Rudin-Shapiro-Like Sequences with Low Correlation

Daniel J. Katz, Sangman Lee, and Stanislav A. Trunov
Department of Mathematics
California State University, Northridge

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Binary Sequences

Binary Sequence: sequence of elements from $\{-1, +1\}$:

$$f = (f_0, f_1, \ldots, f_{\ell-1}) \in \{-1, +1\}^{\ell}$$

Our sequences are finite length: the above has length ℓ

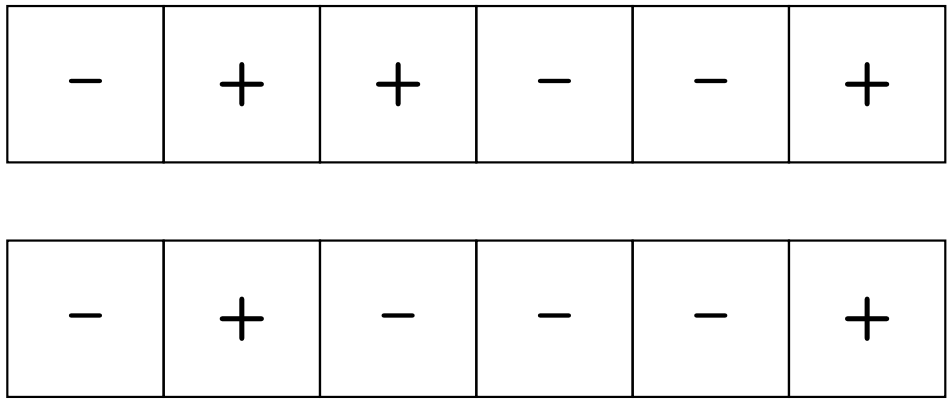
A binary sequence of length 6:

$$\mathbf{(-1,\ +1,\ +1,\ -1,\ -1,\ +1)}$$

We do not write the 1's, just the signs:

−	+	+	−	−	+

Correlation: Inner Product Measuring Similarity



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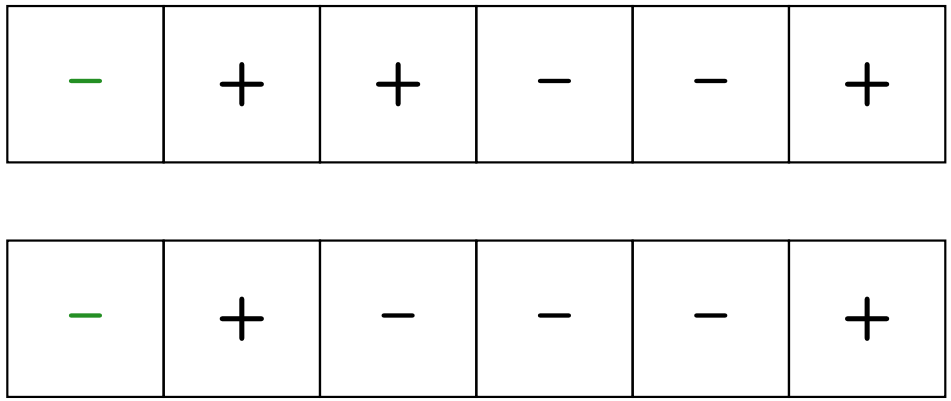

$+1$

Correlation: Inner Product Measuring Similarity

−	+	+	−	−	+

−	+	−	−	−	+

$+1 \quad +1$

Correlation: Inner Product Measuring Similarity

−	+	+	−	−	+

−	+	−	−	−	+

$+1 \quad +1 \quad -1$

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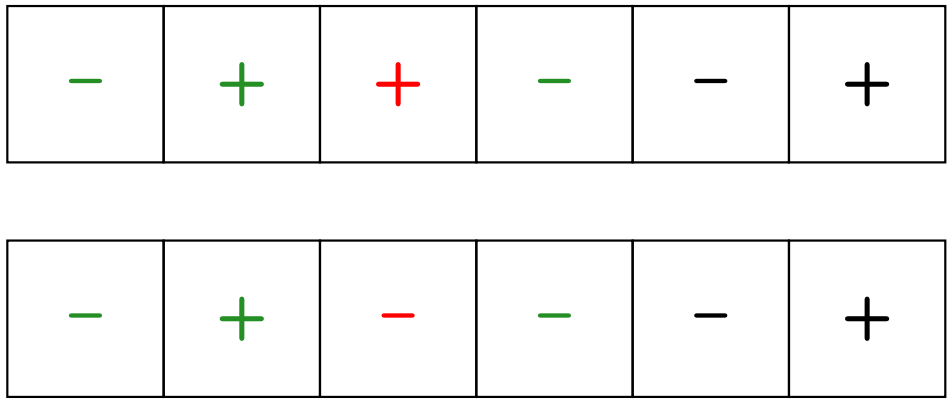

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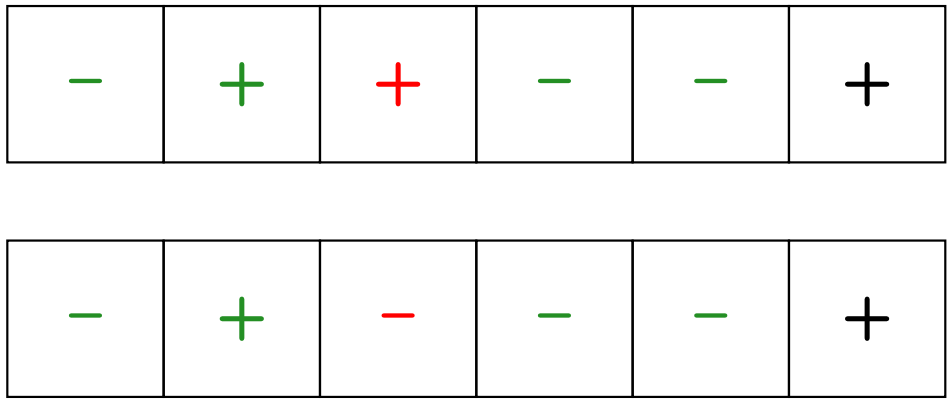

Correlation: Inner Product Measuring Similarity

−	+	+	−	−	+

−	+	−	−	−	+

+1 +1 −1 +1 +1 +1

Correlation: Inner Product Measuring Similarity

−	+	+	−	−	+

−	+	−	−	−	+

$$+1 \quad +1 \quad -1 \quad +1 \quad +1 \quad +1 \quad = \quad +4$$

5 Agreements: score $+5$

1 Disagreement: score -1

Overall Score: $(+5) + (-1) = +4$

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−	+	−	−	−	+

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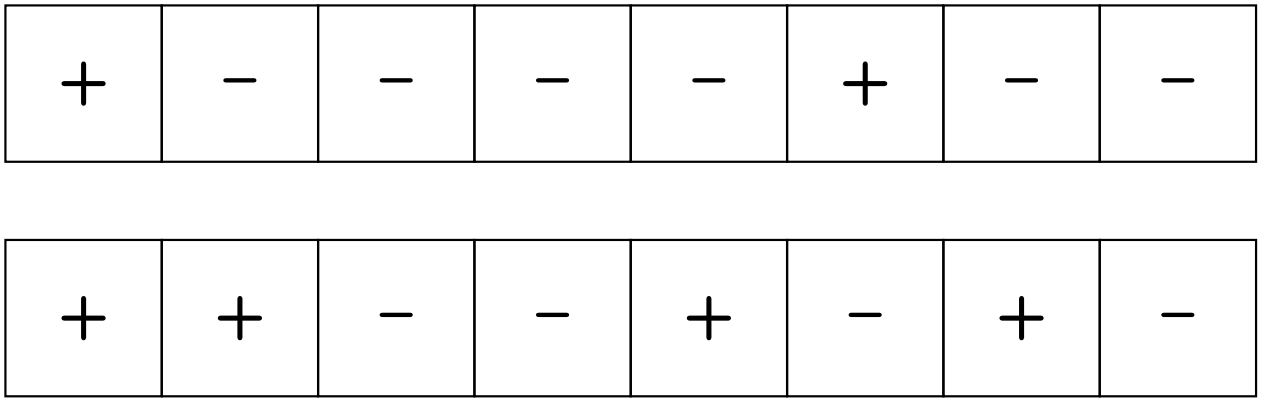
Overall Score: $(+5) + (-1) = +4$

Overall Score $+4$ indicates significant agreement

Shifting



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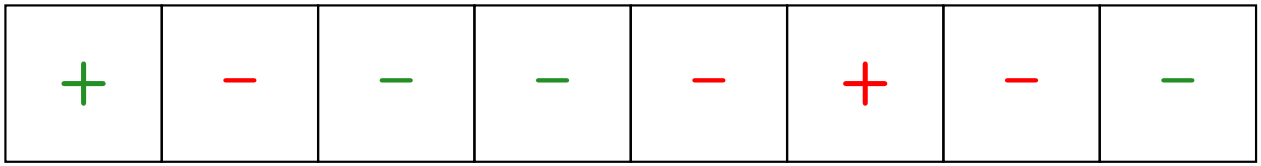
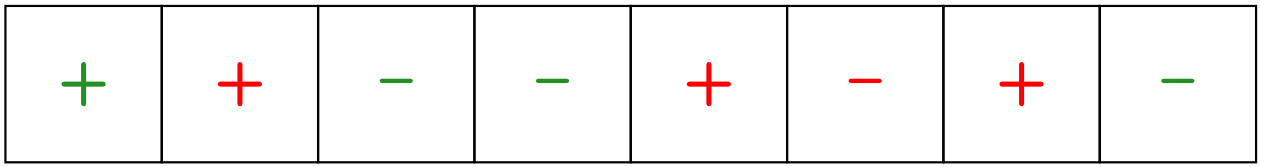




Shifting

+	−	−	−	−	+	−	−

+	+	−	−	+	−	+	−

$$+1 \quad -1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1 \quad +1 \quad = \quad 0$$

Unshifted Correlation: 0 (no resemblance)

Shifting

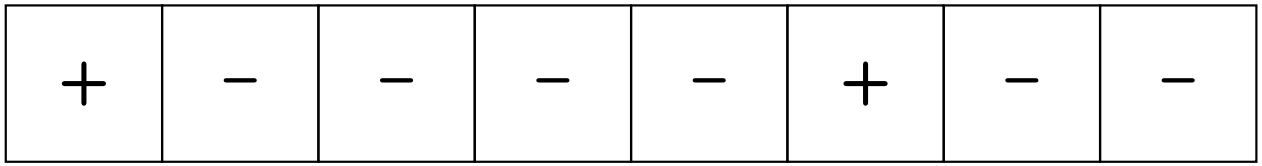
+	−	−	−	−	+	−	−

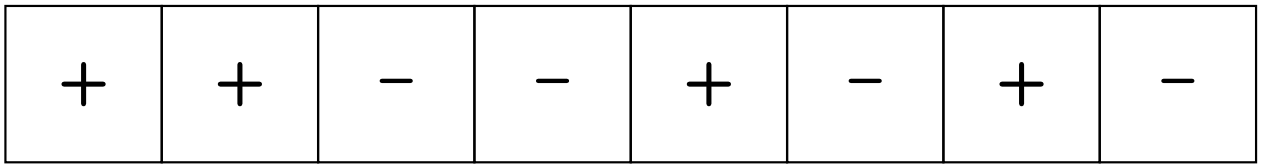
+	+	−	−	+	−	+	−

Unshifted Correlation: 0 (no resemblance)

Shifting

Top sequence is shifted one place to the right





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Aperiodic Shifting: ignore non-overlapping portions

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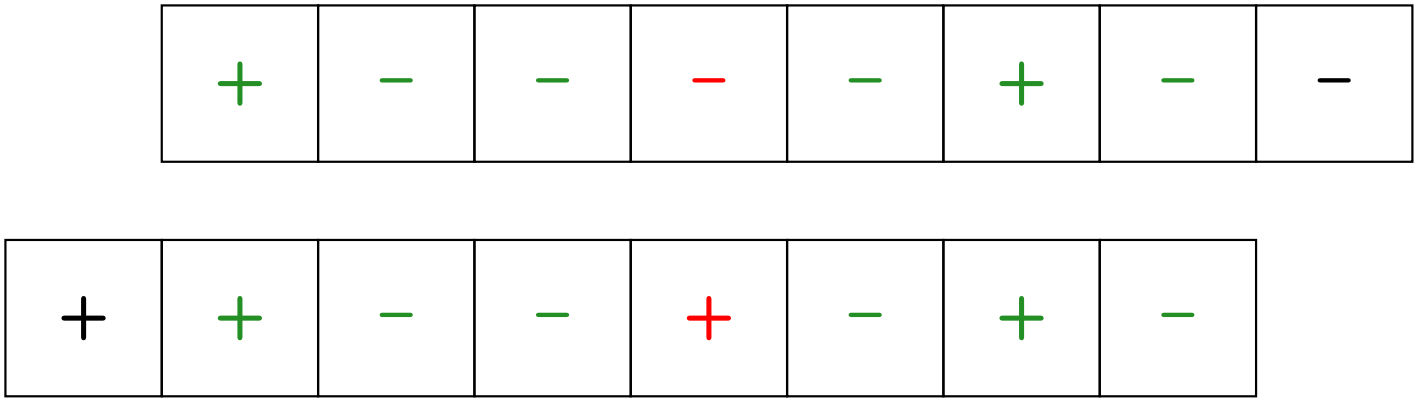

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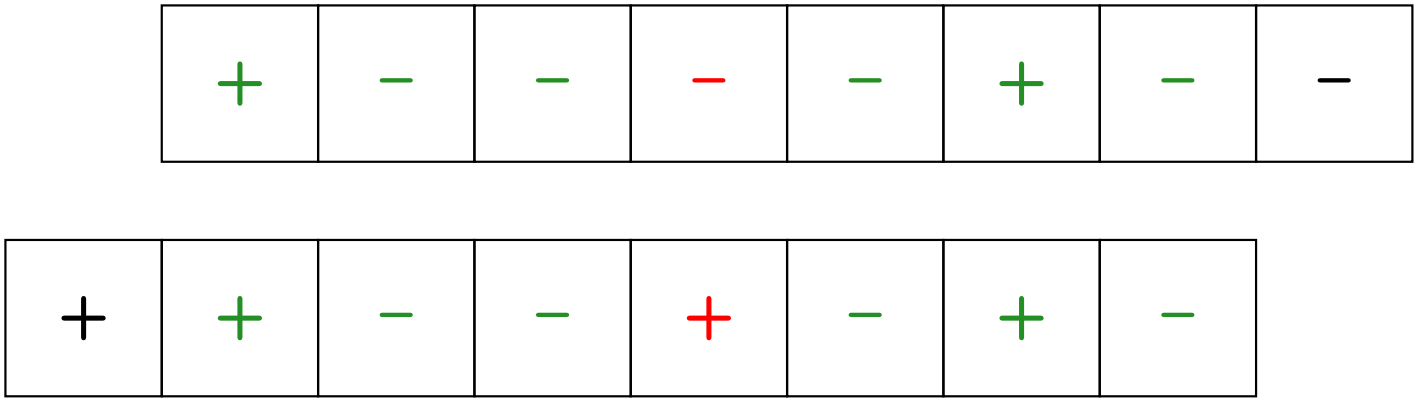

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Unshifted Correlation: 0 (no resemblance)

Aperiodic Shifting: ignore non-overlapping portions

Shifted Correlation: +5 (high resemblance)

Sequences resemble each other when shifted by a shift of +1

Aperiodic Crosscorrelation of Sequences

For two sequences

$$f = (f_0, f_1, \ldots, f_{\ell-1}) \in \{-1, +1\}^\ell$$
$$g = (g_0, g_1, \ldots, g_{\ell-1}) \in \{-1, +1\}^\ell$$

Aperiodic crosscorrelation of f with g at shift s (for any integer s):

$$C_{f,g}(s) = \sum_{j=-\infty}^{\infty} f_j \, g_{j+s}$$

(where we consider $f_j = g_j = 0$ when $j \notin \{0, 1, \ldots, \ell - 1\}$)

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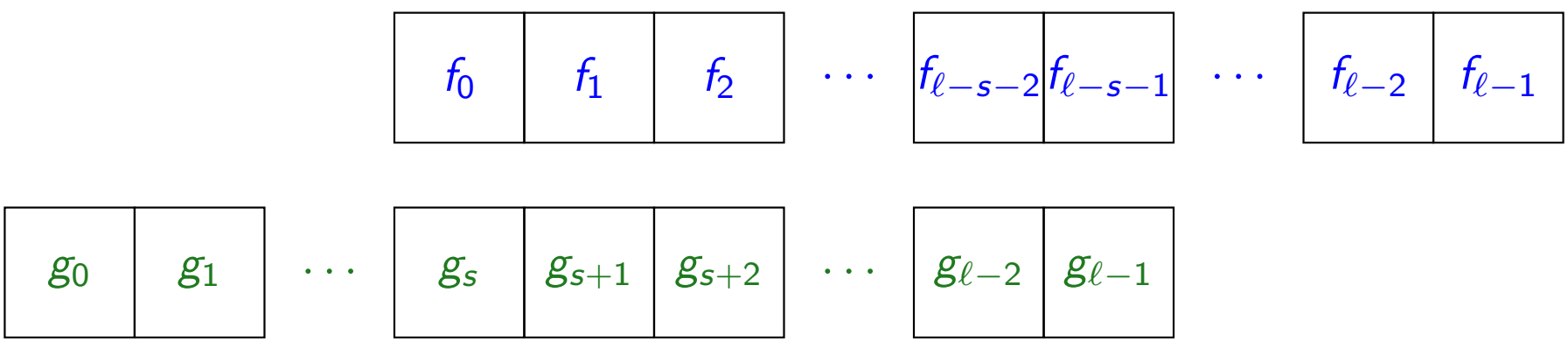

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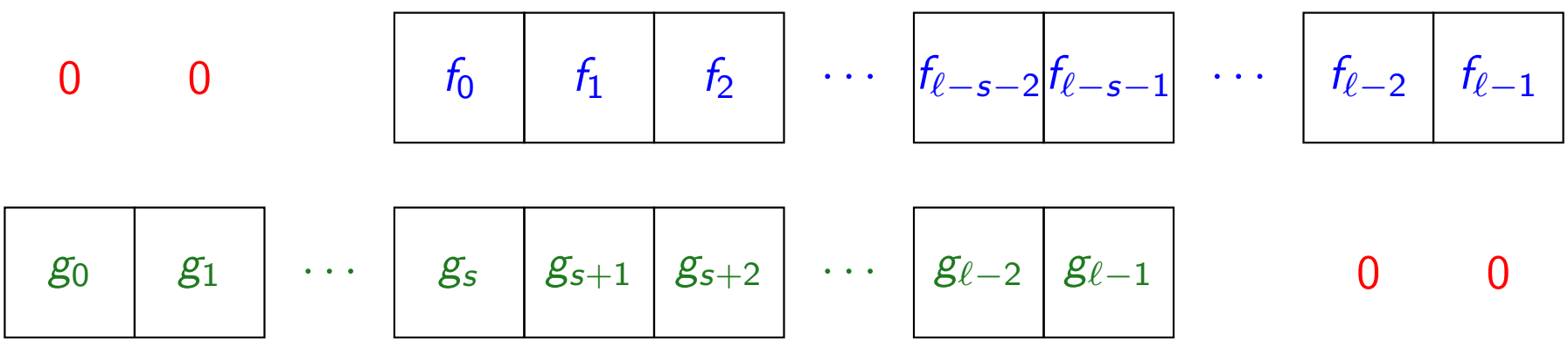

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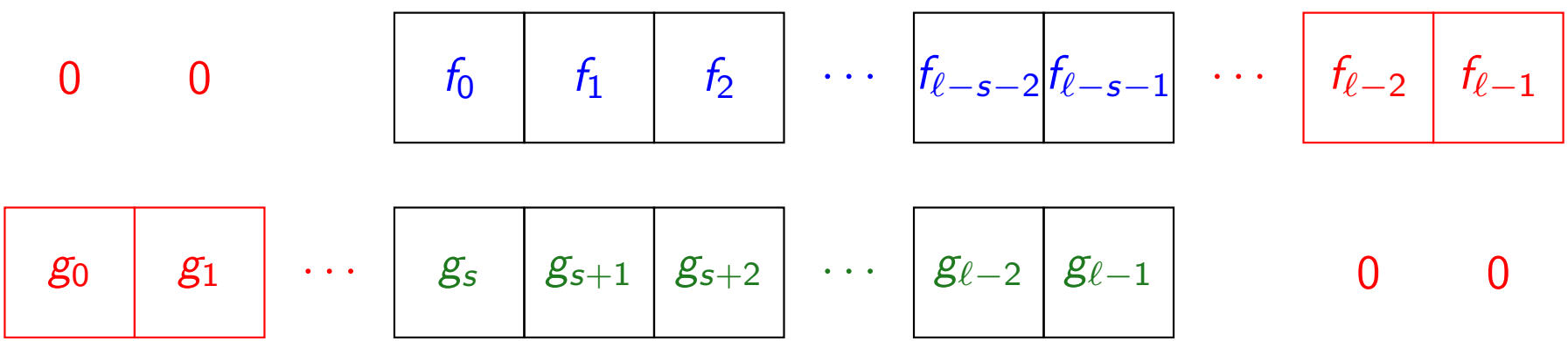

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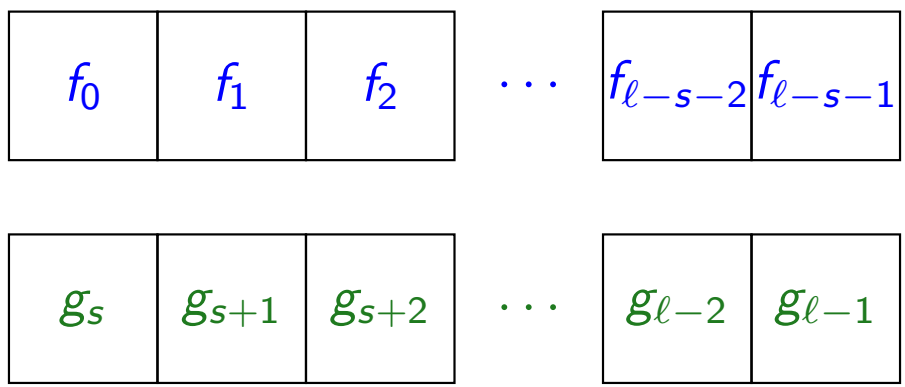


Autocorrelation

Aperiodic Autocorrelation = aperiodic crosscorrelation of a sequence with itself; for

$$f = (f_0, f_1, \ldots, f_{\ell-1}) \in \{+1, -1\}^\ell$$

the autocorrelation of f at shift s is

$$C_{f,f}(s) = \sum_{j=-\infty}^{\infty} f_j \, f_{j+s}$$

(where we consider $f_j = 0$ when $j \notin \{0, 1, \ldots, \ell - 1\}$)

Since our $f_0, \ldots, f_{\ell-1}$ are $+1$'s and -1's, at shift $s = 0$

$$C_{f,f}(0) = \sum_{j=-\infty}^{\infty} {f_j}^2 = \sum_{j=0}^{\ell-1} {f_j}^2 = \ell = \text{ length of sequence } f$$

Design Goal

Our sensors can only measure the absolute value of the correlation

Goal: We want pairs of binary sequences (f, g) with

- Low autocorrelation for f:
  $|C_{f,f}(s)|$ small for all $s \neq 0$
  (recall $|C_{f,f}(0)|$ = length of f = large)

  Assists synchronization for f
- Low autocorrelation for g:
  $|C_{g,g}(s)|$ small for all $s \neq 0$
  (recall $|C_{g,g}(0)|$ = length of g = large)

  Assists synchronization for g
- Low crosscorrelation between f and g:
  $|C_{f,g}(s)|$ small for all s

  Prevents confusion of f with g

Demerit and Merit Factors

Use sum of squared magnitudes compared to sequence length ℓ to measure overall smallness of correlation

Crosscorrelation demerit factor of sequences f and g:

$$\mathrm{CDF}(f,g) = \frac{\sum_{s=-\infty}^{\infty} |C_{f,g}(s)|^2}{\ell^2}$$

Crosscorrelation merit factor

$$\mathrm{CMF}(f,g) = \frac{1}{\mathrm{CDF}(f,g)}$$

Autocorrelation demerit factor of sequence f:

$$\mathrm{ADF}(f) = \frac{\sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} |C_{f,f}(s)|^2}{\ell^2} = \mathrm{CDF}(f,f) - 1$$

Autocorrelation merit factor

$$\mathrm{AMF}(f) = \frac{1}{\mathrm{ADF}(f)}$$

Typical Demerit Factors

Sarwate (1984) calculated expected values of demerit factors

for randomly selected binary sequences f of length ℓ:

Average autocorrelation demerit factor

$$E[\mathrm{ADF}(f)] = 1 - \frac{1}{\ell}$$

Average crosscorrelation demerit factor for sequences f and g of length ℓ:

$$E[\mathrm{CDF}(f, g)] = 1$$

So typical values will be 1 for both autocorrelation and crosscorrelation when ℓ is large.

Pursley-Sarwate Criterion

For binary sequences f and g, Pursley and Sarwate (1976) proved

$$\sqrt{\mathrm{ADF}(f)\,\mathrm{ADF}(g)} + \mathrm{CDF}(f,g) \geq 1$$

The proof is based on the Cauchy-Schwarz inequality

We define the *Pursley-Sarwate Criterion*

$$\mathrm{PSC}(f,g) = \sqrt{\mathrm{ADF}(f)\,\mathrm{ADF}(g)} + \mathrm{CDF}(f,g)$$

For typical randomly selected sequence pairs, we expect

$$\mathrm{PSC}(f,g) \approx 2$$

Goal: make $\mathrm{PSC}(f,g)$ as close to 1 as possible

Rudin-Shapiro-Like Sequences

Rudin-Shapiro-Like Sequences come in families of the form

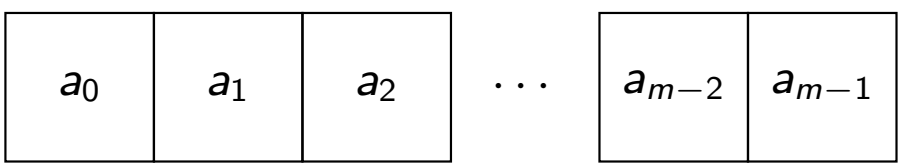
$$f^0, f^1, f^2, \ldots$$

where the length of f^{n+1} is double that of f^n.

- The initial sequence f^0 is called the seed sequence
- The family of sequences $f^0, f^1, f^2, \ldots$ is called the stem

Construction

- Choose a binary sequence f^0 to be the seed
- Once $f^n = (a_0, a_1, \ldots, a_{m-1})$ is defined,



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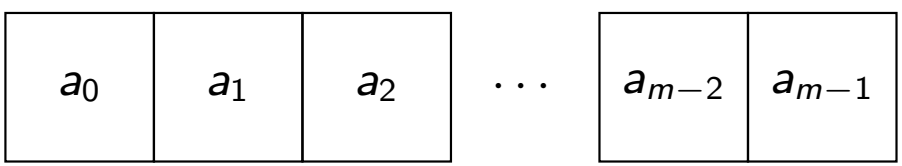

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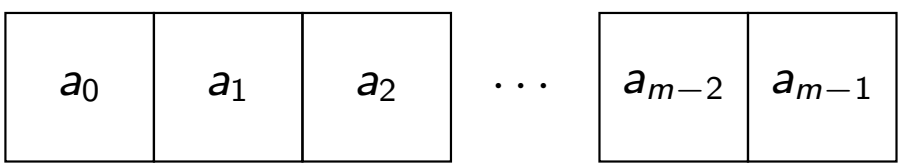

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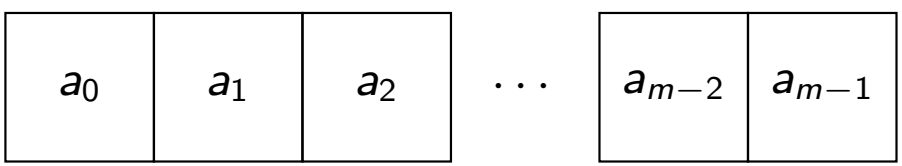

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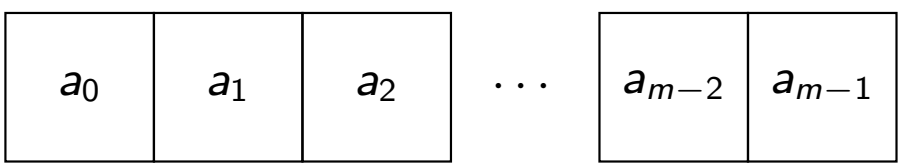

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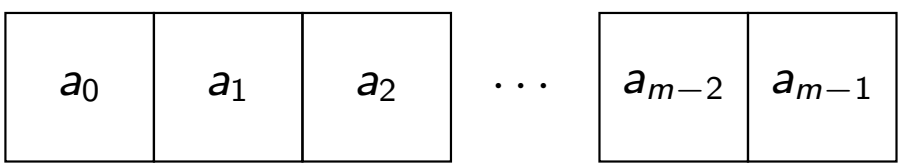

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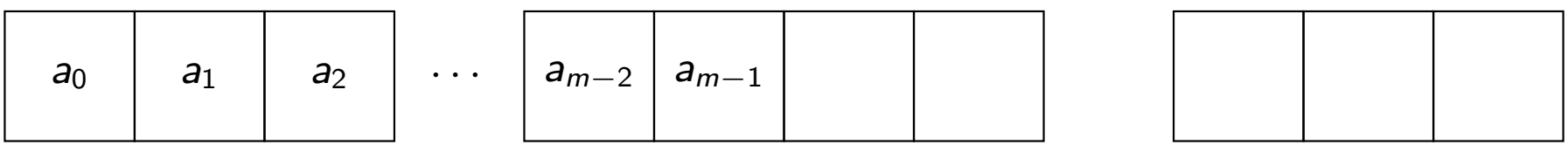

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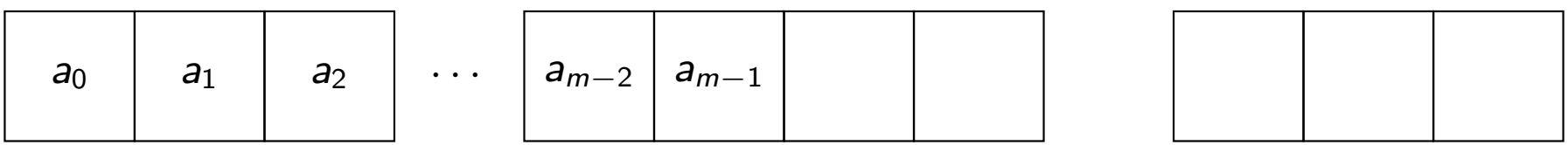

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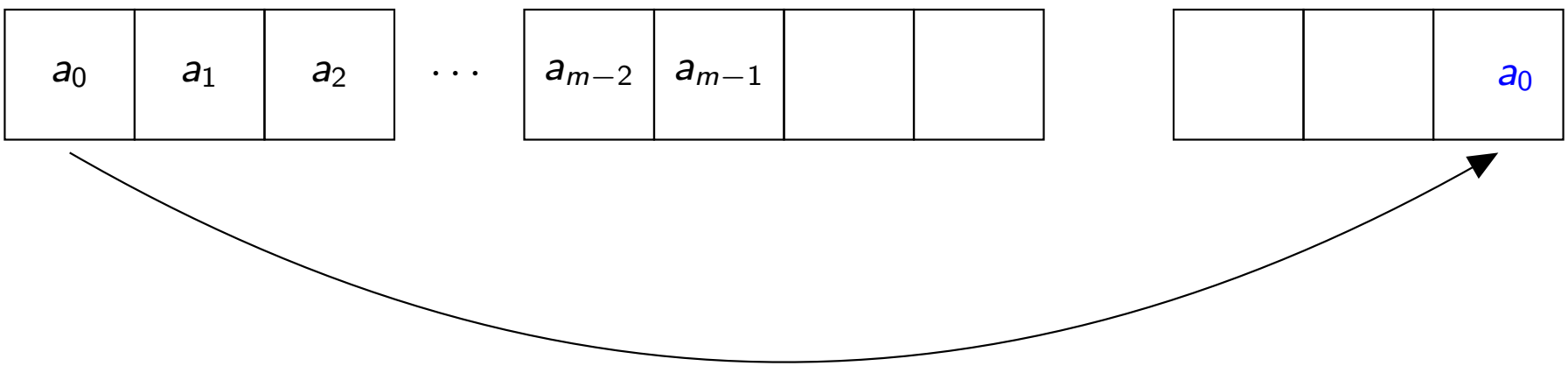

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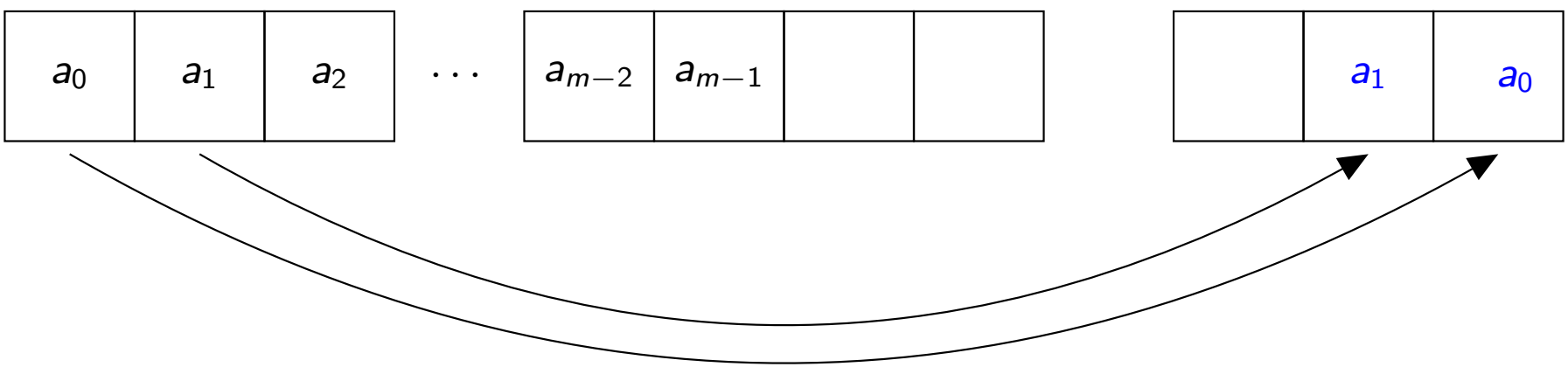

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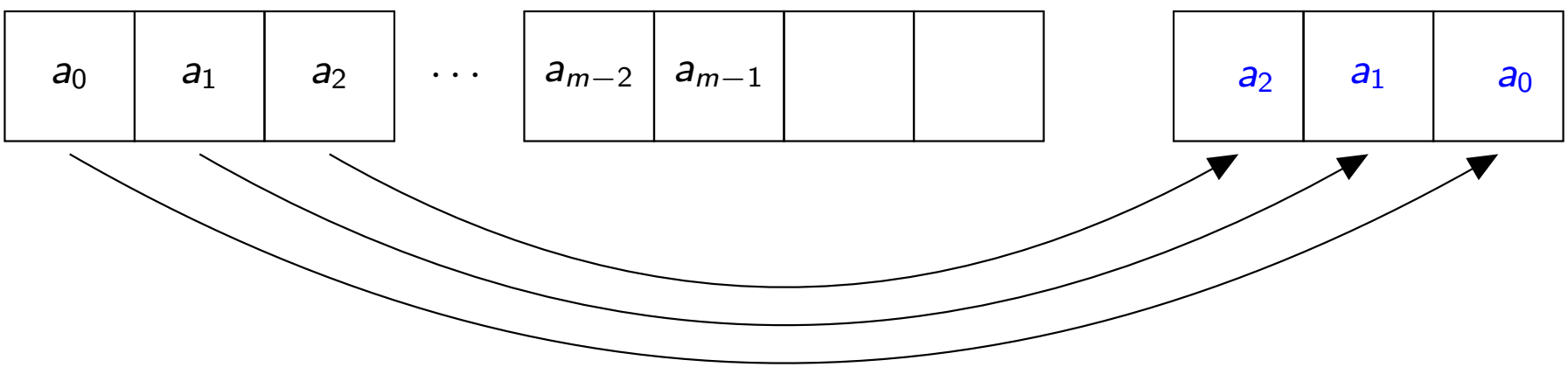

Rudin-Shapiro-Like Sequences

Rudin-Shapiro-Like Sequences come in families of the form

$$f^0, f^1, f^2, \ldots$$

where the length of f^{n+1} is double that of f^n.

- The initial sequence f^0 is called the seed sequence
- The family of sequences $f^0, f^1, f^2, \ldots$ is called the stem

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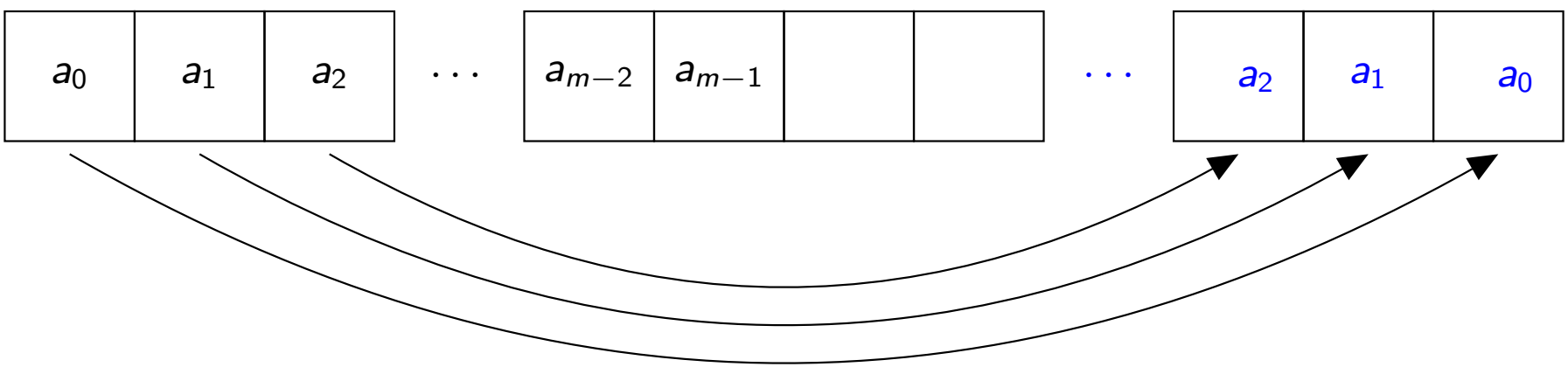

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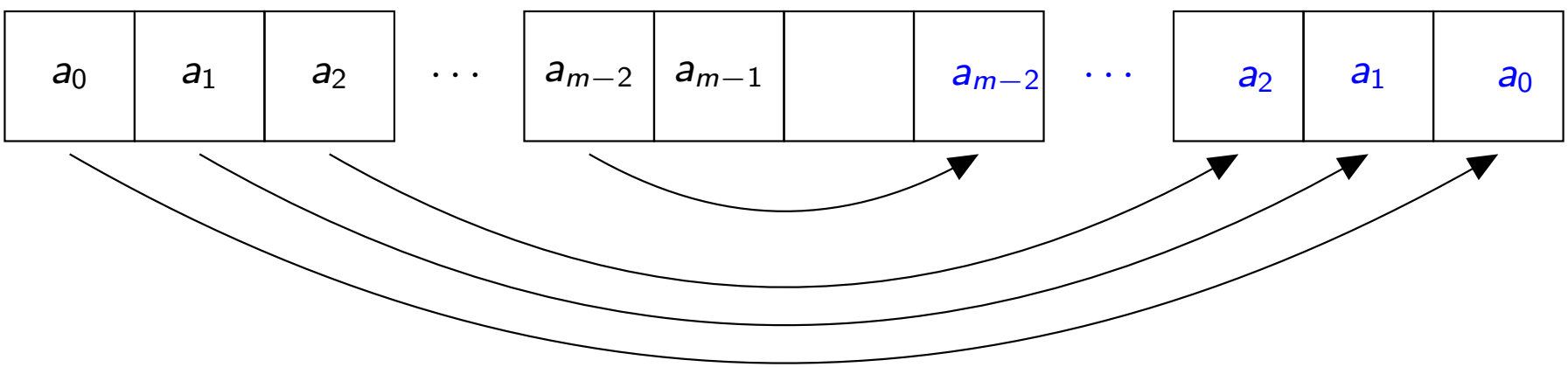

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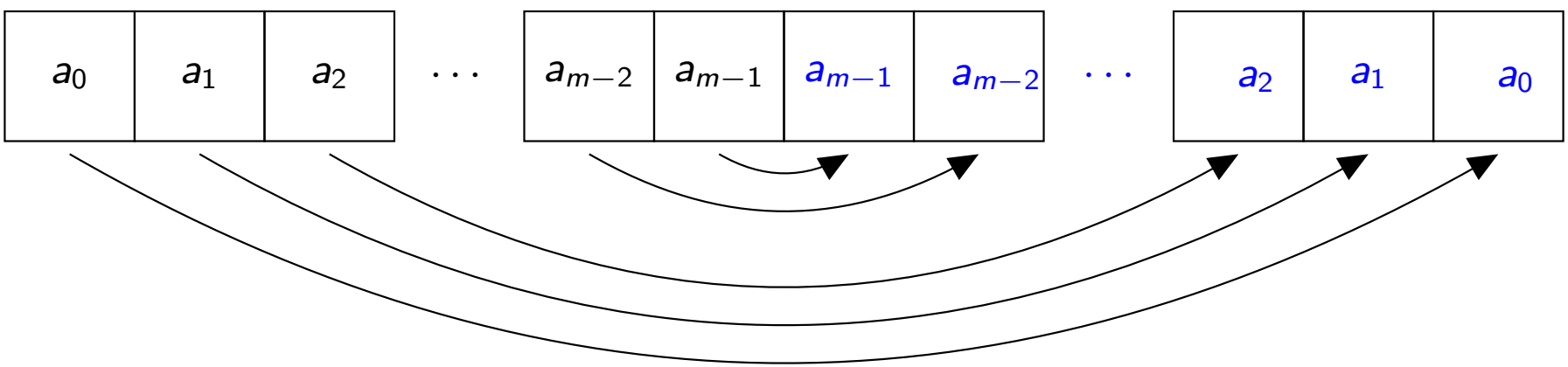

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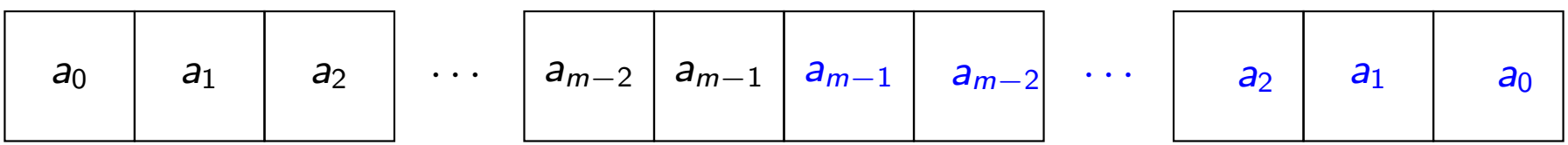

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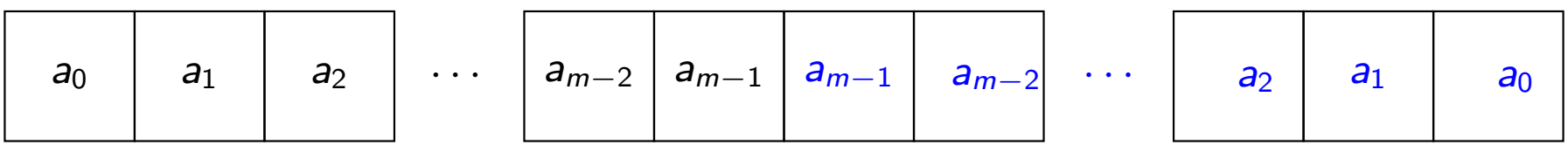

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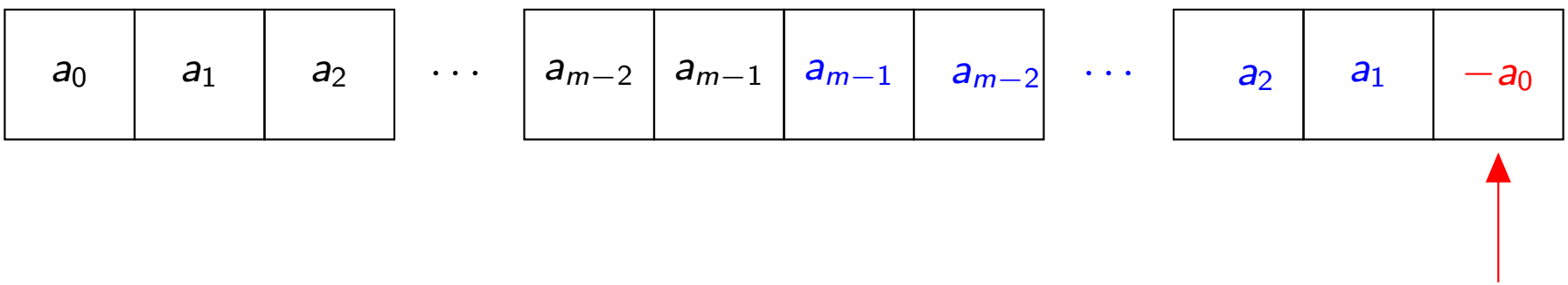

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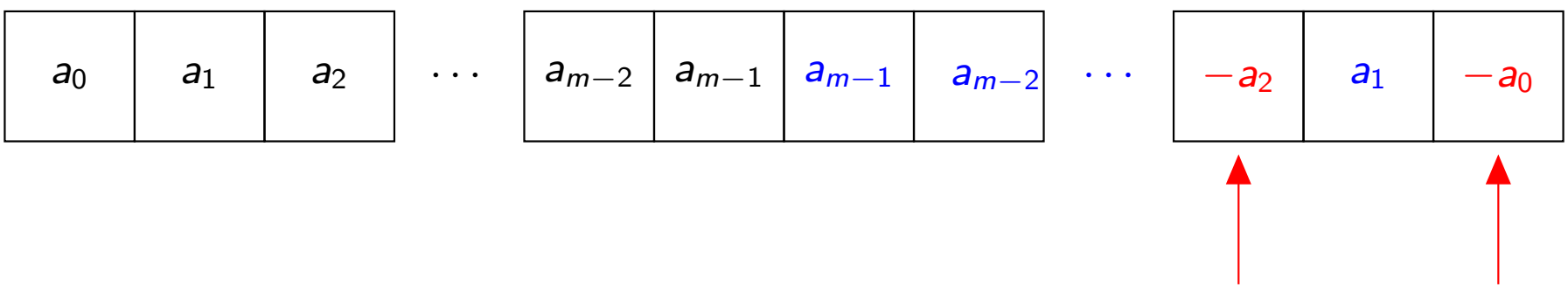

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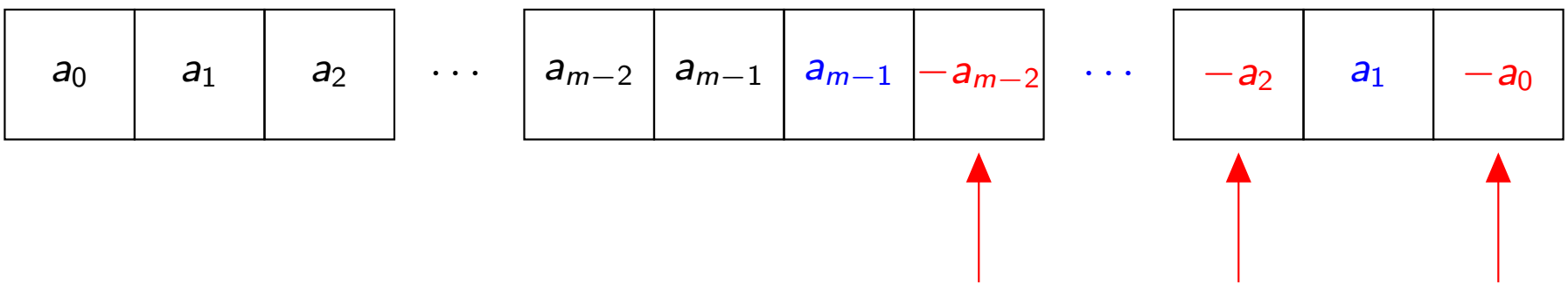
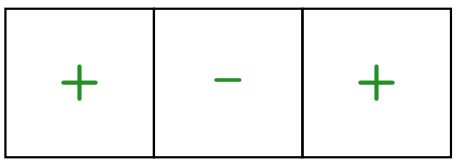


Example of Doubling Construction

Suppose we have the following seed sequence: $f^0 = (+1, -1, +1)$

f^0



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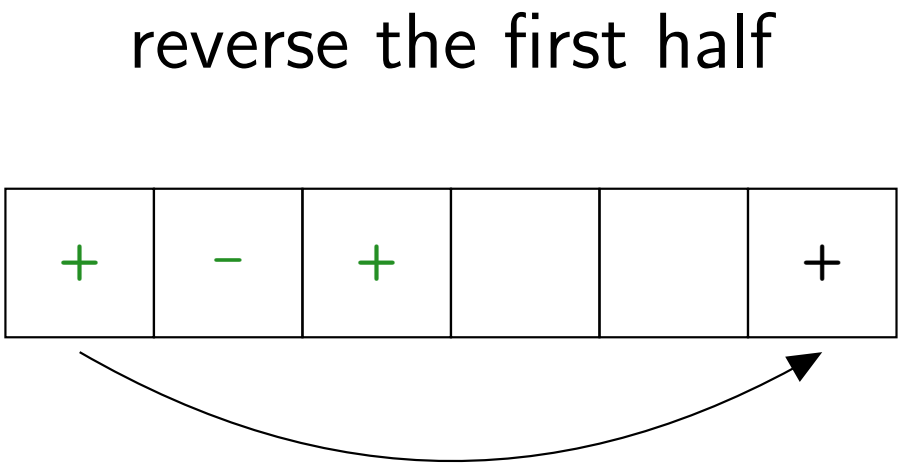

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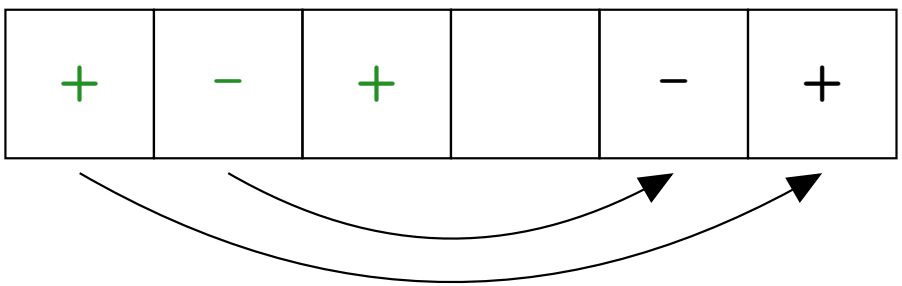

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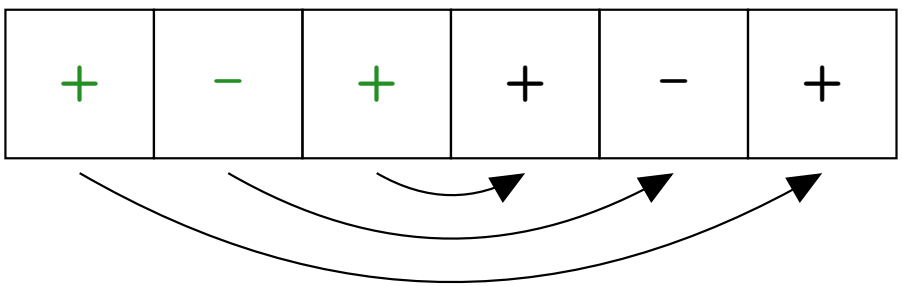

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change half the signs

+	−	+	+	−	+

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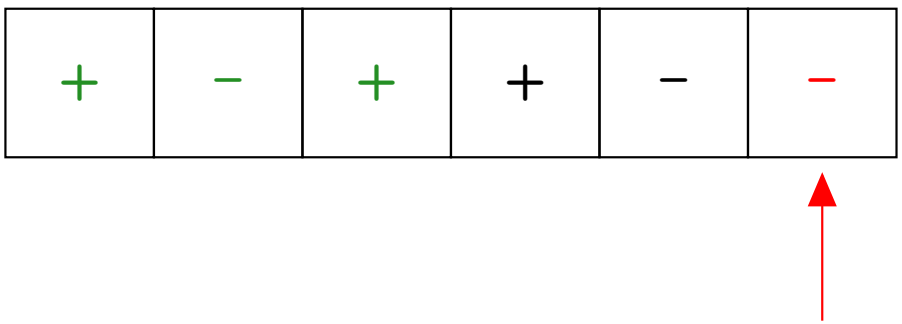

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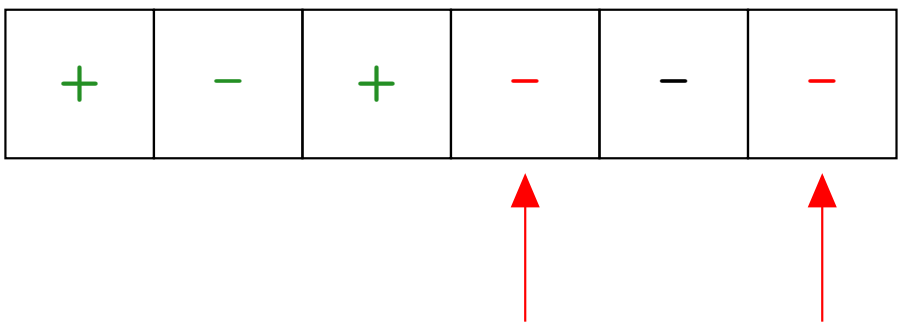

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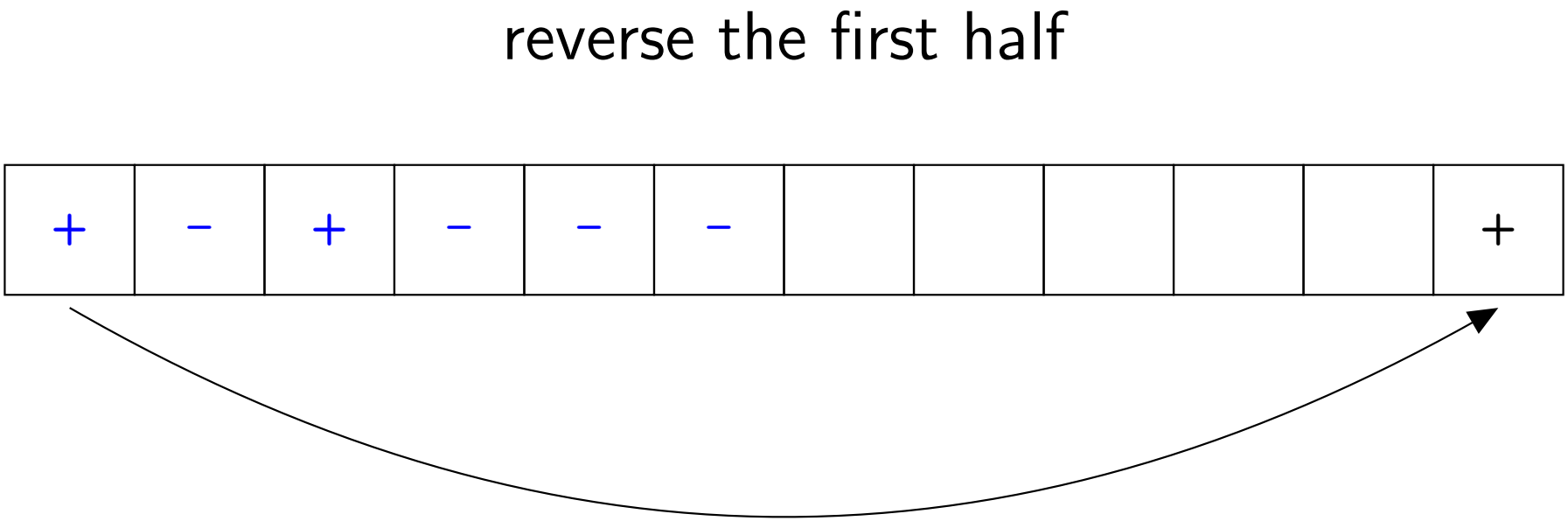

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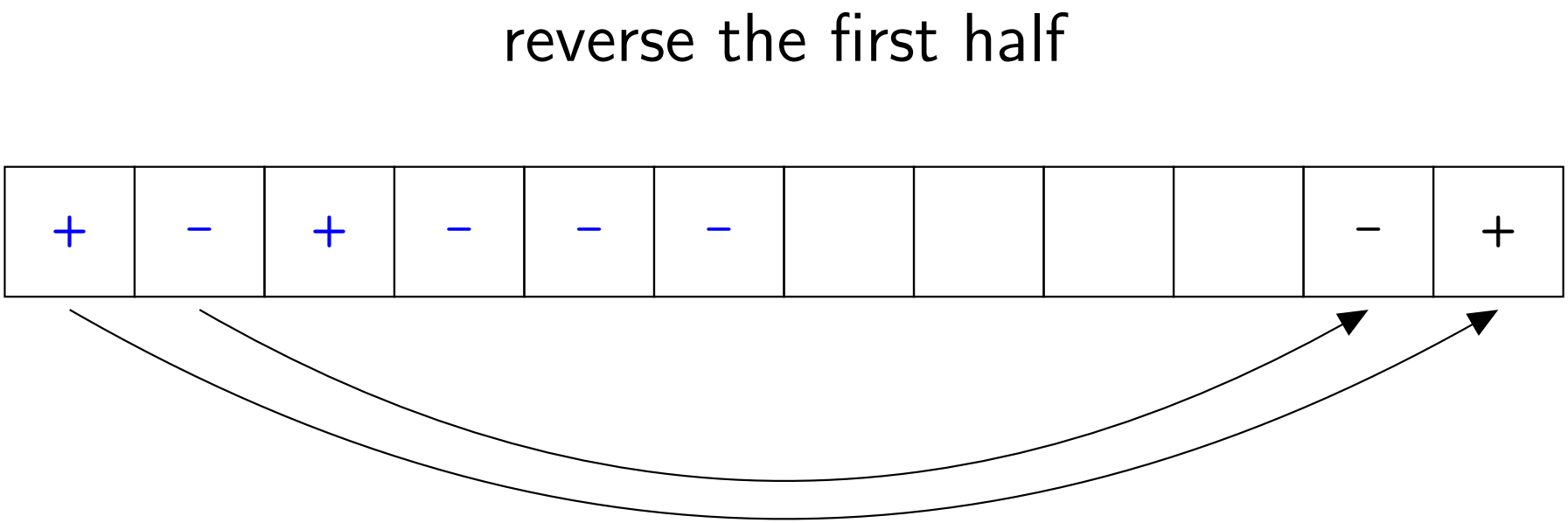

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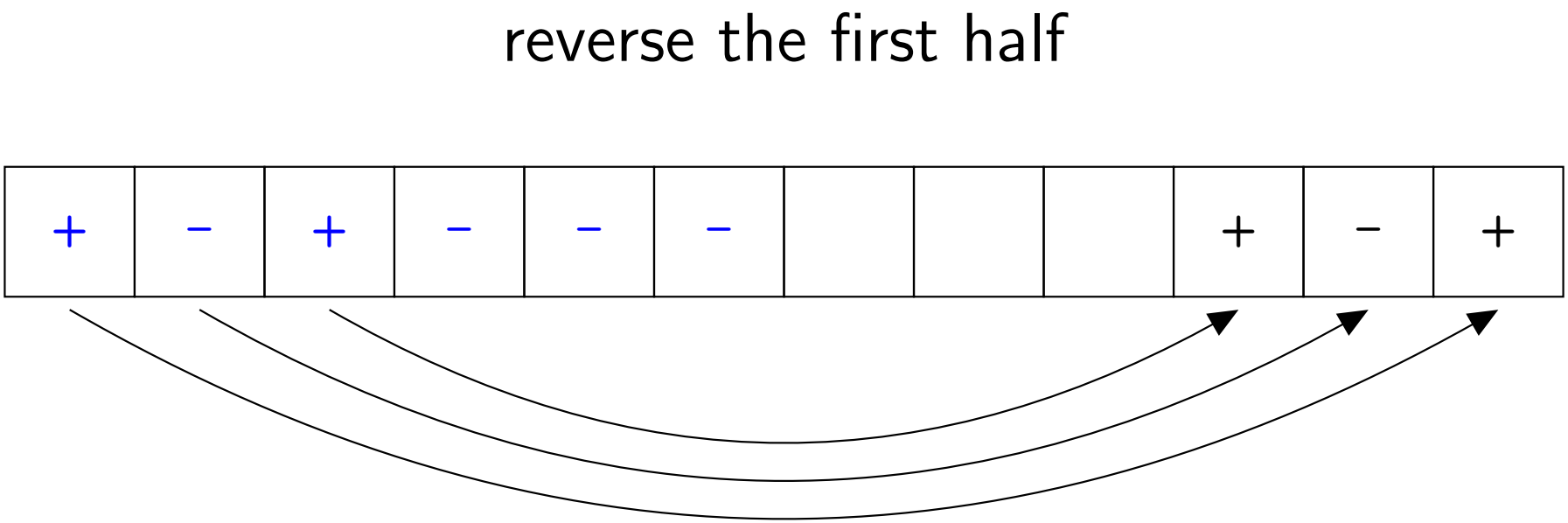

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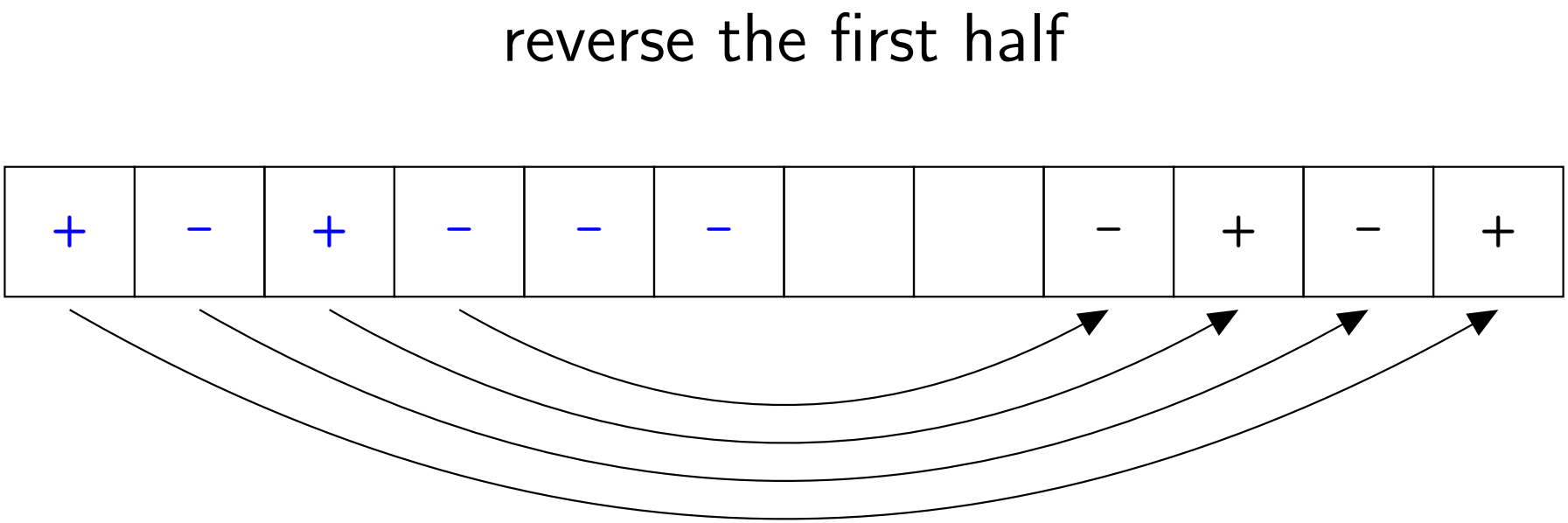

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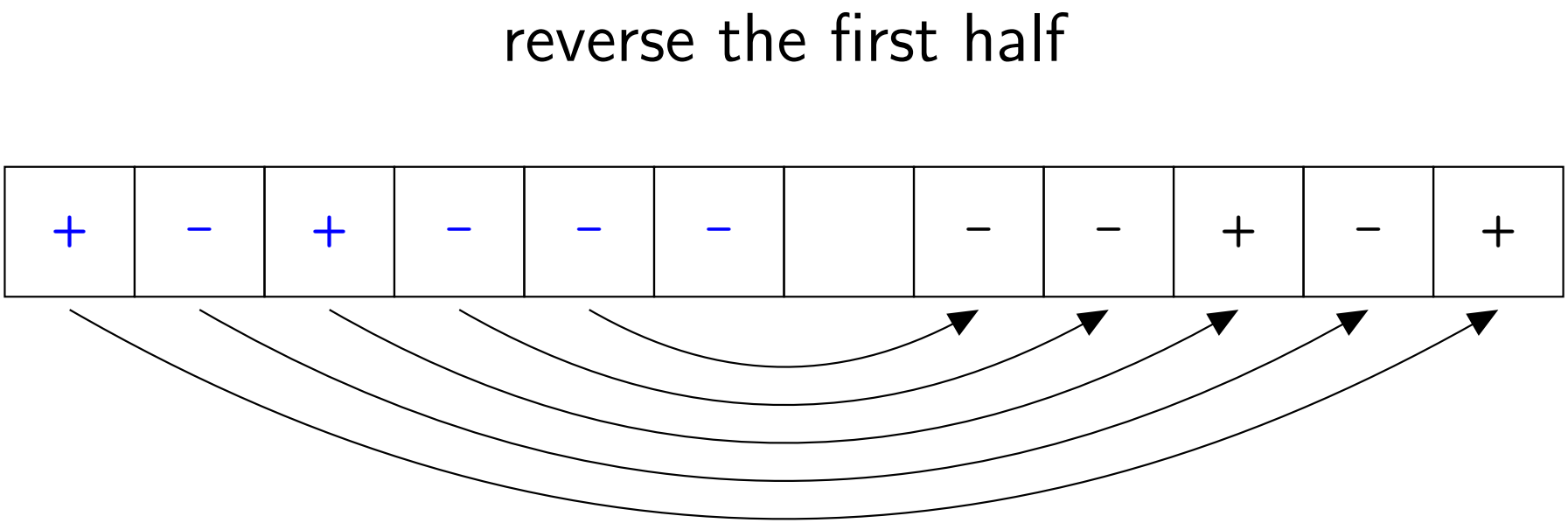

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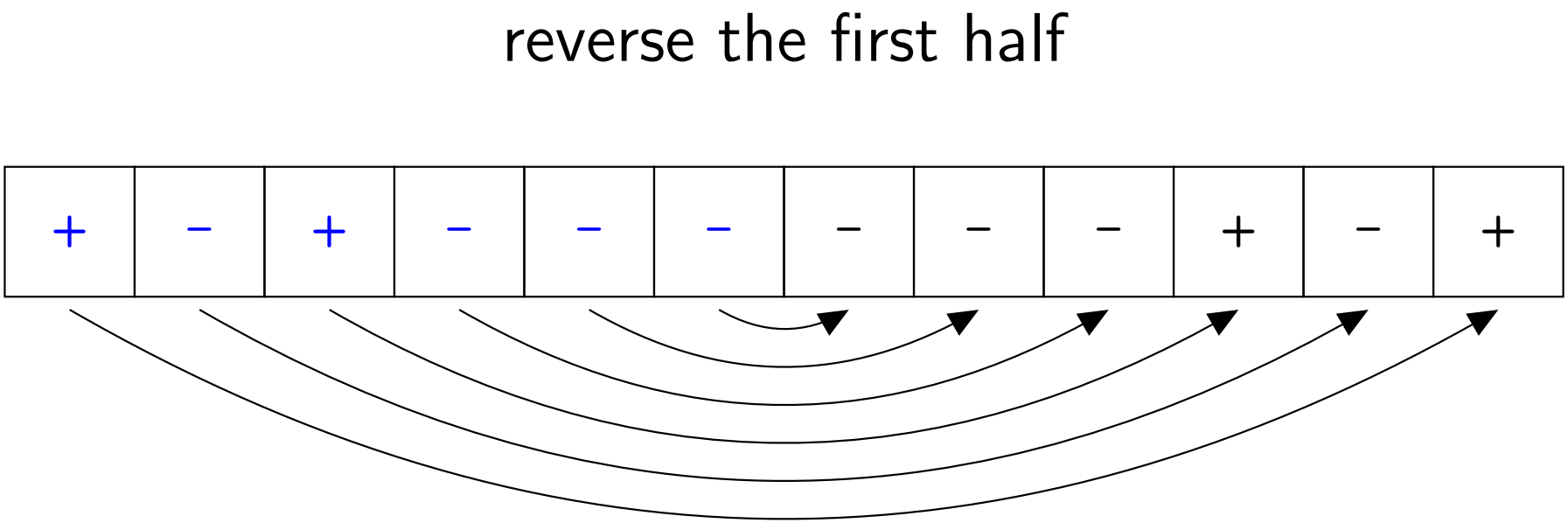

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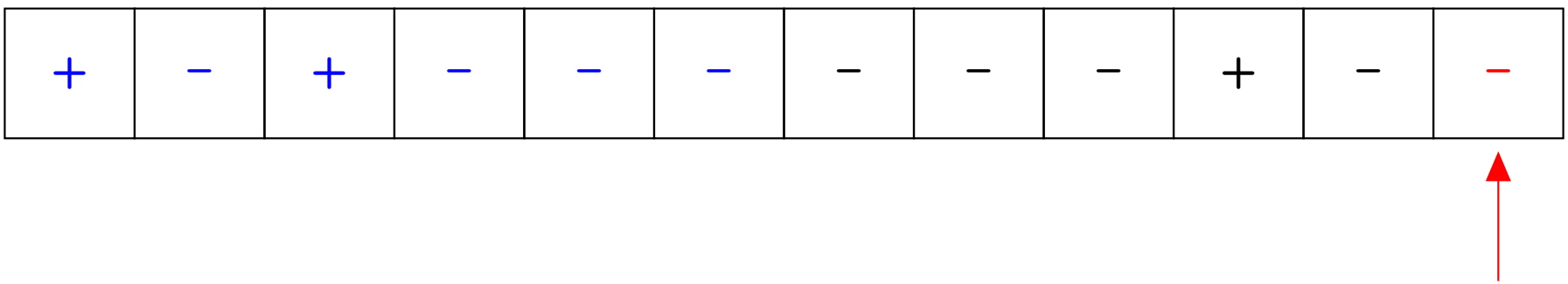

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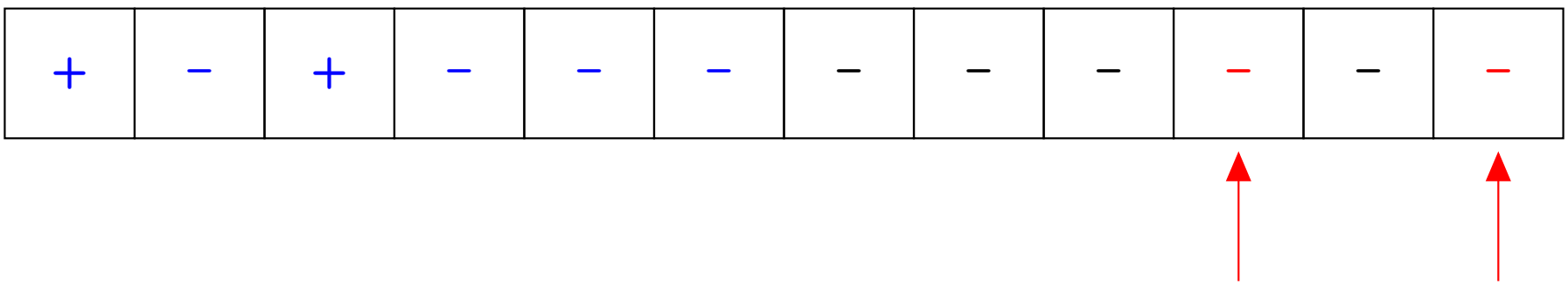

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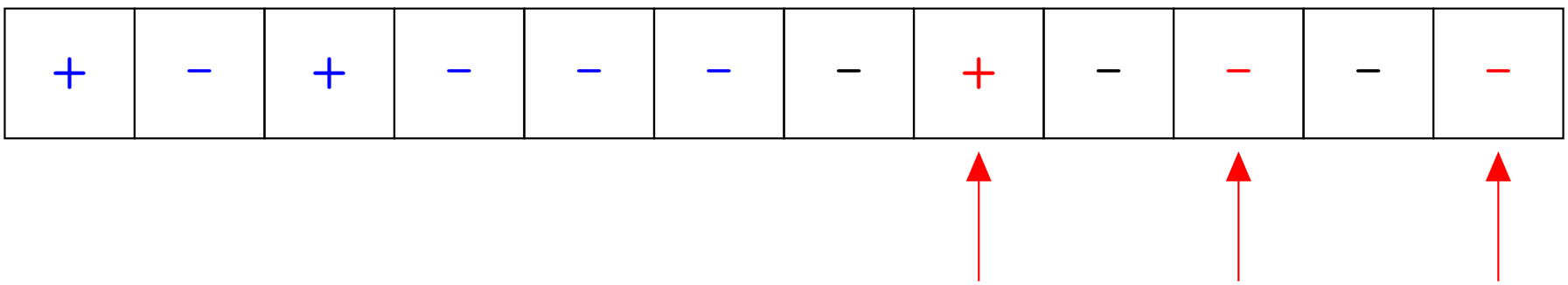

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Our Idea

- Two seed sequences, f^0 and g^0, of equal length
- Grow f^0 into a stem $f^0, f^1, f^2, \ldots$
- Grow g^0 into a stem $g^0, g^1, g^2, \ldots$
- For each n, determine
  - $\mathrm{ADF}(f^n)$ autocorrelation demerit factor of f^n
  - $\mathrm{ADF}(g^n)$ autocorrelation demerit factor of g^n
  - $\mathrm{CDF}(f^n, g^n)$ crosscorrelation demerit factor of f^n and g^n
  - $\mathrm{PSC}(f^n, g^n)$ Pursley-Sarwate criterion for f^n and g^n

$$\mathrm{PSC}(f^n, g^n) = \sqrt{\mathrm{ADF}(f^n)\mathrm{ADF}(g^n)} + \mathrm{CDF}(f^n, g^n)$$

Recall $\mathrm{PSC}(f^n, g^n) \geq 1$ always

We want to make $\mathrm{PSC}(f^n, g^n)$ as close to 1 as possible

Asymptotic Result

If $f^0, f^1, \ldots$ and $g^0, g^1, \ldots$ are stems, then

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If $f^0, f^1, \ldots$ and $g^0, g^1, \ldots$ are stems, then

$$\lim_{n\to\infty} \mathrm{ADF}(f^n) = -1 + \frac{2}{3} \cdot \frac{\|f^0\|_4^4 + \|f^0 \widetilde{f}^0\|_2^2}{\|f^0\|_2^4},$$

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where we identify sequences with polynomials:

$$a = (a_0, \ldots, a_{\ell-1}) \leftrightarrow a(z) = a_0 + a_1 z + \cdots + a_{\ell-1} z^{\ell-1}$$

and $\widetilde{f}^0(z)$ and $\widetilde{g}^0(z)$ are $f^0(-z)$ and $g^0(-z)$, and

$$\|a\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |a(e^{i\theta})|^p d\theta \right)^{1/p}$$

$$\int f^0 \widetilde{f}^0 \overline{g^0 \widetilde{g}^0} = \frac{1}{2\pi} \int_0^{2\pi} f^0(e^{i\theta}) \widetilde{f}^0(e^{i\theta}) \overline{g^0(e^{i\theta}) \widetilde{g}^0(e^{i\theta})} d\theta.$$

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Borwein and Mossinghoff showed that $\lim_{n\to\infty} \mathrm{ADF}(f^n) \geq 1/3$, with equality certain seeds of length 1, 2, 4, 8, 16, 20, 32, 40.

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In our work, we calculate the limiting Pursley-Sarwate Criterion

$$\lim_{n\to\infty} \mathrm{PSC}(f^n, g^n) = \lim_{n\to\infty} \sqrt{\mathrm{ADF}(f^n)\mathrm{ADF}(g^n)} + \mathrm{CDF}(f^n, g^n)$$

Computational Techniques

Gray code: order the sequences so that successive sequences differ in only one position, e.g.

0	+	+	+	4	−	−	+
1	+	+	−	5	−	−	−
2	+	−	−	6	−	+	−
3	+	−	+	7	−	+	+

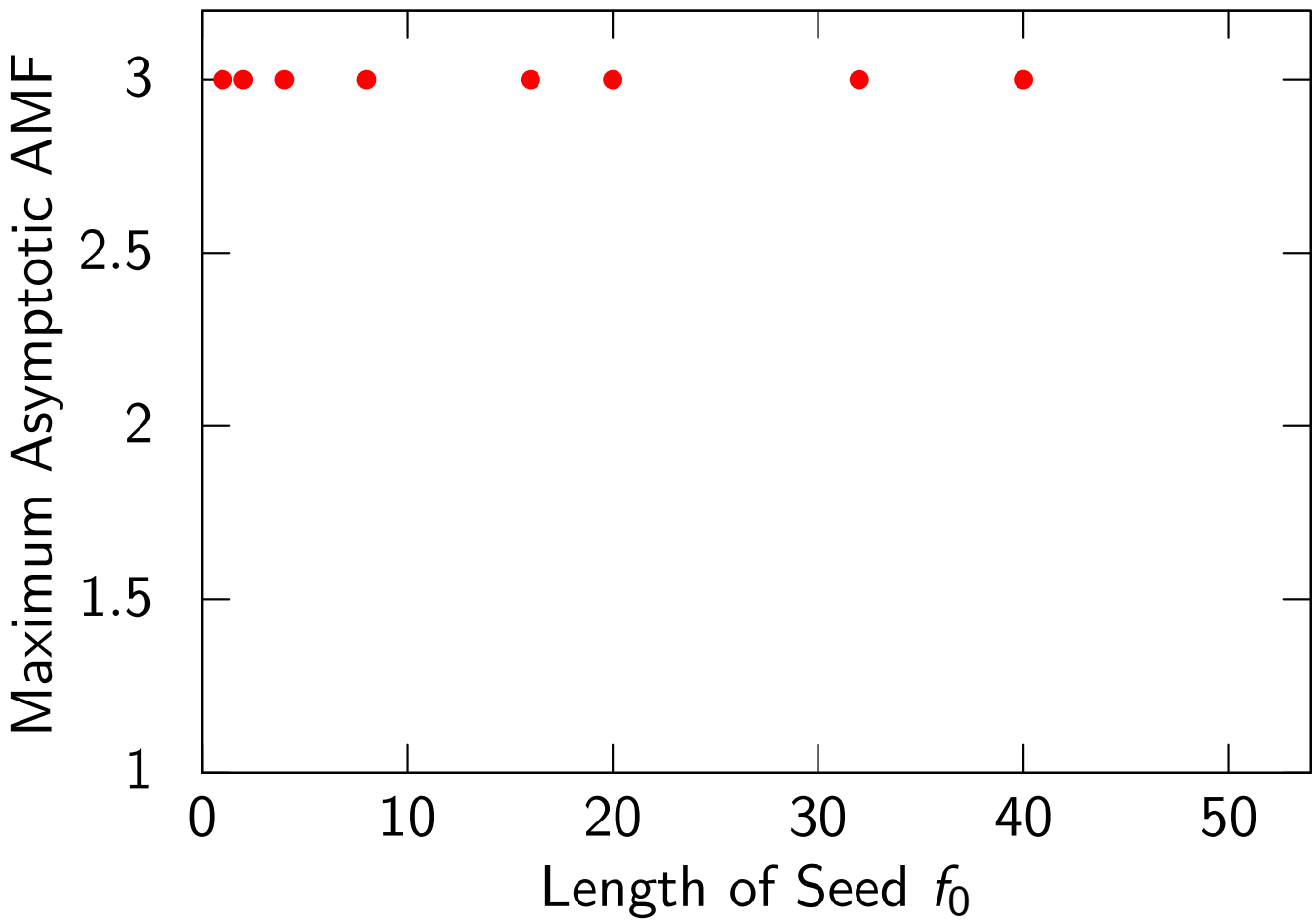
Fourier analysis helps compute the integrals faster

Symmetry helps: exchange sequences, negate either, negate every other term of both, reverse both

- Overall a group G of order 32 of symmetries preserves limiting PSC for seeds of length > 1
- Odd seed lengths: $G \cong D_8 \times C_2 \times C_2$
- Even seed lengths: extraspecial group, order 2^5, type $+$

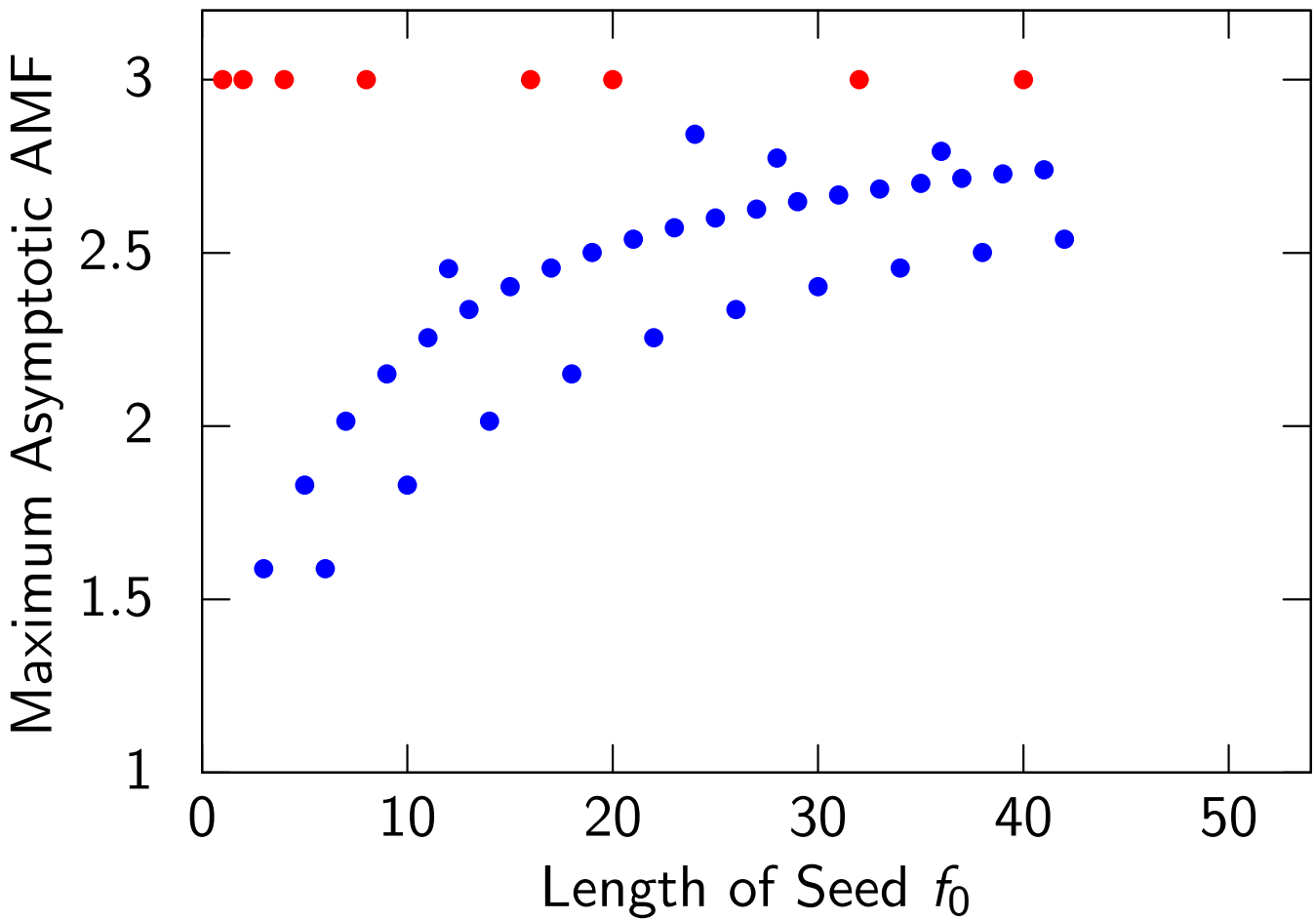
Old Result for Autocorrelation

Recall: Borwein-Mossinghoff showed that $\lim_{n\to\infty} \mathrm{AMF}(f^n) \leq 3$, with equality achievable at lengths 1, 2, 4, 8, 16, 20, 32, 40



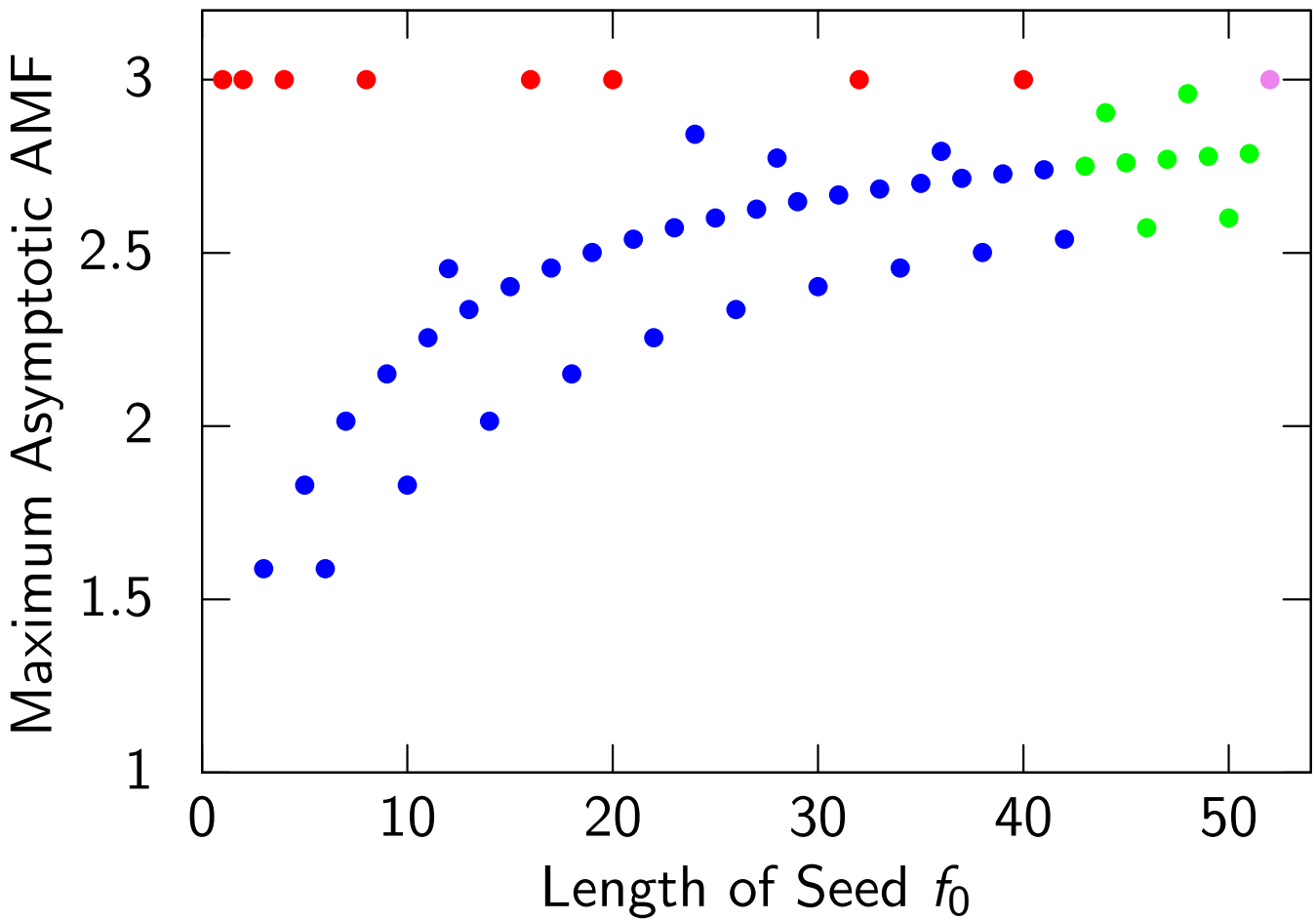
Old Result for Autocorrelation

On our own machines: highest possible $\lim_{n\to\infty} \mathrm{AMF}(f^n)$ with seeds of length ≤ 42. We confirm results of Borwein-Mossinghoff.



New Result for Autocorrelation

We used a massively parallel search using the Open Science Grid to extend the search to length 52



What is Going On?

A special type of sequence pairs called Golay complementary pairs first researched by Golay (1951) in connection with multislit spectrometry

- Golay originally constructed them for lengths 2^a
- Golay later discovered pairs of length 10, excluded various other lengths, and wondered if they might exist at length 26

The Connection

Further constructions have been discovered.

- Known to be Golay sequences of lengths $2^a 10^b 26^c$ for $a, b, c \geq 0$
- Curious: for lengths $2 \leq \ell \leq 52$, asymptotic merit factor 3 is achievable precisely when ℓ is double that of of a Golay pair

Theorem (K.-Trunov, 2017)

A seed f^0 of length > 1 gives rise to a stem $f^0, f^1, \ldots$ with $\lim_{n\to\infty} \mathrm{AMF}(f^n) = 3$ if and only if f^0 is the interleaving of a Golay complementary pair.

What about Crosscorrelation?

Recall: $\lim_{n\to\infty} \mathrm{CDF}(f^n, g^n)$ can be made arbitrarily close to 0, but at the cost of very high autocorrelation.

So better to calculate the limiting Pursley-Sarwate Criterion

$$\lim_{n\to\infty} \mathrm{PSC}(f^n, g^n) = \lim_{n\to\infty} \sqrt{\mathrm{ADF}(f^n)\mathrm{ADF}(g^n)} + \mathrm{CDF}(f^n, g^n)$$

Recall: for typical randomly selected sequence pairs with large ℓ, we expect

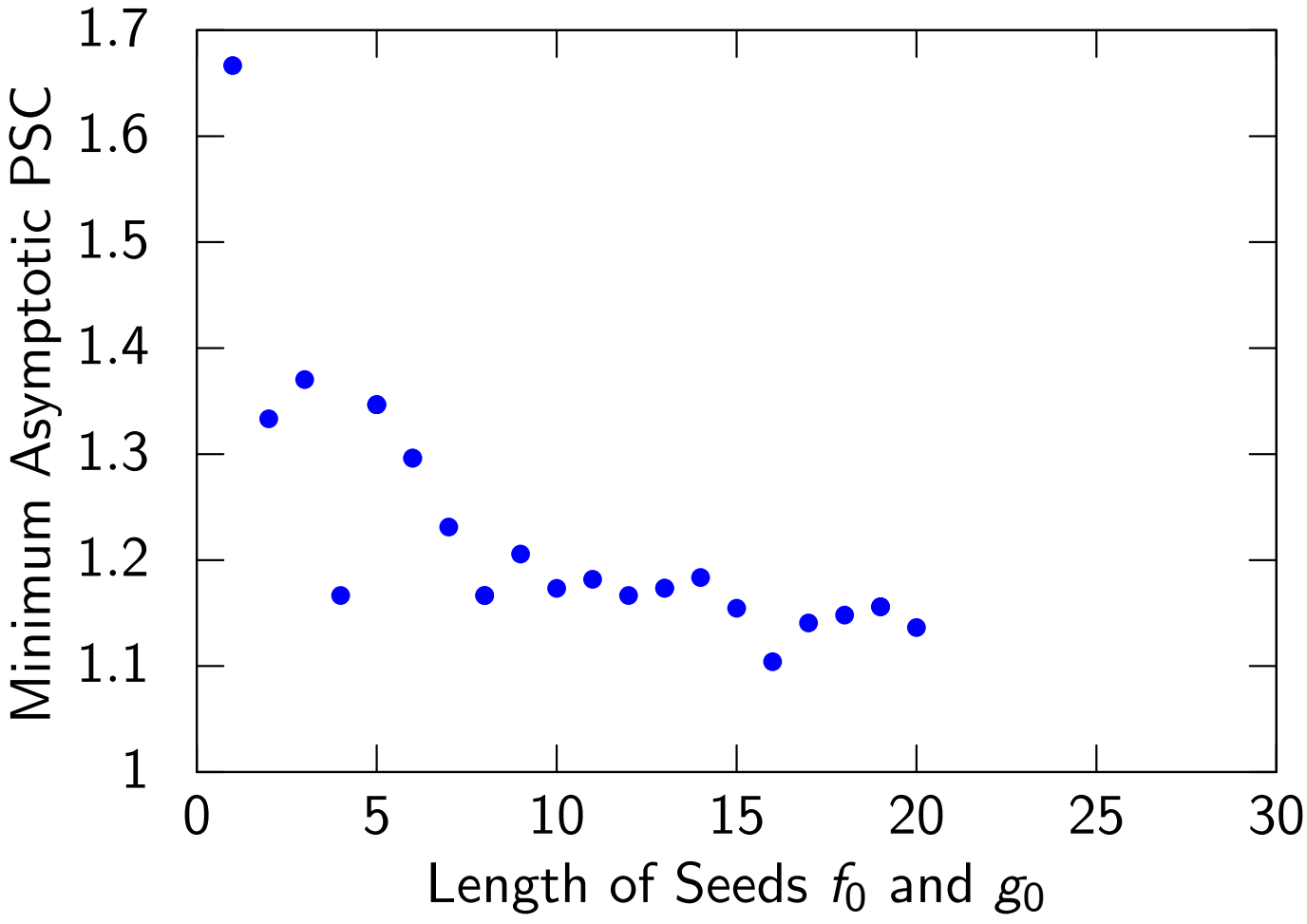
$$\mathrm{PSC}(f, g) \approx 2$$

Recall: we always have

$$\mathrm{PSC}(f, g) \geq 1$$

Crosscorrelation Result

Big Question: How close to 1 do Rudin-Shapiro-like sequences get?



Crosscorrelation Result

Big Question: How close to 1 do Rudin-Shapiro-like sequences get?