Math 150B Semester Review Solutions

1. a) $f(x) = \dfrac{e^{4x}}{x^2+1} = \dfrac{u}{v}$

$f'(x) = \dfrac{u'v - uv'}{v^2} = \boxed{\dfrac{4e^{4x}(x^2+1) - e^{4x}(2x)}{(x^2+1)^2}}$

b) $f(x) = \ln(1+e^x) = \ln w$

$w = 1+e^x \quad f'(x) = \dfrac{1}{w} \cdot dw$

$dw = e^x \quad = \boxed{\dfrac{e^x}{1+e^x}}$

c) $f(x) = x^2 \ln x$

$f'(x) = u'v + uv' = 2x\ln x + x^{\cancel{2}} \cdot \dfrac{1}{\cancel{x}}$

$= \boxed{2x\ln x + x}$

d) $f(x) = \sin^{-1}(\sqrt{x}) = \sin^{-1} w$

$w = \sqrt{x} \quad f'(x) = \dfrac{1}{\sqrt{1-w^2}}\, dw$

$dw = \frac{1}{2}x^{-1/2} \quad = \dfrac{1}{2\sqrt{1-(\sqrt{x})^2}} \cdot \dfrac{1}{\sqrt{x}}$

$= \boxed{\dfrac{1}{2\sqrt{x(1-x)}}}$

e) $f(x) = (1+\tan^{-1}x)^{10}$

$f'(x) = 10(1+\tan^{-1}x)^9 \cdot \left(\dfrac{1}{1+x^2}\right)$

$= \boxed{\dfrac{10(1+\tan^{-1}x)^9}{(1+x^2)}}$

f) $f(x) = e^{x\ln(x^2+3)} \Rightarrow f'(x) = e^{x\ln(x^2+3)}\left(1\cdot\ln(x^2+3) + x\cdot\dfrac{1}{x^2+3}\cdot 2x\right)$

$= \boxed{(x^2+3)^x\left(\ln(x^2+3) + \dfrac{2x^2}{x^2+3}\right)}$

2) $8x = 2y\dfrac{dy}{dx} + \dfrac{1}{3x-y}\cdot 3 - \dfrac{dy}{dx}$ @ $\overset{x,y}{(1,2)}$

$8x = 2y\dfrac{dy}{dx} + \dfrac{3-\frac{dy}{dx}}{3x-y} \Rightarrow 8(1) = 2(2)\dfrac{dy}{dx} + \dfrac{3-\frac{dy}{dx}}{3(1)-(2)}$

$8 = 4\dfrac{dy}{dx} + 3 - \dfrac{dy}{dx}$

$-3 \qquad -3$

$5 = 4\dfrac{dy}{dx} - \dfrac{dy}{dx}$

$5 = \dfrac{dy}{dx}(4-1) \Rightarrow \dfrac{5}{3} = \dfrac{\frac{dy}{dx}(\cancel{3})}{\cancel{3}}$

$\dfrac{dy}{dx} = \dfrac{5}{3}$

$y = y_1 + y'(x-x_1)$

$y = 2 + \frac{5}{3}(x-1)$

$y = 2 + \frac{5}{3}x - \frac{5}{3}$

$\boxed{y = \frac{5}{3}x + \frac{1}{3}}$

3) a) $f'(x) = 2xe^{-2x} + x^2\cdot -2e^{-2x} = 2xe^{-2x}(1-x)$

f': dec. <0 | 0 ($=0$) | inc. >0 | 1 ($=0$) | dec. <0

$f''(x) = 2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 4x^2e^{-2x}$

$= 2e^{-2x}(1-2x-2x+2x^2)$

$= 2e^{-2x}(2x^2-4x+1)$

$2x^2-4x+1$

$\dfrac{4 \pm \sqrt{16-4(2)(1)}}{4}$

$\dfrac{4\pm\sqrt{8}}{4} \quad x = 1.707 \quad x = .2929$

f'': c.u. >0 | $.2929$ ($=0$) | c.d. <0 | 1.707 ($=0$) | c.u. >0

$f(x)$:

increasing: $(0,1)$

decreasing: $(-\infty,0) \cup (1,\infty)$

concave up: $(-\infty,.2929) \cup (1.707,\infty)$

concave down: $(.2929, 1.707)$

local max: $x=1$

local min: $x=0$

inflection points: $x=.2929$, $x=1.707$

#3b No solution provided.

#4 (a) $\int \frac{x^2+1}{x^2-3x+2}\,dx$

$$= \int \frac{x^2+1}{(x-2)(x-1)}\,dx$$

$$\frac{x^2+1}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

$$x^2+1 = A(x-1) + B(x-2)$$

When $x=1$. $2=-B$ $B=-2$

When $x=2$. $5=A$ $A=5$

$$\rightarrow = \int \frac{5}{x-2}\,dx + \int \frac{-2}{x-1}\,dx$$

$$= 5\ln|x-2| - 2\ln|x-1| + C$$

4) b) $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx = -\int (1-w^2)\,dw$

$\cos^2 x + \sin^2 x = 1$

$w = \cos x$

$-dw = \sin x\,dx$

$$= \int -1\,dw + \int w^2$$

$$= -w + \frac{w^3}{3} + C = \boxed{-\cos x + \frac{\cos^3 x}{3} + C}$$

4 (c)

$$\int \frac{x}{x^2+2x+2}\,dx = \int \left(\frac{x}{(x^2+2x+1)+1}\right)dx$$

$$= \int \frac{x}{(x+1)^2+1}\,dx$$

$$\boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}}$$

$$= \int \frac{u-1}{u^2+1}\,du$$

$$= \int \frac{u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$$\boxed{\begin{aligned} v &= u^2+1 \\ dv &= 2u\,du \end{aligned}}$$

$$= \frac{1}{2}\int \frac{2u\,du}{(u^2+1)} - \tan^{-1}(u) + c$$

$$= \frac{1}{2}\int \frac{dv}{v} - \tan^{-1}(u) + c$$

$$= \frac{1}{2}\ln|v| - \tan^{-1}(u) + c$$

$$= \frac{1}{2}\ln(u^2+1) - \tan^{-1}(u) + c$$

$$= \frac{1}{2}\ln[(x+1)^2+1] - \operatorname{Arctan}(x+1) + c$$

4. (d) $\int x^3 \ln x \, dx$ $\quad u = \ln x \Rightarrow du = \frac{1}{x} dx$

$\quad dv = x^3 dx \Rightarrow v = \frac{1}{4} x^4$

$= \frac{1}{4} x^4 \cdot \ln x - \frac{1}{4} \int x^{\cancel{4}\,3} \cdot \frac{1}{x} dx$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \cdot \frac{1}{4} x^4 = \boxed{\frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + C}$

4) c) $\int \underbrace{x}_{u} \underbrace{(2x+1)^{1/2}}_{v'} dx = \frac{1}{3} x (2x+1)^{3/2} - \frac{1}{3} \int (2x+1)^{3/2} dx$

$u' = 1$

$v = \frac{1}{2} \cdot \frac{2}{3} (2x+1)^{3/2} = \frac{1}{3} (2x+1)^{3/2}$

$= \frac{1}{2} \cdot \frac{2}{5} (2x+1)^{5/2} = \frac{1}{5} (2x+1)^{5/2}$

$= \boxed{\frac{1}{3} x (2x+1)^{3/2} - \frac{1}{15} (2x+1)^{5/2} + C}$

$\frac{1}{3} \cdot \frac{1}{5}$

4f) Find $\int \frac{x^2}{(x^2+1)^2}\,dx$ Trig Sub $x=\tan\theta$

$dx=\sec^2\theta\,d\theta$

$$\int \frac{\tan^2\theta\cdot\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} = \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta} = \int \frac{\tan^2\theta\,d\theta}{\sec^2\theta}$$

Change $\frac{\tan^2\theta}{\sec^2\theta}$ to $\frac{\sin^2\theta}{\cos^2\theta} \div \frac{1}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta = \sin^2\theta$

$=\int \sin^2\theta\,d\theta$ Use trig identity $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$\frac{1}{2}\int (1-\cos(2\theta))\,d\theta = \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)$ $\Big\}$ Replace θ's with x's

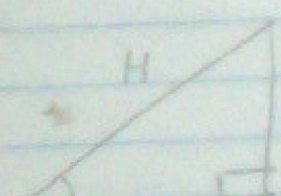
$x=\tan\theta \to \arctan(x)=\theta$

Changed $\sin(2\theta) \to 2\sin\theta\cos\theta$

$$=\frac{1}{2}\left(\arctan(x) - \frac{2\sin(\arctan(x))\cos(\arctan(x))}{2}\right)$$

$$=\frac{1}{2}\left(\arctan(x) - \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right)$$

$$\boxed{=\frac{1}{2}\left(\arctan(x) - \frac{x}{x^2+1}\right) + C}$$



$x^2+1^2=H^2$

$H=\sqrt{x^2+1}$

so $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$

$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$

g) $\int \frac{x+5}{x^3-2x^2+x}\,dx = \int \frac{x+5}{x(x^2-2x+1)}\,dx = \int \frac{x+5}{x(x-1)(x-1)}\,dx$

$x(x-2)^2$ $\qquad\qquad x(x-1)^2$

$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{x+5}{x(x-1)^2}$

$x+5 = A(x-1)^2 + Bx(x-1) + Cx$

@$x=1$: $6 = C$

@$x=0$: $5 = A$

@$x=2$: $7 = 5(1) + 2B + 6(2) \Rightarrow B = -5$

$\int \left(\frac{5}{x} - \frac{5}{x-1} + \frac{6}{(x-1)^2}\right) dx = \boxed{5\ln|x| - 5\ln|x-1| - \frac{6}{(x-1)} + C}$

h) No solution provided.

4. (i) $\int \sec^3 x \tan^3 x \, dx$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$= \int \sec^3 x \tan^2 x \cdot \tan x \, dx$

$= \int \sec^3 x \, (\sec^2 x - 1) \tan x \, dx$

$= \int (\sec^5 x - \sec^3 x) \tan x \, dx$

$= \int \left(\frac{1}{\cos^5 x} - \frac{1}{\cos^3 x} \right) \frac{\sin x}{\cos x} \, dx$

$= \int \left(\frac{\sin x}{\cos^6 x} - \frac{\sin x}{\cos^4 x} \right) dx$

Let $w = \cos x$

$dw = -\sin x \, dx$

$= -\int \frac{1}{w^6} \, dw + \int \frac{1}{w^4} \, dw$

$= \frac{1}{5} w^{-5} - \frac{1}{3} w^{-3} + C$

$= \frac{1}{5} \cos^{-5} x - \frac{1}{3} \cos^{-3} x + C$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$

#4

j) $\int_{-\pi/4}^{\pi/4} x^2 \sin(x)\, dx$

$u = x^2 \qquad dv = \sin(x)dx$

$du = 2x\,dx \qquad v = -\cos(x)$

$-x^2\cos(x) - \int -\cos(x)\, 2x\, dx$

$-x^2\cos(x) + \int 2x\cos(x)\, dx$

$u = 2x \qquad dv = \cos(x)$

$du = 2 \qquad v = \sin(x)$

$2x\sin(x) - \int 2\sin(x)$

$$\left[-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\right]\Big|_{-\pi/4}^{\pi/4}$$

$$\left[-\pi/4\cos(\pi/4) + \frac{2\pi}{4}\sin(\pi/4) + 2\cos(\pi/4)\right]$$

$$-\left[-\pi/4\cos(-\pi/4) - \frac{2\pi}{4}\sin(-\pi/4) + 2\cos(-\pi/4)\right]$$

$= \underline{\underline{0}}$ We could also know it $\underline{\underline{0}}$

since it odd function and we are integrating from $-\pi/4$ to $\pi/4$.

4.k) $\int \underbrace{e^x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} = I \qquad du = e^x dx \quad v = \sin x$

$I = \sin x \cdot e^x - \int \underbrace{e^x}_{\bar{u}} \cdot \underbrace{\sin x \, dx}_{d\bar{v}} \qquad d\bar{u} = e^x dx, \quad \bar{v} = -\cos x$

$I = \sin x \cdot e^x + \cos x \cdot e^x - \underbrace{\int \cos x \cdot e^x dx}_{I}$

$\Rightarrow 2I = e^x(\sin x + \cos x) \Rightarrow \boxed{I = \frac{e^x(\sin x + \cos x)}{2} + C}$

4.l) $\int x\cos^2 x \, dx = \int x\left(\frac{1+\cos(2x)}{2}\right)dx = \int\left(\frac{x}{2} + \frac{x\cos(2x)}{2}\right)dx$

$\int \frac{x}{2}dx = \frac{x^2}{4} \qquad \frac{1}{2}\int \underbrace{x}_{u}\underbrace{\cos(2x)dx}_{v'} = \frac{x\sin(2x)}{2} - \underbrace{\int \frac{\sin(2x)}{2}dx}_{= +\frac{\cos(2x)}{4}}$

$u' = 1$

$v = \frac{\sin(2x)}{2}$

$= \frac{x^2}{4} + \frac{1}{2}\left(\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right) = \boxed{\frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C}$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x}{2} = \frac{2\sin\theta}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\theta = \sin^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

5) $\int_1 \sqrt{4-x^2}\, dx = \int_{\pi/6} \sqrt{4-(2\sin\theta)^2} \cdot 2\cos\theta \, d\theta = \int_{\pi/6} \sqrt{4(1-\sin^2\theta)} \cdot 2\cos\theta \, d\theta$

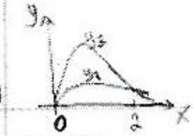
$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$= 2\int_{\pi/6}^{\pi/2} \sqrt{\cos^2\theta} \cdot 2\cos\theta \, d\theta = 2\int_{\pi/6}^{\pi/2} 2\cos^2\theta \, d\theta$

$= 4\int_{\pi/6}^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta = 4\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\Bigg|_{\pi/6}^{\pi/2} = 4(.785 + 0 - .262 - .217)$

$= \boxed{1.228}$

6) 

$y_1 = \frac{x}{x^2+1}$ $\quad y_2 = \frac{5x}{(x^2+1)^2}$

$\frac{x}{x^2+1} = \frac{5x}{(x^2+1)^2}$ how to solve?

$x(x^2+1)^2 = 5x(x^2+1)$

intersect @ $x=2$

$$\int_0^2 \left[\left(\frac{5x}{(x^2+1)^2}\right) - \left(\frac{x}{x^2+1}\right)\right]dx = 5\int_0^2 \frac{x}{(x^2+1)^2}\,dx - \int_0^2 \frac{x}{x^2+1}\,dx$$

$w = x^2+1 \qquad w = x^2+1$

$\frac{1}{2}dw = x\,dx \qquad \frac{1}{2}dw = x\,dx$

$$5\cdot\frac{1}{2}\int_1^5 \frac{1}{w^2}\,dw = \frac{5}{2}\cdot -\frac{1}{w}\Big|_1^5 = -\frac{5}{2w}\Big|_1^5 = -\frac{1}{2}+\frac{5}{2} = 2$$

$$\frac{1}{2}\int_1^5 \frac{1}{w}\,dw = \frac{1}{2}\ln|w|\Big|_1^5 = .8047$$

$2 - .8047 = \boxed{1.195}$

7. $left(4) = 2^0+2^1+2^2+2^3 = 15$

$right(4) = 2^1+2^2+2^3+2^4 = 30$

$$Trap(4) = \frac{left(4)+right(4)}{2} = \frac{15+30}{2} = \boxed{22.5}$$

$Mid(4) = 2^{0.5}+2^{1.5}+2^{2.5}+2^{3.5} \approx 21.21$

$$Simp(4) = \frac{2}{3}Mid(4) + \frac{1}{3}Trap(4)$$
$$= \frac{2}{3}\times 21.21 + \frac{1}{3}\times 22.5 \approx \boxed{21.64}$$

8. (a) $\lim_{x\to 0} \frac{1-\cos 2x}{1-\cos 3x}$ as $x\to 0 \Rightarrow \begin{cases} 1-\cos 2x \to 0 \\ 1-\cos 3x \to 0 \end{cases}$

So LHR: $\lim_{x\to 0} \frac{\sin 2x\cdot 2}{\sin 3x \cdot 3}$ as $x\to 0 \Rightarrow \begin{cases} 2\sin 2x \to 0 \\ 3\sin 3x \to 0 \end{cases}$

So LHR: $\lim_{x\to 0} \frac{\cos 2x\cdot 2\cdot 2}{\cos 3x\cdot 3\cdot 3}$ as $x\to 0$ $\begin{cases} \cos 2x \to 1 \\ \cos 3x \to 1 \end{cases}$

$= \boxed{\frac{4}{9}}$

8(b) $\lim_{x\to\infty} (x^2+1)^{1/\ln x} = y$

$\ln y = \lim_{x\to\infty} \ln\left[(x^2+1)^{\frac{1}{\ln x}}\right]$

$\ln y = \lim_{x\to\infty} \dfrac{\ln(x^2+1)}{\ln x}$ as $x\to\infty \Rightarrow \begin{cases} \ln(x^2+1)\to\infty \\ \ln x\to\infty \end{cases}$

so LHR: $\ln y = \lim_{x\to\infty} \dfrac{\frac{1}{x^2+1}\cdot 2x}{1/x} = \lim_{x\to\infty} \dfrac{2x^2}{x^2+1}$

as $x\to\infty \Rightarrow \begin{cases} 2x^2\to\infty \\ x^2+1\to\infty \end{cases}$ so LHR: $\ln y = \lim_{x\to\infty} \dfrac{4x}{2x}$

$\Rightarrow \ln y = 2 \Rightarrow y = \boxed{e^2}$

9. (a) $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^{\frac{1}{3}}}\,dx \to$ P-series $p = \frac{1}{3} < 1$

$= \lim_{a\to 0^+} \frac{3}{2} x^{2/3}\Big|_a^1 = \frac{3}{2} - 0 = \boxed{\frac{3}{2}}$ $\boxed{\text{converges}}$

(b) $\displaystyle\int_0^6 \frac{1}{(9-2x)^2}\,dx$ $\quad 9-2x \neq 0 \Rightarrow x \neq \frac{9}{2}$

$= \displaystyle\lim_{b\to\frac{9}{2}^-} \int_0^b \frac{1}{(9-2x)^2}\,dx + \lim_{a\to\frac{9}{2}^+} \int_a^6 \frac{1}{(9-2x)^2}\,dx$

$= \displaystyle\lim_{b\to\frac{9}{2}^-} \int_0^b (9-2x)^{-2}\,dx + \lim_{a\to\frac{9}{2}^+} \int_a^6 (9-2x)^{-2}\,dx$

$= \displaystyle\lim_{b\to\frac{9}{2}^-} \frac{1}{2x-9}\Big|_0^b + \lim_{a\to\frac{9}{2}^+} \frac{1}{2x-9}\Big|_a^6 \Rightarrow \boxed{\text{diverge}}$

c) $\displaystyle\lim_{b\to\infty} \int_4^b \frac{1}{x^2-1}\,dx = \lim_{b\to\infty} \int_4^b \frac{1}{(x-1)(x+1)}\,dx$ $\quad \frac{A}{x-1} + \frac{B}{x+1}$

$1 = A(x+1) + B(x-1)$ $\quad$ converges

@ $x=1$: $1 = 2A$ $\quad A = 1/2$

@ $x=-1$: $1 = -2B$ $\quad B = -1/2$

$\displaystyle\lim_{b\to\infty} \int_4^b \left[\frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right] dx =$ $\quad \infty - \infty$ indeterminate form

$= \displaystyle\lim_{b\to\infty} \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\Big|_4^b = \lim_{b\to\infty} \frac{1}{2}(\ln|b-1| - \ln|b+1| - \ln 3 + \ln 5)$

$= \displaystyle\lim_{b\to\infty} \frac{1}{2}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln 3 + \ln 5\right) = \lim_{b\to\infty} \frac{1}{2}\left(\ln\left|\frac{\frac{b}{b} - \frac{1}{b}}{\frac{b}{b} + \frac{1}{b}}\right| - \ln 3 + \ln 5\right)$

$= \frac{1}{2}(\ln 1 - \ln 3 + \ln 5) = \frac{1}{2}(0 - \ln 3 + \ln 5) = \boxed{\frac{1}{2}(\ln 5 - \ln 3)}$

9

d) $\int_3^\infty \frac{dx}{9+x^2}$

$$\lim_{b\to\infty} \int_3^b \frac{dx}{9+x^2}$$

and we know $\int \frac{dx}{1+x^2} = \arctan(x)$

So $\frac{1}{3} \cdot \lim_{b\to\infty} \int_3^b \frac{dx}{1+(\frac{x}{3})^2} \Rightarrow \frac{1}{3} \lim_{b\to\infty} \arctan(\frac{x}{3}) \Big|_3^b$

$$\Rightarrow \frac{1}{3} \lim_{b\to\infty} \left[\arctan(\frac{b}{3}) - \arctan(\frac{3}{3})\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{12}$$

So it Converge to $\boxed{\pi/12}$

9(e) $\int_4^\infty \frac{x}{x^2-1}\,dx = \lim_{b\to\infty^-} \int_4^b \frac{x}{x^2-1}\,dx$ $\quad w = x^2-1$, $dw = 2x\,dx$, $\frac{1}{2}dw = x\,dx$

$= \lim_{b\to\infty^-} \frac{1}{2}\int_{15}^{b^2-1} \frac{1}{w}\,dw \rightarrow p=1 \Rightarrow \boxed{\text{diverges}}$

10. (a) $y = x^4$ for $1 \le x \le 2$ $\quad y' = 4x^3$

$$L = \int_1^2 \sqrt{1 + (4x^3)^2}\,dx = \boxed{\int_1^2 \sqrt{1 + 16x^6}\,dx}$$

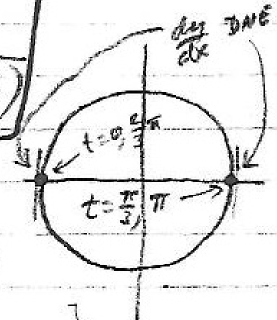
11. $x = 4\cos(3t + \pi)$

$$\frac{dx}{dt} = -4\sin(3t+\pi)\cdot 3 = -12\sin(3t+\pi)$$

$y = 4\sin(3t+\pi)$

$$\frac{dy}{dt} = 4\cos(3t+\pi)\cdot 3 = 12\cos(3t+\pi)$$

$$\Rightarrow \frac{dy}{dx} = \frac{12\cos(3t+\pi)}{-12\sin(3t+\pi)} = \boxed{-\frac{1}{\tan(3t+\pi)}}$$



$\because 0 \le t \le \pi \Rightarrow 0 \le 3t + \pi \le 4\pi$

not exist when $\tan(3t+\pi) = 0$

$\Rightarrow 3t + \pi \ne \{0, \pi, 2\pi, 3\pi, \dots\}$

$\Rightarrow t \ne \{0, \frac{\pi}{3}, \frac{2}{3}\pi, \pi\}$

10b) Express the length of the following curve as an integral (Do not evaluate).

$x = t^2$, $y = \cos(\pi t)$ for $0 \le t \le 1$

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -\pi\sin(\pi t)$, $\displaystyle\int_0^1 \sqrt{(2t)^2 + (-\pi\sin(\pi t))^2}\,dt$

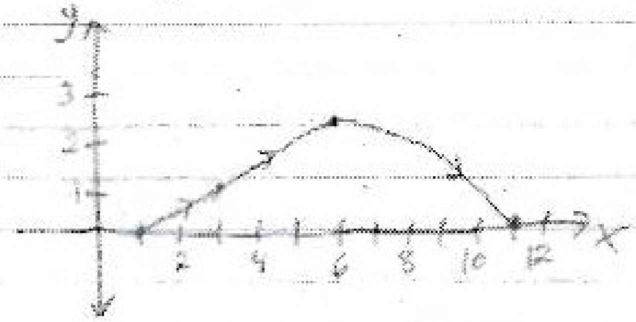
$$= \boxed{\int_0^1 \sqrt{4t^2 + \pi^2\sin^2(\pi t)}\,dt}$$

#11 No solution provided.

12) $x = t^3 + t + 1$ $y = 2t^4 - t^5$ for $0 \le t \le 2$

a)

t	x	y
0	1	0
1	3	1
1.5	5.9	2.5
2	11	0



b) $(3,1)$
x y

plug in 3 for
x + 1 for y +
solve for t

$\left.\dfrac{dx}{dt}\right|_{t=1} = 3(1)^2 + 1 = 4$

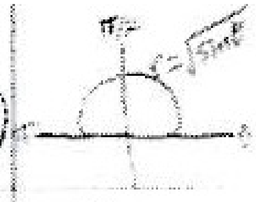
$\left.\dfrac{dy}{dt}\right|_{t=1} = 8(1)^3 - 5(1)^4 = 3$

$\dfrac{dy}{dx} = \dfrac{3}{4}$

$y = y_1 + m(x - x_1)$ $\Rightarrow$ $\boxed{y = 1 + \frac{3}{4}(x-3)}$

#12c No solution provided.

13)



$\sqrt{\sin\theta} = 0 \Rightarrow \sin\theta = 0$

$\theta = 0, \pi$

$$\frac{1}{2}\int_0^{\pi} \left(\sqrt{\sin\theta}\right)^2 d\theta = \frac{1}{2}\int_0^{\pi} \sin\theta \, d\theta = -\frac{1}{2}\cos\theta \Big|_0^{\pi} = -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0)$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

#14 No solution provided.

#15 No solution provided.

16) a) $a_n = \frac{2n^3 - n^2 - 2n^2 + n}{3n^3 + 9n + n^2 + 3} \Rightarrow a_n = \frac{2n^3 - 3n^2 + n}{3n^3 + n^2 + 9n + 3}$

$\lim_{n\to\infty} \frac{2n^3 - 3n^2 + n}{3n^3 + n^2 + 9n + 3} = \lim_{n\to\infty} \frac{\frac{2n^3}{n^3} - \frac{3n^2}{n^3} + \frac{n}{n^3}}{\frac{3n^3}{n^3} + \frac{n^2}{n^3} + \frac{9n}{n^3} + \frac{3}{n^3}}$

$= \lim_{n\to\infty} \frac{2 - \underbrace{\frac{3}{n}}_{\to 0} + \underbrace{\frac{1}{n^2}}_{\to 0}}{3 + \underbrace{\frac{1}{n}}_{\to 0} + \underbrace{\frac{9}{n^2}}_{\to 0} + \underbrace{\frac{3}{n^3}}_{\to 0}} = \boxed{\frac{2}{3}}$ $\boxed{\text{converges}}$

16 (b) $a_n = \frac{\ln(n^2+2n+1)}{n+1} = \frac{\ln(n+1)^2}{n+1} = \frac{2\ln(n+1)}{n+1}$

$\lim_{n\to\infty} 2\frac{\ln(n+1)}{n+1}$ as $n \to \infty \Rightarrow \begin{cases} \ln(n+1) \to \infty \\ n+1 \to \infty \end{cases}$

LHR: $\lim_{n\to\infty} 2\frac{\frac{1}{n+1}}{1} = \lim_{n\to\infty} \frac{2}{n+1}$ as $n \to \infty$ $\frac{2}{n+1} \to 0$

$= \boxed{0}$ $\boxed{\text{Converges}}$

c) $\lim_{n\to\infty} \frac{n!}{5^n}$ $\boxed{\text{diverges}}$ because factorials dominate exponentials, so the numerator increases faster than the denominator.

#16d No solution provided.

17. a) $\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k$ $\quad \frac{a}{1-x} = \frac{\frac{9}{16}}{1-\frac{3}{4}} = \frac{9}{16} \cdot \frac{4}{1} = \frac{36}{16} = \boxed{\frac{9}{4}}$

17b) $\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$ Find the sum

Use partial fractions: $\frac{2}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2} \rightarrow 2 = A(k+2) + Bk$

$2 = k(A+B) + 2A$ $\quad A+B=0$

$2A = 2 \rightarrow \boxed{A=1}$ so $1+B=0$

$\boxed{B=-1}$

$$\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$$

Listing the terms shows that this series is a telescoping series

$(1 - \cancel{1/3}) + (1/2 - \cancel{1/4}) + (\cancel{1/3} - 1/5) + (\cancel{1/4} - 1/6)$

The only terms that don't cancel: 1 and $\frac{1}{2}$

So the sum $= 1 + 1/2 = \boxed{\frac{3}{2}}$

17c. Find the sum of $\sum_{k=0}^{\infty} \frac{(-2)^k}{k!}$

This series behaves like the e^x taylor series:

$$e^x \approx 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

In order to get the original series $\sum_{k=0}^{\infty} \frac{(-2)^k}{k!}$, we can plug in -2 into x in the e^x series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$, which would equal $\boxed{e^{-2}}$

#18 No solution provided.

19. a) $\sum_n \frac{n}{n+10} \rightarrow \frac{n/n}{n/n + 10/n} \rightarrow \frac{1}{1+\frac{10}{n}} \rightarrow 1$ ($\frac{10}{n} \rightarrow 0$)

- terms don't go to zero. therefore the series $\boxed{\text{diverges}}$

b). $\sum_k \frac{1}{k^2+3k+2} < \sum_k \frac{1}{k^2}$

- $\sum_k \frac{1}{k^2}$ converges because $p > 1$: p-series
- therefore, using the comparison test the original $\boxed{\text{converges}}$ as well.

19 c) $\sum \frac{2^k}{k^4}$ take limit of terms

$\Rightarrow \lim_{k\to\infty} \frac{2^k}{k^4}$ $\left\{\begin{array}{l}\text{Use L'Hospital's}\\ \text{Rule}\end{array}\right\}$ $= \lim_{k\to\infty} \frac{\ln(2)2^k}{4k^3}$

$= \ldots$ Using L'Hospital's Rule $\ldots = \lim_{k\to\infty} \frac{(\ln(2))^4 2^k}{4!} = \infty$ DNE

Since terms do not tend to zero $\Rightarrow \sum \frac{2^k}{k^4}$ $\boxed{\text{Diverges}}$

19) d) $\sum \frac{k!}{k^k}$ Ratio Test

$L = \lim_{k\to\infty} \left| \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{k^k}{k!} \right| = \lim_{k\to\infty} \frac{(k+1)\cdot k^k}{(k+1)^{(k+1)}} = \lim_{k\to\infty} \frac{k^k}{(k+1)^k}$

$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e}$ (Definition of $e = \lim_{k\to\infty}(1+\frac{1}{k})^k$)

$\therefore$ Series $\boxed{\text{Converges}}$ $(L = \frac{1}{e} < 1)$

19) e) $\frac{1}{e} + \frac{4}{e^2} + \frac{9}{e^3} + \frac{16}{e^4} + \ldots + = \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ Ratio Test

$\Rightarrow \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{en^2} = \frac{1}{e}$ $\therefore$ $\boxed{\text{Converges}}$ $(L = \frac{1}{e} < 1)$

f) $\sum_{n=1}^{\infty} \frac{(\ln n)^5}{n}$ Integral Test

$\Rightarrow \int_1^{\infty} \frac{(\ln x)^5}{x} dx$ $\quad w = \ln x$, $dw = \frac{1}{x} dx$ $\Rightarrow \lim_{b\to\infty} \int w^5 dw = \lim_{b\to\infty} \frac{w^6}{6} \Big|$

$= \lim_{b\to\infty} \frac{(\ln x)^6}{6} \Big|_1^{\infty} = \infty$ DNE $\therefore$ Series $\boxed{\text{Diverges}}$

20) a) $\sum \frac{(-1)^k k}{2^k}$ Test for absolute convergence

$\Rightarrow \sum \frac{k}{2^k}$ Ratio Test

$\Rightarrow L = \lim_{k\to\infty} \left| \frac{(k+1)}{2^{(k+1)}} \cdot \frac{2^k}{k} \right| = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2}$, Converges$(L = \frac{1}{2} < 1)$

$\therefore$ Series is $\boxed{\text{absolutely Convergent}}$

b) $\sin 1 + \sin\frac{1}{4} + \sin\frac{1}{9} + \ldots + = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ Limit Comparison Test

Compare to $\sum \frac{1}{n^2}$ which Converges (p-series, $p=2>1$)

$L = \lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{1/n^2} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{\cos(\frac{1}{n^2})(-2n^{-3})}{(-2n^{-3})} = \lim_{n\to\infty} \cos(\frac{1}{n^2}) = 1 > 0$

$\therefore$ Series $\boxed{\text{Converges}}$

c) $\sum \frac{(-1)^k k}{2k-1}$ Test abs

$\Rightarrow \sum \frac{k}{2k-1}$ which Diverges $\left(\lim_{k\to\infty} \frac{k}{2k-1} = \frac{1}{2} \neq 0\right)$

Test conditional convergence

Let $a_k = \frac{k}{2k-1}$

$\lim_{k\to\infty} \frac{k}{2k-1} = \frac{1}{2} \therefore \boxed{\text{Diverges}}$ (terms don't tend to zero)

d) $\sum \frac{(-1)^k}{3k-2}$ Test abs $\Rightarrow \sum \frac{1}{3k-2} > \frac{1}{3}\sum \frac{1}{k}$ (Diverges, harmonic series)

Test conditional Convergence

let $a_k = \frac{1}{3k-2}$, $a_{k+1} = \frac{1}{3(k+1)-2} = \frac{1}{3k+1} < \frac{1}{3k-2} \; \forall k > 2$

$\lim_{k\to\infty} \frac{1}{3k-2} = 0 \therefore \boxed{\text{Conditionally Convergent}}$ (Alt. series test)

20e) $\sum (-1)^n \frac{1}{1+n^{3/2}}$ Check Abs

$\Rightarrow \sum \frac{1}{1+n^{3/2}} < \sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2 > 1$)

$\therefore$ Series is $\boxed{\text{Absolutely Convergent}}$

21. (a) $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$

Binomial: $P_3(x) = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\times\left(-\frac{2}{3}\right)}{2!}x^2 + \frac{\left(-\frac{2}{9}\right)\left(-\frac{5}{3}\right)}{3!}x^3$

$$= \boxed{1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3}$$

Math 150 B Semester Review question # 21 Part B

Find the taylor Polynomial of order 3 based at $a=0$ for each of the following:

B) $\sin(x^2)$

A) first thing to realize is we know the taylor series for $\sin(x)$ which is similar so lets write its first three terms: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

B) we are trying to find the first three terms of $\sin(x^2)$ so we can relate it to the basic taylor series for $\sin(x)$:

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} = x^2 - \frac{x^6}{3\cdot 2\cdot 1} + \frac{x^{10}}{(5\cdot 4\cdot 3\cdot 2\cdot 1)}$$

$$= \boxed{x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}}$$

22

a) Since

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$$

$$\int \frac{d}{dx}\ln(1+x)\,dx = \int \frac{1}{1+x}\,dx$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

Maclaurin series for $\frac{1}{1+x}$ using the fact that $\frac{1}{1+x}$ is a binomial series with $p=-1$ then:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$$\ln(1+x) = \int (1 - x + x^2 - x^3 + x^4 + \ldots)\,dx$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + C}$$

where $C=0$, plug in $x=0$ in both sides of the equation $\therefore$ $\ln(1) = C$

b) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\left(\frac{x^n}{n}\right) + \ldots$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{x^n}{n}\right)$$

Ratio Test

$$L = \lim_{n\to\infty} \frac{|x^{n+1}/n+1|}{|x^n/n|}$$

$$= \lim_{n\to\infty} \frac{|(x^{n+1})(n)|}{|(x^n)(n+1)|}$$

$$= \lim_{n\to\infty} \frac{|(x)(n)|}{|n+1|}$$

$$\lim_{n\to\infty} \frac{|x\cancel{n}|}{|n/n + 1/n|}$$

$$\lim_{n\to\infty} \frac{|x|}{1+\frac{1}{n}} \quad \frac{1}{n}\to 0 \quad = |x| < 1$$

$\boxed{\text{Radius of convergence is } 1}$

23) a) $\sum \frac{(-1)^k x^k}{3k}$ Ratio Test

$$L = \lim_{k\to\infty} \left|\frac{x^{(k+1)}}{3(k+1)} \cdot \frac{3k}{x^k}\right| = \lim_{k\to\infty} \frac{|x|k}{k+1} = |x| < 1 \text{ (ROC)}$$

Check endpoints:

$x=1 \to \sum \frac{(-1)^k 1^k}{3k}$ Converges AST

$x=-1 \to \sum \frac{(-1)^k(-1)^k}{3k} = \sum\frac{(-1)^{2k}}{3k} = \sum \frac{1}{3k}$ (Defenition of even number)

$= \frac{1}{3} \sum \frac{1}{k}$ Diverges (Harmonic Series)

$\therefore$ IOC is $\boxed{(-1,1]}$

23 (b) $\sum_k 2^k (x-3)^{2k}$

$$L = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{2^{k+1}(x-3)^{2k+2}}{2^k(x-3)^{2k}}\right|$$

$$= |2(x-3)^2|$$

$$2|(x-3)^2|<1$$

$$|(x-3)^2| < \frac{1}{2}$$

$$|x-3|<\sqrt{\frac{1}{2}}$$

$$-\sqrt{\frac{1}{2}}+3 < x < \sqrt{\frac{1}{2}}+3$$

Check Endpoints:

When $x=\sqrt{\frac{1}{2}}+3$

$$\sum_k 2^k\left(-\sqrt{\frac{1}{2}}\right)^{2k} = \sum_k 2^k\cdot\left(\frac{1}{2}\right)^k = \sum_k 2^k\cdot 2^{-k} = \sum_k 1$$

Diverge

When $x=-\sqrt{\frac{1}{2}}+3$

$$\sum_k 2^k\left(\sqrt{\frac{1}{2}}\right)^{2k} = \sum_k 2^k\cdot\left(\frac{1}{2}\right)^k = \sum_k 2^k\cdot 2^{-k} = \sum_k 1$$

Diverge

(24)

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^2 + x^4 - x^5 + \cdots$$

$$(1+x)^{-1} = \sum_{i=1}^{\infty} (-1)^{(i-1)} x^{(i-1)}$$

$$\frac{d}{dx}\left[(1+x)^{-1}\right] = \frac{d}{dx} \sum_{i=1}^{\infty} (-1)^{(i-1)} \cdot x^{(i-1)}$$

$$-1 \cdot (1+x)^{-2} = \sum_{i=1}^{\infty} (-1)^{(i-1)} \cdot \frac{d}{dx}\left(x^{i-1}\right)$$

$$-1 \cdot \frac{1}{(1+x)^2} = \sum_{i=1}^{\infty} (-1)^{(i-1)} (i-1) \cdot x^{i-2}$$

$$\frac{1}{(1+x)^2} = \sum_{i=1}^{\infty} (-1)^{(i-1)} \cdot (-1)^1 \cdot (i-1) \cdot x^{i-2}$$

$$\frac{1}{(1+x)^2} = \sum_{i=1}^{\infty} (-1)^{i} \cdot (i-1) x^{i-2}$$

$$= 0 + 1$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 +$$

#25: No solution provided.

#26

26. Use a Maclaurin polynomial to approximate $e^{-1/3}$ with an error less than 0.001. Show how you determined the accuracy.

$$|E_n(x)| \le \frac{\max f^{(n+1)}(x)}{(n+1)!} \cdot |x-a|^{n+1}$$

$$|E_n(x)| \le \frac{1}{(n+1)!} \left|-\frac{1}{3}\right|^{n+1} < \frac{1}{1{,}000}$$

Holds for $n = 3$

$$e^{(x)} = \sum_{i=0}^{n} \frac{x^i}{i!}$$

$$e^{-\frac{1}{3}} = \sum_{i=0}^{n} \frac{\left(-\frac{1}{3}\right)^i}{i!} = 1 - \frac{1}{3} + \frac{1}{2}\cdot\left(\frac{1}{9}\right) - \frac{1}{6}\cdot\left(\frac{1}{27}\right) + \frac{1}{24}\left(\frac{1}{81}\right) \ldots$$

$$e^{\left(-\frac{1}{3}\right)} = \sum_{i=0}^{n} \frac{(-1)^i}{i!\cdot 3^i} = 1 - \frac{1}{3} + \frac{1}{2\cdot 9} - \frac{1}{6\cdot 27} + \ldots$$

$$e^{-\frac{1}{3}} \approx \underbrace{1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162}}$$

$$0.71653 \approx 0.71605$$

$$\therefore e^{-\frac{1}{3}} - 0.71605 = 0.00048 < 0.0010$$