p and q increase, the chosen strategies become more and more covert and demanding, involving more investigations and aiming at higher level arrests, so that Q_A and Q_D approach Q_I. Also, note that under strategy $Q_A(1)$ law enforcement will remove the node directly above the street criminal selected as the initial investigative point. This is the same result that would arise from strategy $Q_D(1)$, since all nodes above street criminals have at least one linked criminal, *i.e.* the street criminal itself.

We can evaluate each strategy's performance for several recruitment indices k using numerical simulations. Runs were continued until either the kingpin was arrested, eradicating the criminal network, or the kingpin was not arrested and the network exceeded a total, given population size n^*. The latter scenario represents the case of a vast organization permeating all society. In Fig. 9 we plot the network eradication probability for $Q_A(p), Q_D(q)$ and Q_I as a function of the recruitment index k, for various choices of p, q and for various thresholds of criminal population n^*. In all cases, as can be expected, the probability of capturing the kingpin decreases with k, as the rate of adding new criminals becomes faster than any disruption attempts by law enforcement. We also define $\text{Beat}(Q)$, the "beat number" of strategy Q, as the maximum recruitment index k of the network for which law enforcement will reach the kingpin with unit probability across the simulations performed. The eradication probabilities and $\text{Beat}(Q)$ depend on the total size of the network n^* as can be seen from Fig. 9. $\text{Beat}(Q)$ increases with n^* since, for small and intermediate values of k, growth occurs at a relatively slow rate and although increasing n^* allows for more criminals to join the organization, there will also be a relatively large number of police pursuits during the slow dynamics, allowing for greater eradication probabilities. Given a fixed value of n^*, we can compare different strategies and their results. From the left hand panels of Fig. 9 we note that for large k values the eradication probability increases with the number of investigations p. In this case the network is rapidly increasing and allowing more investigations before an arrest is made also allows for the possibility of arresting senior, highly nested criminals with many underlings, greatly undermining network structure and size. For small values of k few criminals are added at each time step and each node will have few underlings. Small values of p restrict law enforcement to activity close to street level, where nodes have low hierarchical value and allow for modest but progressive node removal. Increasing p, when p is small, is beneficial as can be seen by comparing the $p = 1, 2$ curves. However, when k is small, increasing p to larger values may not be the best strategy: since each node has few underlings, higher values of p increase the possibility that the self-avoiding random walk performed by law enforcement reaches a dead-end, effectively leaving the network untouched. This is the case for example for $Q_A(p = 6)$ and $Q_A(p = 8)$ for which the eradication probability is smaller than for $Q_A(p = 1)$

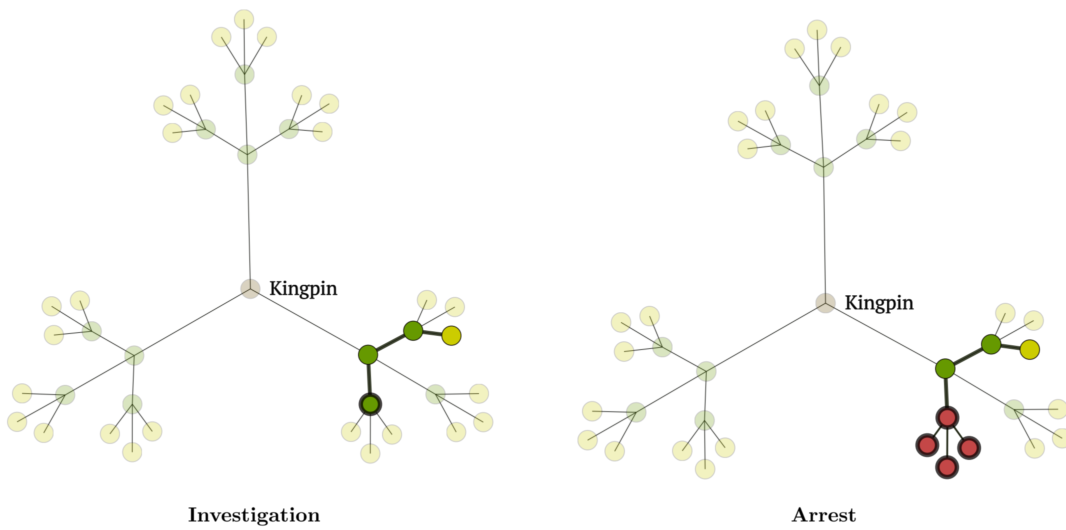

Figure 8. The pursuit process as described in our text. We start the network as a full ternary tree of height three with forty criminals, as shown by the light colored nodes. (Left) At time $t = 1$ a law enforcement agent begins an "investigation" from the highlighted yellow criminal and without having full knowledge of the network. The investigative trail involves three other nodes, highlighted in green and linked by a self-avoiding random walk, marked by a solid line. The last node, surrounded by a dark ring, is the criminal that is arrested. (Right) Once a criminal is arrested and removed from the network all related underlings in the hierarchy are removed as well. In this case, all removed nodes are depicted in red and are have a darker boundary. We note that not all criminals arrested were investigated and vice-versa.

and $Q_A(p = 2)$ for small k values. Similar trends arise for $Q_D(q)$ as shown in the right hand panels of Fig. 9. Here, for large values of k the best strategy is to set a target of relatively high out degree q before performing an arrest, while for lower values of k moderate q values are preferred, due the possibility of reaching dead-ends during the pursuit. Note that the intermediate panels in Fig. 9 depicting the results stemming from strategy Q_I represent limiting values for $Q_A(p \to \infty)$ and $Q_D(q \to \infty)$. Our results indicate that the optimal approach for law enforcement, whether engaged in the fixed investigation number pursuit pursuit $Q_A(p)$ or in the minimum out degree strategy $Q_D(q)$, is to adjust arrest criteria to an optimal p or q depending on the recruitment rate k, if this variable is known or estimates are available.

The results discussed so far depend on initial conditions, which can be chosen to be any full or complete tree with b branches and height h. Full trees have a total number of $(b^{h+1} - 1)/(b - 1)$ criminals; the case of a full ternary tree of height three discussed above corresponds to $b = h = 3$ with forty initial criminals. For a fixed number of criminals, the larger b, the smaller h and the less hierarchical the initial tree is. The choice of $b = 1$ is the limiting case of a linear chain: here, strategy $Q_D(q > 1)$ will lead to a unitary eradication probability since all nodes will have out degree one and at the onset of the pursuit phase law enforcement will proceed until the kingpin is reached, regardless of the number of total criminals and of h. The eradication probability decreases on a binary initial tree with $b = 2$, since now dead-ends may be encountered. Further increasing b leads to diminishing eradication probabilities due to the possibility of more unsuccessful pursuits. Once a sufficiently large value for b is reached however, h will be small enough, that reaching the kingpin may become more feasible. We find the critical value of b at the threshold between the two trends to depend on the initial number of criminals. In the left hand panel of Fig. 10 for example we show the eradication probability for $Q_D(q = 3)$ on an initial network of forty criminals with varying b and with a fixed value of $k = 30$. The complete trees we created were as regular as possible, with all second-to-last nodes sharing the same number of underlings as possible. The choice $b = 3$ is the full ternary tree case we have analyzed in detail above. Note the minimum for intermediate values of b. Similar trends arise also for $Q_A(p)$ although the decrease in the eradication probability is less pronounced for small b. In the middle and right hand panels of Fig. 10 we consider initial networks made of full trees with fixed b and varying h and vice-versa. We expect increases in b or h to lead to lower eradication probabilities since the number of initial criminals will be larger. In the middle panel of Fig. 10 we show an initial network of height $h = 4$ with branching $b = 1$, representing a chain of criminals, with $b = 2$, representing a binary tree, and with $b = 4$ representing a quaternary one. As expected, the eradication probability decreases with k for all b, and with b for fixed k. On the right hand panel of Fig. 10 we find similar results for an initial full binary tree with $b = 2$ and with heights $h = 2, 4, 6$. The eradication probability

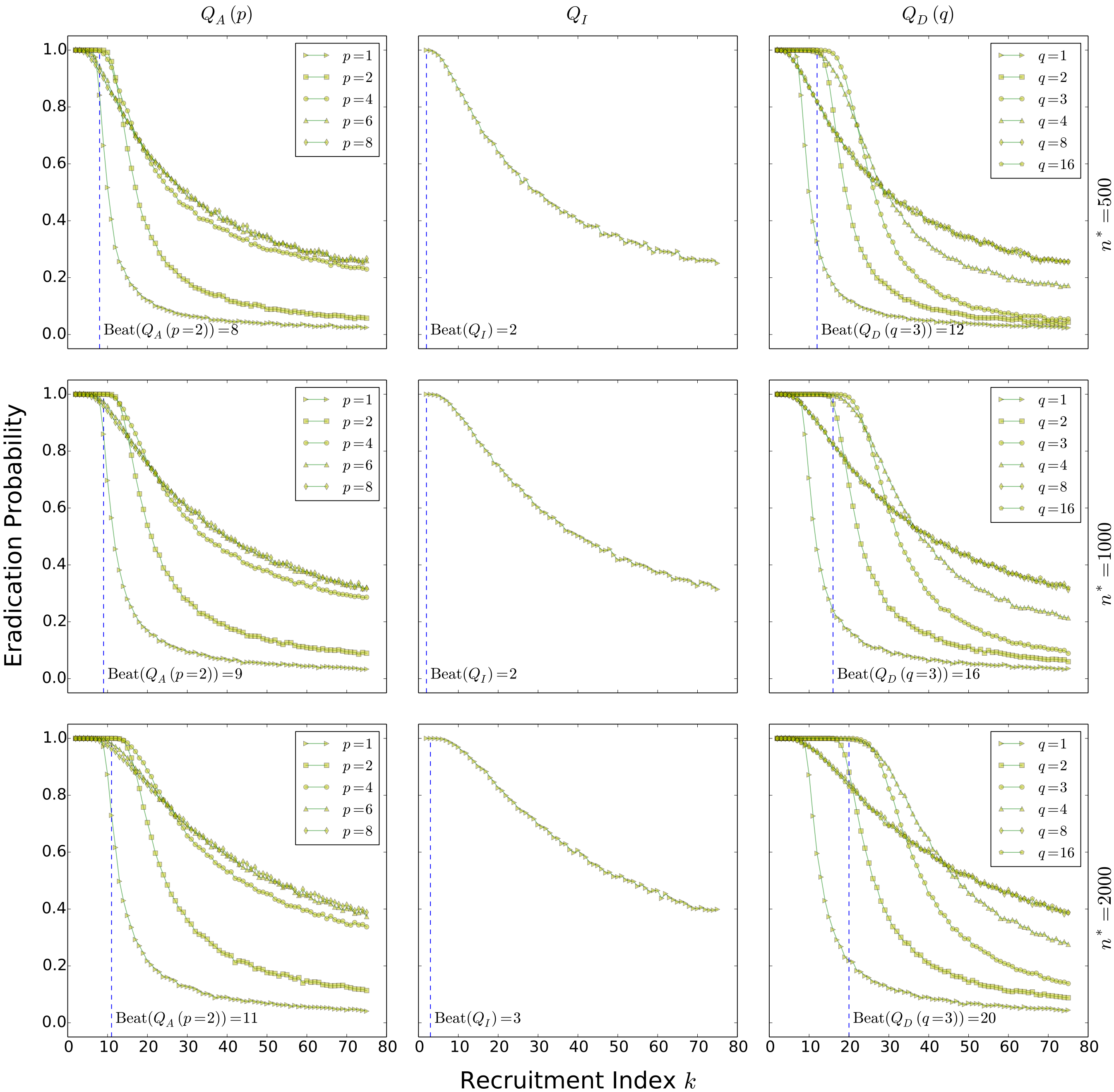

Figure 9. The network eradication probability as a function of the recruitment index k, obtained by averaging over 10,000 simulations for different strategies. We consider a total population of $n^* = 500, 1000, 2000$ individuals and halt our simulations when the criminal network reaches this size. We specify the Beat(Q) of each strategy as the maximum value of the recruitment index k for which the network is eradicated with probability one, over all simulations. Note that Q_I is the limiting strategy for $Q_A(p \to \infty)$ and $Q_D(q \to \infty)$. Our results reveal that the optimal strategy for fast growing networks with large k is to use investigative strategies with large values of p, q while for slower growing networks with small values of k, moderate values of p, q yield higher probabilities of eradicating the network. Note that curves for $Q_A(p = 1)$ and $Q_D(q = 1)$ are the same and that the Q_I curve is the limit for $Q_A(p \to \infty) = Q_D(q \to \infty)$.

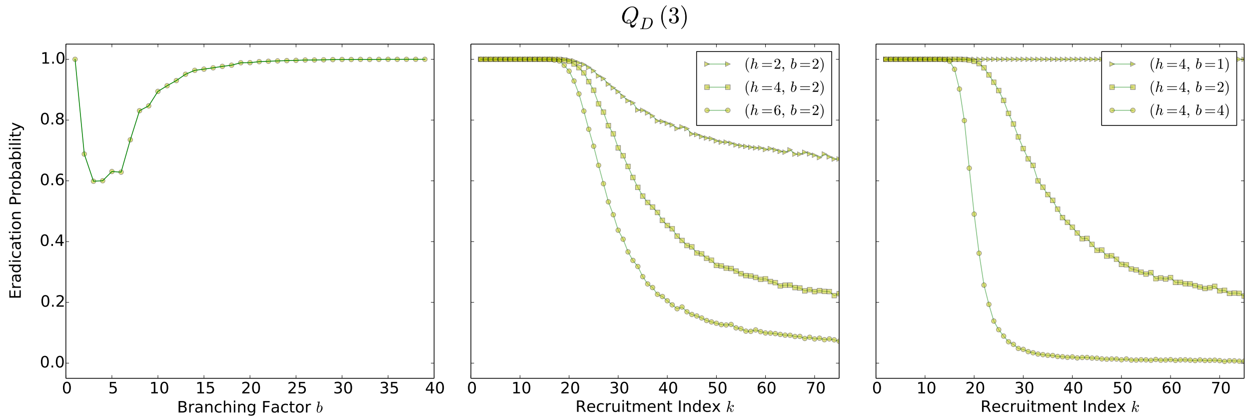

Figure 10. (Left) Eradication probabilities on a complete initial tree with b branches and forty criminals for $k = 30$ and law enforcement strategy $Q_D(q = 3)$. On a linear chain ($b = 1$) the eradication probability is one, since no dead-ends will be reached. Increasing b allows for more dead-ends to be encountered so that the eradication probability decreases until a threshold value of b when the height of the network becomes small enough to allow for easier access to the kingpin. Here $b = 3$. (Right) Eradication probabilities using full initial trees with b branches and height h as initial conditions and using strategy $Q_D(q = 3)$. The initial number of criminals is $(b^{h+1} - 1)/(b - 1)$. For all values of k eradication is higher for lower values of b, h indicating that best results will be obtained with law enforcement intervening on initially contained networks, as can be expected. Qualitatively similar results arise for other strategies $Q_D(q)$ and $Q_A(p)$.

decreases with k for all h, and with h for fixed k as well.

B. Strategic Costs and Time to Eradication

In the previous subsection, we implicitly assumed that all pursuit strategies could be conducted with any value of p, q indistinctly. Here we quantify the efficiency of various investigative methods by associating a cost measure to each of them. We assume that investigating criminals requires the expenditure of societal resources, while arresting criminals can be considered a gain, since an arrest will weaken the criminal network and the prospect of future crime will lessen. We thus evaluate cost as the number of investigations performed by law enforcement within a given pursuit phase minus the number of criminals removed within the same pursuit, if this difference is positive. Otherwise, if the number of investigations is lower than the number of arrests, we let costs be zero. For example, in the left hand panel of Fig. 8 four nodes are investigated, the street criminal (depicted in yellow), and three senior members (depicted in green). During the arrest phase, in the right hand panel of Fig. 8, four criminals are eliminated (depicted in red). At this iteration, the costs associated with investigating four criminals are balanced by the benefits associated with removing four criminals, so we tally costs as zero. The total cost is then calculated as the cumulative cost required to reach the kingpin throughout the simulation, assuming the network is eradicated. If the network is not eradicated, we can still record the cumulative costs until the entire population n^* has joined the criminal organization. However, we do not discuss cases where the kingpin is not reached, since here costs will be proportional to the number of rounds until the n^* criminals are incorporated in the network. Note that dead-ends in this context are very expensive, since there are no net gains in a pursuit that leads to no arrests. In the upper panel of Fig. 11 we plot the total cost incurred by authorities conditioned on the network being dismantled for a total population size $n = 1000$ and for various pursuit methods $Q_A(p)$ and $Q_D(q)$ starting on an initial full ternary tree. We also depict the probability of kingpin capture and criminal network eradication as shades in the data points. Costs are identically zero for $Q_A(p = 1)$ and $Q_D(q = 1)$ since in these cases there will be two investigations and two criminals removed at every time step. Similarly, costs stay low for low values of p, q across both strategies. For example for both $Q_A(p = 2)$ and $Q_D(q = 3)$ total expenditures are very small, indicating that the number of investigations is at most comparable to the number of criminals removed throughout the entire network evolution.

For intermediate values of p, q we note that curves in the upper panels of Fig. 11 are monotonically decreasing, as can be seen for $Q_A(p = 3), Q_A(p = 4)$ and $Q_D(q = 4)$. To understand this behavior, we note that several distinct trends arise upon increasing k for small values of k. On one hand, the likelihood of reaching costly dead-ends increases. Here, larger values of k represent faster network growth, with more nodes being added at every time step. Each criminal is thus linked to a greater number of underlings and the probability of moving higher up in the hierarchy decreases. In this case, it is very likely for law enforcement to eventually reach a dead-end, and for costs to increase. On the other hand, if dead-ends are not reached and arrests are performed, larger k values

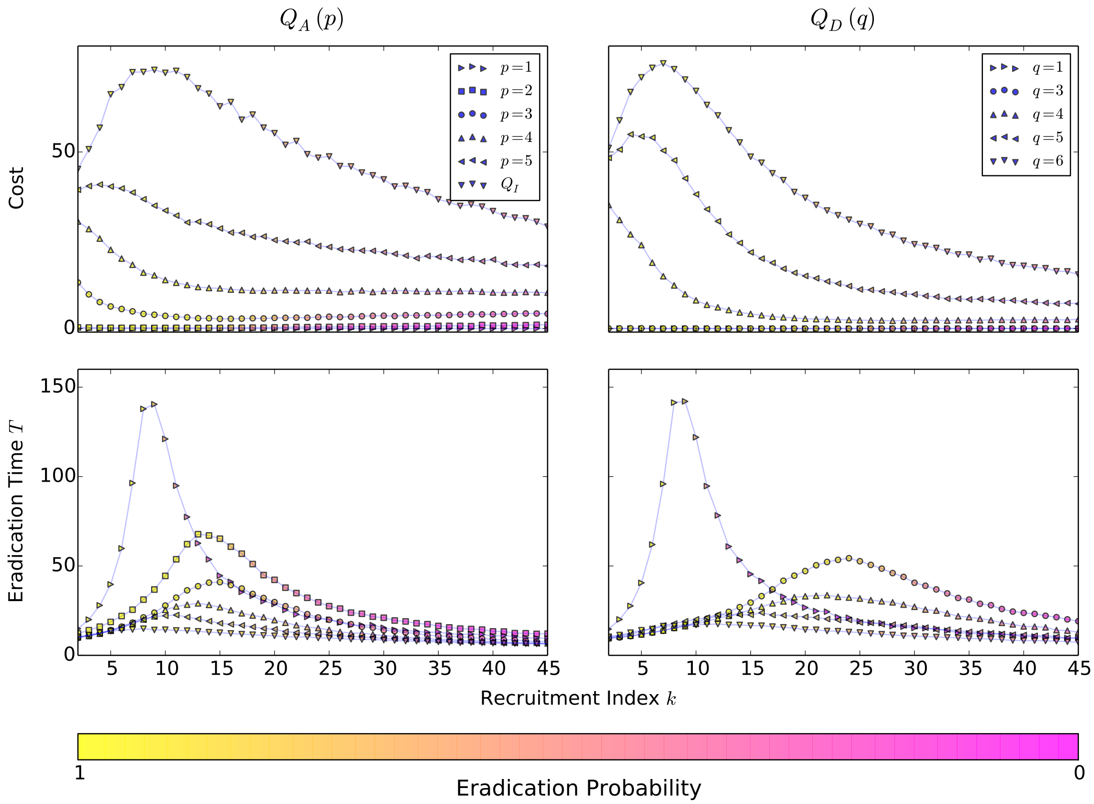

Figure 11. (Top) Costs incurred by law enforcement conditioned on kingpin capture as a function of the recruitment index k. Shades in the data points represent the probabilities of network eradication. We considered 10000 simulations and allowed the network to grow to $n^* = 1000$. Note the emergence in maxima for all curves, due to the conditional nature of cost evaluations. As k increases, the network grows more rapidly and so does the number of investigations necessary for eradication. Upon reaching a threshold in k, eradication becomes less likely especially in the latter growth stages so that for large enough k either the kingpin is captured in the early stages of network growth, with little costs, or it will never will be. The requirement for quick capture with growing k for large k is associated with decreases in the cost function, leading to the maxima in k. (Bottom) The mean eradication time as a function of k for various strategies. Similarly as for the above panel, the conditional nature of the process is manifest from the emergence of maxima in all curves. Note that curves for $Q_A(p = 1)$ and $Q_D(q = 1)$ are the same and that the Q_I curve is the limit for $Q_A(p \to \infty)$.

imply that more underlings will be eliminated per pursuit, leading to decreasing costs. Of these two trends, for intermediate values of p, q, the most important is the greater elimination of criminals with increasing k. Here, pursuits are short so that the self-avoiding random walks performed by law enforcement have a slightly lower chance of incurring in dead-ends, compared to higher values of p, q. As a result, the number of criminals arrested at each time step is contained, but almost always will criminals be arrested, leading to the monotonically decreasing curves in the upper panels of Fig. 11. Conversely, as p, q increase further, pursuits are longer with dead-ends becoming more likely and more costly, failing to limit network growth. Here, increasing k for small k leads to higher costs, as can be seen from the $Q_A(p = 5)$ and $Q_D(p = 5), Q_D(p = 6)$ curves. Note that the same trend arises for Q_I, which is the limiting behavior for $Q_A(p \to \infty)$ and $Q_D(p \to \infty)$. As k increases even further, although the likelihood of reaching dead-ends increases, the gains in eliminating more criminals prevails and all curves show decreasing costs with increasing k leading to a maximum for intermediate p, q.

These behaviors are valid for small and intermediate values of k, when the likelihood of eradication is almost unitary. Beyond a certain threshold in k however, the eradication probability decreases for all choices of p, q and the probability that the entire society is taken over by organized crime increases. In this case, either the network is eradicated at its early stages of growth, or it will never be. Compounded with the above considerations, the decrease in the cost curves for all values of p, q for large k are indicative of the conditional nature of the process: as k increases beyond a certain threshold, it becomes less and less likely to be able capture the kingpin and the process must occur more and more swiftly with less investigations, so that costs decrease as a function of k regardless of p, q. In the lower panel of Fig. 11 we plot the time of first eradication of the network as a function of recruitment index k and find trends that support these considerations. The time of first eradication is always non-monotonic: it increases in k before decreasing again, with the non-monotonicity stemming from the conditional manner in which eradication times are evaluated. For small k, increasing k requires more time steps in capturing the kingpin but beyond a certain threshold, eradication must be quick or it will never occur at all, leading to decreasing first eradication times. Indeed, peaks in the eradication time curves roughly coincide with drops in the eradication probability, as can be seen from the shaded colors in the lower panels of Fig. 11. Note that for small to intermediate k values increasing eradication times may be coupled with lower costs, indicating that while more time steps are required to reach the kingpin, criminals are being eliminated from the network in a more efficient way.

It is important to note that these results are highly dependent on the network configuration on which the pursuit was initiated: different initial conditions will yield different eradication probabilities, costs and eradication times. We simulated different initial tree configurations and found that given an initial number of criminals, here set at forty, more hierarchical structures (complete linear or binary trees, with lower b values) allow for better results in terms of maximizing eradication probabilities and minimizing costs and eradication times as discussed in the previous subsection. In general, initial networks a lower number of initial criminals, with either lower b or h allow for shorter eradication times and lower costs, as can be expected.

C. Best strategies

From the above results we can try to identify a best strategy for network eradication. As discussed above, eradication probabilities decrease with k. Law enforcement cannot influence k values, as this is an intrinsic feature of the criminal organization and may depend, for example, on kingpin charisma or on rewards offered by the network to its members. Indeed, on dark networks, law enforcement may only have best guesses for k. We thus assume that law enforcement agencies may only select which strategy to use given a preset value of $p = q = q^*$ that is not exceedingly large since for $q^* \to \infty$ all strategies are the same. Fig. 9 shows that for very small and very large values of k, given a full ternary tree as initial condition, eradication probabilities do not change significantly across strategies. However, for intermediate values of k, the minimum out degree strategy $Q_D(q^*)$ is associated with slightly larger eradication probabilities compared to the fixed investigation number strategy $Q_A(q^*)$. This can be seen for example by comparing curves for $Q_A(p = 3)$ and $Q_D(q = 3)$ and in particular by noting that $\text{Beat}(Q_D(p = 3)) > \text{Beat}(Q_A(q = 3))$ for all values of n^*. From this perspective, given that typical k values are not known to law enforcement, it is optimal to utilize strategy $Q_D(q^*)$ for a given value of q^*. The choice of what q^* to select, if there is any information known on the rate of growth of the criminal organization is to use lower values of q^* for slowly growing organizations, with lower values of k, and larger q^* in the opposite case. If one is interested in lowering costs, for example when the criminal organization is not engaged in activities that are deemed to be especially dangerous for the community, the best intervention method is to use the less sophisticated investigative methods associates with lower values of q^*, since these are associated with lower costs, albeit to longer eradication times as well. Costs are lower for $Q_D(q^*)$ than for $Q_A(q^*)$ as can be seen from Fig. 11. Similarly, if one is interested in lowering eradication times, the minimum out degree strategy always yields better results for $Q_D(q^*)$ than for $Q_A(q^*)$.

V. CONCLUSION

The simple network model we presented provides insight about the hierarchal formation and active disruption of criminal networks. Our work can be useful as a basic yet fundamental template to study the disruption of growing dark networks [18, 46, 47], such as the terrorist group ISIS where new members are recruited rapidly, in a distributed manner, while the network itself endures repeated eradication attempts by external disruptors.

The recruitment mechanism we used is a variation on standard preferential attachment models, though there are some important distinctions. For example, the resulting out degree distribution we find is not heavy tailed [48, 49] and appears to be independent of the recruitment index k. We also found the distribution of criminal position relative to the kingpin is well approximated by a shifted gamma distribution with parameters depending on the initial configuration, on the recruitment index k, and on the maximum network size n^*. Other fitting functions could be possible as well, and determining the exact mathematical form of the distribution is an open problem. Our model yielded a linear relationship between the number of street criminals and the recruitment rate k, for which we provided a heuristic justification. A more rigorous framework could be useful to determine the relationship between criminals of higher degree than street criminals and k. This is a more difficult task than what presented in this work for street criminals, since the distribution would depend on the exact topography of the network.

Law enforcement pursuit and arrest was modeled as a dark network disruption problem. We introduced and analyzed the efficacy of several strategies that could be used, beginning on a full ternary tree of height three. For a preset value of $p = q = q^*$ we can heuristically determine the most effective strategy on a dark network to be $Q_D(q^*)$, when the pursuit ends upon reaching criminals with at least q^* connections and for moderate values of q^*. Indeed, strategy $Q_D(q^*)$ yields comparable or larger eradication probabilities than $Q_A(q^*)$. The optimal value of the arrest parameter q^* will depend on the recruitment rate k and on the overall population size n^*. For example, from Fig. 9 it appears that $Q_D(8)$ is more effective than $Q_D(3)$ only when $k \gtrsim 40$ for all values of n^* with the opposite being true for smaller values of k. Also in terms of minimizing costs and decreasing eradication times strategy $Q_D(q^*)$ is more efficient than strategy $Q_A(q^*)$ as can be seen from Fig. 11. These results are consistent with previous network models where the optimal disruptor strategy involved seeking the nodes of highest detectable degree on the network [14, 15].

We also find that as k increases and the network grows at a faster rate, the eradication probability decreases and either the kingpin is captured in the early stages of network growth or it will never be. This result is in agreement with previous results on attempting to dismantle operating criminal networks. For example, in the network model of Dujin *et al.* discussed above every single disruption strategy proposed was largely ineffective except when performed during the nascency phase of the criminal organization [13] . While the disruption and recruitment processes analyzed in the latter work are different from ours, we can draw a similar conclusion on the importance of "proactively" attacking organized crime networks before they become too entrenched within society and eradication proves more and more elusive.

In this work we have only modeled a professional network of criminals. However, a social network may be more useful to law enforcement officials [18], with the possible inclusion of geo-spatial constraints. The disruption strategies presented here could also be modified to be adaptive so that at some time threshold, strategy $Q_D(q)$ could be abandoned in favor of strategy $Q_A(p)$ or, for example, the number of investigations p when following strategy $Q_A(p)$ could be changed based on previous experiences and seeking to optimize arrests and efficiency.

Finally, we only looked in detail at pursuit methods for a single initial configuration, that of a full ternary tree of height three with forty initial criminals. Although not shown here in full detail, simulations performed with forty initial criminals on extremely hierarchical structures, such as linear or binary trees, lead to higher eradication probabilities once law enforcement pursuits are introduced. Similarly extremely low hierarchical initial structures lead to higher eradication probabilities. These results indicate that the simplest networks to eradicate are those with initial low b where once investigations begin, at the early stages of the pursuit process, almost certainly the kingpin will be reached, or those that grow in a non-centralized manner where the many initial street criminals provided by very high b allow for easy access to the kingpin who is just one or two levels removed from street activity. Extremely prudent (low b) or ambitious (high b) kingpins are thus the most vulnerable. On the contrary, the most robust initial networks are those that grow enough levels and with enough members at each of them to effectively shield the kingpin from arrest in the early stages of the pursuit process.

VI. CODE

We used NetworkX, NumPy, and SciPy for all the simulations. d3.js was used for the network diagrams. The code for the preferential attachment tree, the dynamic game, and the simulations can be found at `https://github.com/cmarshak/GameOfPablos`.

VII. ACKNOWLEDGEMENTS

We acknowledge fruitful discussions with James von Brecht, Ryan Compton and Greg Klar. This work was supported by ARO MURI grant W911NF-11-1-0332, AFOSR MURI grant FA9550-10-1-0569, NSF grant DMS-0968309, ONR grant N000141210838.

[1] B. K. Chakrabarti, A. Chakraborti, and A. Chatterjee, *Econophysics and Sociophysics: Trends and Perspectives* (John Wiley & Sons, 2007).
[2] C. Castellano, S. Fortunato, and V. Loreto, Reviews of Modern Physics **81**, 591 (2009).
[3] M. R. D'Orsogna and M. Perc, Physics of Life Reviews **12**, 1 (2014).
[4] M. B. Short, M. R. D'Orsogna, V. B. Pasour, G. E. Tita, P. J. Brantingham, A. L. Bertozzi, and L. Chayes, Mathematical Models and Methods in Applied Science **18**, 1249 (2008).
[5] M. B. Short, A. L. Bertozzi, and P. J. Brantingham, SIAM Journal on Applied Dynamical Systems **9**, 462 (2010).
[6] S. Chaturapruek, J. Breslau, D. Yazdi, T. Kolokolnikov, and S. G. McCalla, SIAM Journal on Applied Mathematics **73**, 1703 (2013).
[7] M. R. D'Orsogna, M. McBride, R. Kendall, and M. B. Short, PloS One **8**, e61458 (2013).
[8] N. Bellomo, F. Colasuonno, D. Knopoff, and J. Soler, arXiv preprint arXiv:1412.2159 (2014).
[9] A. Stomakhin, M. B. Short, and A. L. Bertozzi, Inverse Problems **27**, 115013 (2011).
[10] B. Berenji, T. Chou, and M. R. D'Orsogna, PLoS One **9**, e85531 (2014).
[11] J. R. Wagley, in *CRS Report for Congress* (2006).

[12] S. Forbes, M. Susan, M. McBride, B. Allison, and N. Candelo (2014), preprint.
[13] P. A. Duijn, V. Kashirin, and P. M. Sloot, Scientific Reports **4** (2014).
[14] M. McBride and D. Hewitt, Journal of Economic Behavior & Organization **93**, 32 (2013).
[15] M. McBride and M. Caldara, Social Networks **35**, 406 (2013).
[16] C. Ballester, A. Calvó-Armengol, and Y. Zenou, Econometrica **74**, 1403 (2006).
[17] S. P. Borgatti, Computational & Mathematical Organization Theory **12**, 21 (2006).
[18] C. Morselli, *Inside Criminal Networks*, vol. 8 (Springer, 2008).
[19] M. Kenney, Survival **45**, 187 (2003).
[20] W. L. Reed and S. H. Decker, *Responding to Gangs Evaluation and Research* (Education Resource Information Center, 2002).
[21] G. Mastrobuoni and E. Patacchini, Carlo Alberto Notebooks **152** (2010).
[22] G. Mastrobuoni and E. Patacchini, Review of Network Economics **11** (2012).
[23] C. Morselli, Journal of Contemporary Criminal Justice **26**, 382 (2010).
[24] V. Krebs, Connections **24**, 43 (2002).
[25] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[26] H. Jeong, Z. Néda, and A.-L. Barabási, Europhysics Letters **61**, 567 (2003).
[27] M. Boguñá, R. Pastor-Satorras, A. Díaz-Guilera, and A. Arenas, Physical Review E **70**, 056122 (2004).
[28] J. H. Jones and M. S. Handcock, Proceedings of the Royal Society of London: Biological Sciences **270**, 1123 (2003).
[29] T. Antunović, E. Mossel, and M. Z. Racz, arXiv preprint arXiv:1307.2893 (2013).
[30] R. T. Smythe and H. M. Mahmoud, Theory of Probability and Mathematical Statistics **51**, 1 (1995).
[31] H. M. Mahmoud, Journal of Computational and Applied Mathematics **41**, 237 (1992).
[32] J. P. Gleeson, D. Cellaia, J.-P. Onnelab, M. A. Porter, and R.-T. Felix, Proceedings of the National Academy of Sciences **111**, 10411 (2014).
[33] M. Catanzaro, M. Boguñá, and R. Pastor-Satorras, Physical Review E **71**, 027103 (2005).
[34] B. Bollobás and O. Riordan, Internet Mathematics **1**, 1 (2004).
[35] A. E. Motter and Y.-C. Lai, Physical Review E **66**, 065102 (2002).
[36] A. E. Motter, Physical Review Letters **93**, 098701 (2004).
[37] A. Allard, P.-A. Noël, L. J. Dubé, and B. Pourbohloul, Physical Review E **79**, 036113 (2009).
[38] W. L. Shew, H. Yang, T. Petermann, R. Roy, and D. Plenz, The Journal of Neuroscience **29**, 15595 (2009).
[39] D. B. Larremore, M. Y. Carpenter, E. Ott, and J. G. Restrepo, Physical Review E **85**, 066131 (2012).
[40] D. B. Larremore, W. L. Shew, and J. G. Restrepo, Physical Review Letters **106**, 058101 (2011).
[41] A. Gutfraind, PloS One **5**, e13448 (2010).
[42] J. A. Densley, Social Problems **59**, 301 (2012).
[43] N. Berger, C. Borgs, J. T. Chayes, A. Saberi, et al., The Annals of Probability **42**, 1 (2014).
[44] M. Newman, *Networks: An Introduction* (Oxford University Press, 2010).
[45] R. Durrett, *Random Graph Dynamics*, vol. 200 (Cambridge University Press, 2007).
[46] R. M. Bakker, J. Raab, and H. B. Milward, Journal of Policy Analysis and Management **31**, 33 (2012).
[47] S. F. Everton, *Disrupting Dark Networks*, 34 (Cambridge University Press, 2012).
[48] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, Proceedings of the National Academy of Sciences **101**, 3747 (2004).
[49] S. H. Strogatz, Nature **410**, 268 (2001).