9 GEOMETRY AND TOPOLOGY OF POLYGONAL LINKAGES

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INTRODUCTION

There is a long and involved history of linkages starting at least in the nineteenth century with the advent of complicated and intricate mechanical engineering. Some of these practical problems led to interesting, nontrivial geometric problems, and in modern mathematics there has been significant progress on even very basic questions. Over the years, several different points of view have been taken by various groups of people. We will attempt to survey these different perspectives and results obtained in the exploration of linkages.

9.1 MATHEMATICAL THEORY OF LINKAGES

The underlying principles and definitions are mathematical and in particular geometric. Despite the long history of kinematics, even theoretical kinematics (see, e.g., Bottema and Roth [BR79a]), only since the 1970s does there seem to be any systematic attempt to explore the mathematical and geometric foundations of a theory of linkages.

We begin with some definitions, some of which follow those in rigidity theory described in Chapter 61. The rough, intuitive notions are as follows. A *linkage* is a graph with assigned edge lengths, and we distinguish three special types of linkages: arcs, cycles, and trees. A *configuration* realizes a linkage in Euclidean space, a *reconfiguration* (or *flex*) is a continuum of such configurations, and the *configuration space* embodies all configurations, with paths in the space corresponding to reconfigurations. The configuration space can be considered either allowing or disallowing bars to intersect each other.

GLOSSARY

Bar linkage or ***linkage:*** A graph $G = (V, E)$ and an assignment $\ell : E \to \mathbb{R}^+$ of positive real *lengths* to edges.

Vertex or ***joint:*** A vertex of a linkage.

Bar or ***link:*** An edge e of a linkage, which has a specified fixed length $\ell(e)$.

Polygonal arc: A linkage whose underlying graph is a single path. (Also called an ***open chain*** or a ***ruler***.)

Polygonal cycle: A linkage whose underlying graph is a single cycle. (Also called a ***closed chain*** or a ***polygon***.)

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FIGURE 9.1.1
Different types of linkages, according to whether the underlying graph is a path, cycle, or tree, or the graph is arbitrary.

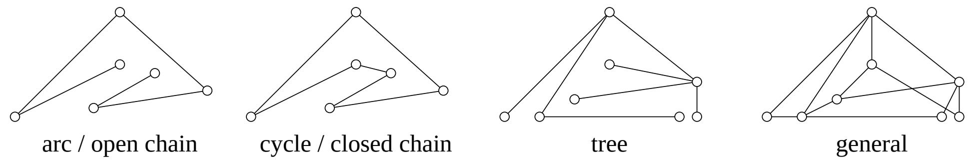
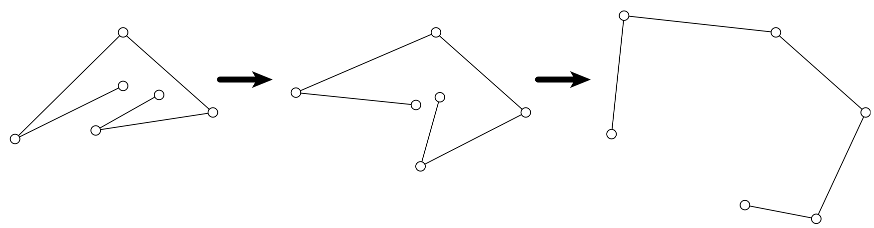


FIGURE 9.1.2
Snapshots of a reconfiguration of a polygonal arc.



Polygonal tree: A linkage whose underlying graph is a single tree.

Configuration of a linkage in d-space: A mapping $p : V \to \mathbb{R}^d$ specifying a point $p(v) \in \mathbb{R}^d$ for each vertex v of the linkage, such that each bar $\{v, w\} \in E$ has the desired length $\ell(e)$, i.e., $|p(v) - p(w)| = \ell(e)$.

A configuration can be viewed as a point p in $\mathbb{R}^{d|V|}$ by arbitrarily ordering the vertices in V, and assigning the coordinates of the ith vertex ($0 \leq i < |V|$) to coordinates $id + 1, id + 2, \ldots, id + d$ of p.

Framework or ***bar framework:*** A linkage together with a configuration.

Reconfiguration or ***motion*** or ***flex*** of a linkage: A continuous function $f : [0, 1] \to \mathbb{R}^{d|V|}$ specifying a configuration of the linkage for every moment in time between 0 and 1.

Configuration space or ***Moduli space of a linkage:*** The set $\mathcal{M}$ of all configurations (treated as points in $\mathbb{R}^{d|V|}$) of the linkage.

Self-intersecting configuration: A configuration in which two bars intersect but are not incident in the underlying graph of the linkage.

Reconfiguration avoiding self-intersection: A reconfiguration f in which every configuration $f(t)$ does not self-intersect.

Configuration space of a linkage, disallowing self-intersection: The subset $\mathcal{F}$ of the configuration space $\mathcal{M}$ in which every configuration does not self-intersect. (Also called the ***free space*** of the linkage.)

Paths in the configuration space of a linkage capture the key notion of reconfiguration (either allowing or disallowing self-intersection as appropriate). Many important questions about linkages can be most easily phrased in terms of the configuration space. For example, we are often interested in whether the configu-

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ration space is connected (every configuration can be reconfigured into every other configuration), or in the topology of the configuration space.

9.2 CONFIGURATION SPACES OF ARCS AND CYCLES WITH POSSIBLE INTERSECTIONS

One fundamental problem is to compute the topology of the configuration space of planar polygonal cycles (polygons), allowing possible self-intersections. There is a long list of results in increasing generality for computing information about the algebraic topological invariants of this configuration space. One approach is Morse Theory, which reveals some of the basic information, in particular, the connectivity and some of the easier invariants such as the Euler characteristic.

CONNECTIVITY

The following is an early result possibly first due to [Hau91], but rediscovered by [Jag92], and then rediscovered again or generalized considerably by many others, in particular, [Kam99, KT99, MS00, KM95, LW95].

THEOREM 9.2.1 *Connectivity for planar polygons* [Hau91]

Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be the cyclic sequence of bar lengths in a polygon, and let $s = s_1 + s_2 + \cdots + s_n$. Then the following occurs:

i) The configuration space is nonempty if and only if $s_n \leq s/2$.

ii) The configuration space, modulo orientation-preserving congruences, is connected if and only if $s_{n-2} + s_{n-1} \leq s/2$. If the space is not connected, there are exactly two connected components, where each configuration in one component is the reflection of a configuration in the other component.

The configuration space is a smooth manifold if and only if there is some configuration p with all its vertices on a line, which in turn is determined by the edge lengths as described above. Also, the configuration space remains congruent no matter how we permute the cyclic sequence of bar lengths. When the linkage is not allowed to self-intersect, it is common to consider the configuration space modulo all congruences of the plane (including reflections); but when self-intersections are allowed, and condition ii) above is satisfied, it is possible to move the linkage from any configuration to its mirror image.

For polygons in dimensions higher than two, the situation is simpler:

THEOREM 9.2.2 *Connectivity for nonplanar polygons* [LW95]

The configuration space of a polygon in d-dimensional space, for $d > 2$, is always connected.

HOMOLOGY, COHOMOLOGY, AND HOMOTOPY

After connectivity, there remains the calculation of the higher homology groups, cohomology groups, and the homotopy type of the configuration space. Here is one special case as an example:

THEOREM 9.2.3 *Configuration space of equilateral polygons* [KT99]
Let $\mathcal{M}$ be the configuration space of a polygon with n equal bar lengths, modulo congruences of the plane. The homology of $\mathcal{M}$ is a torsion-free module given explicitly in [KT99]. When n is odd, $\mathcal{M}$ is a smooth manifold; and when $n = 5$, $\mathcal{M}$ is the compact, orientable two-dimensional manifold of genus 4 (originally shown in [Hav91], as well as in [Jag92]).

See also especially [KM95] for some of the basic techniques. For calculating the configuration space of graphs other than a cycle, see in particular the article [TW84], where a particular linkage, with some pinned vertices, has a configuration space that is an orientable two-dimensional manifold of genus 6.

Another case that has been considered is an equilateral polygon in 3-space with angles between incident edges fixed. This fixed-angle model arises in chemistry [CH88] and in particular in protein folding (see Section 9.7). Alternatively, a fixed angle can be simulated by adding bars between vertices of distance two along the polygon. The configuration space behaves similarly to the planar case:

THEOREM 9.2.4 *Fixed-angle equilateral 3D polygons* [CJ]
Let $\mathcal{M}$ be the configuration space of an equilateral polygon with $n \geq 6$ equal bar lengths and fixed equal angles, modulo congruences of $\mathbb{R}^3$. Suppose further that every turn angle is within an additive ϵ of $2\pi/n$ for ϵ sufficiently small (i.e., configurations are forced nearly planar). Then $\mathcal{M}$ has at most two components. When n is odd, $\mathcal{M}$ is a smooth manifold of dimension $n - 6$. When n is even, $\mathcal{M}$ is singular.

When $n = 6$, the underlying graph is the graph of an octahedron, and there are cases when it is rigid and cases when it is not. This linkage corresponds to cyclohexane in chemistry, and its flexibility was studied by [Bri96] and [Con78].

The restriction of the polygon configurations being almost planar leads to the following problem:

PROBLEM 9.2.5 *General equilateral equi-angular 3D polygons* [Cri92]
How many components does $\mathcal{M}$ have in the theorem above if ϵ is allowed to be large?

9.3 CONFIGURATION SPACES WITHOUT SELF-INTERSECTIONS

When the linkage is not permitted to self-intersect, the main question that has been studied is when it can be locked. Three main classes of linkages have been studied in this context: arcs, cycles, and trees. When the linkage is planar and has cycles, we assume that the clockwise/counterclockwise orientation is given and fixed, for otherwise the linkage is trivially locked: no cycle can be "flipped over" in the plane without self-intersection.

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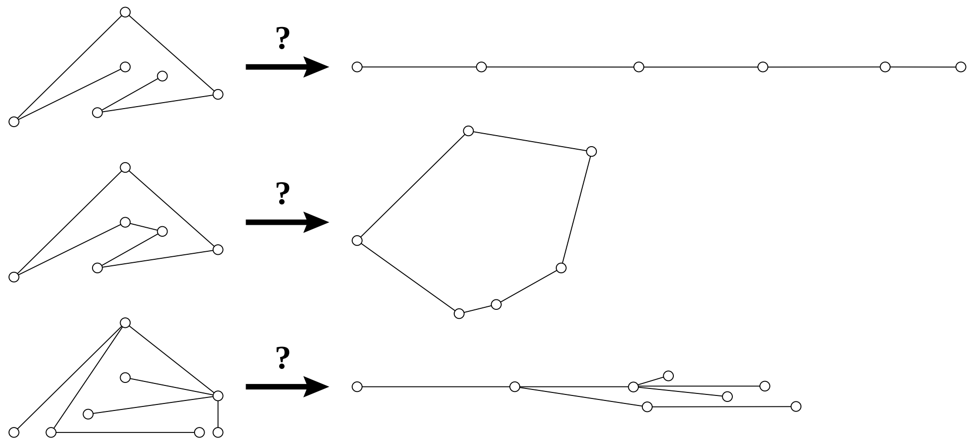
GLOSSARY

Locked linkage: A linkage whose configuration space has multiple connected components when self-intersections are disallowed.

Lockable class of linkages: There is a locked linkage in the class.

Unlockable class of linkages: No linkage in the class is locked.

FIGURE 9.3.1
The problems of arc straightening, cycle convexifying, and tree flattening.



Straightening an arc: A motion bringing a polygonal arc from a given configuration to its ***straight configuration*** in which every joint angle is π.

Convexifying a cycle: A motion bringing a polygonal arc from a given configuration to a ***convex configuration*** in which every joint angle is at most π.

Flattening a tree: A motion bringing a polygonal tree from a given configuration to a ***flat configuration*** in which every joint angle is either 0, π, or 2π, and every bar points "away" from a designated root node.

WHICH LINKAGES ARE LOCKED?

Which of the main classes of linkages can be locked is summarized in Table 9.3.1. In short, the existence of locked arcs and locked unknotted cycles is equivalent to the existence of knots in that dimension: just in 3D. However, this equivalence is by no means obvious, especially in 2D, as evidenced by the existence of locked trees in 2D.

One main approach for determining whether a linkage is locked is to consider the equivalent problem of finding a motion from any configuration to a ***canonical configuration***. Because linkage motions are reversible and concatenable, if every configuration can be canonicalized, then every configuration can be brought to any other configuration, routing through the canonical configuration. Conversely, if

TABLE 9.3.1 Summary of what types of linkages can be locked.

	ARCS AND CYCLES	TREES
2D	Not lockable [CDR03, Str00, CDIO04]	Lockable [BDD$^+$02, CDR02]
3D	Lockable [CJ98, BDD$^+$01, Tou01]	Lockable [arcs are a special case]
4D$^+$	Not lockable [CO01]	Not lockable [CO01]

some configuration cannot be canonicalized, then we know a pair of configurations that cannot reach each other, and therefore the linkage is locked.

This idea leads to the notions of straightening arcs, convexifying cycles, and flattening trees, as defined above. There is only one straight configuration of an arc, but there are multiple convex configurations of cycles and flat configurations of trees; fortunately, it is fairly easy to reconfigure between any pair of convex configurations of a cycle [ADE$^+$01] or between any pair of flat configurations of a tree [BDD$^+$02].

LOCKED LINKAGES

The first results along these lines were negative (see Figure 9.3.2): polygonal arcs in 3D and unknotted polygonal cycles in 3D can be locked [CJ98], and planar polygonal trees can be locked [BDD$^+$02]. Since these results, other examples of unknotted but locked 3D polygonal cycles [BDD$^+$01, Tou01] and locked 2D polygonal trees [CDR02, BCD$^+$09] have been discovered.

More generally, Alt, Knauer, Rote, and Whitesides [AKRW03] constructed a large family of locked 2D trees and 3D arcs in which it is PSPACE-hard to determine whether one configuration can reach another configuration via a continuous motion that avoids self-intersection. Their construction combines several gadgets, many of which resemble the examples in Figure 9.3.2, as well as the "interlocked" linkages of [DLOS03, DLOS02]. However, this work leaves open a closely related problem, deciding whether *every* pair of configurations can reach each other:

PROBLEM 9.3.1 *Complexity of testing whether a linkage is locked* [BDD$^+$01]

What is the complexity of deciding whether a linkage is locked? Particular cases of interest are 3D arcs, 3D cycles restricted to unknotted configurations, and 2D trees.

UNLOCKED LINKAGES

Unlockability was first established in 4D and higher [CO01], where one-dimensional arcs, cycles, and trees have so much freedom that they can never lock. Intuitively, the barriers (self-intersecting configurations) that might prevent e.g., straightening the vertex between the first two bars of an arc have dimension at least 2 lower than the configuration space of that vertex, and hence all barriers can be avoided. Thus, the only problem with straightening an arc vertex-by-vertex is that the configuration that results from straightening one extreme vertex might have self-intersections; in this case, the linkage can be perturbed to remove the problem. Convexifying cycles in 4D and higher is more difficult, but follows a similar idea.

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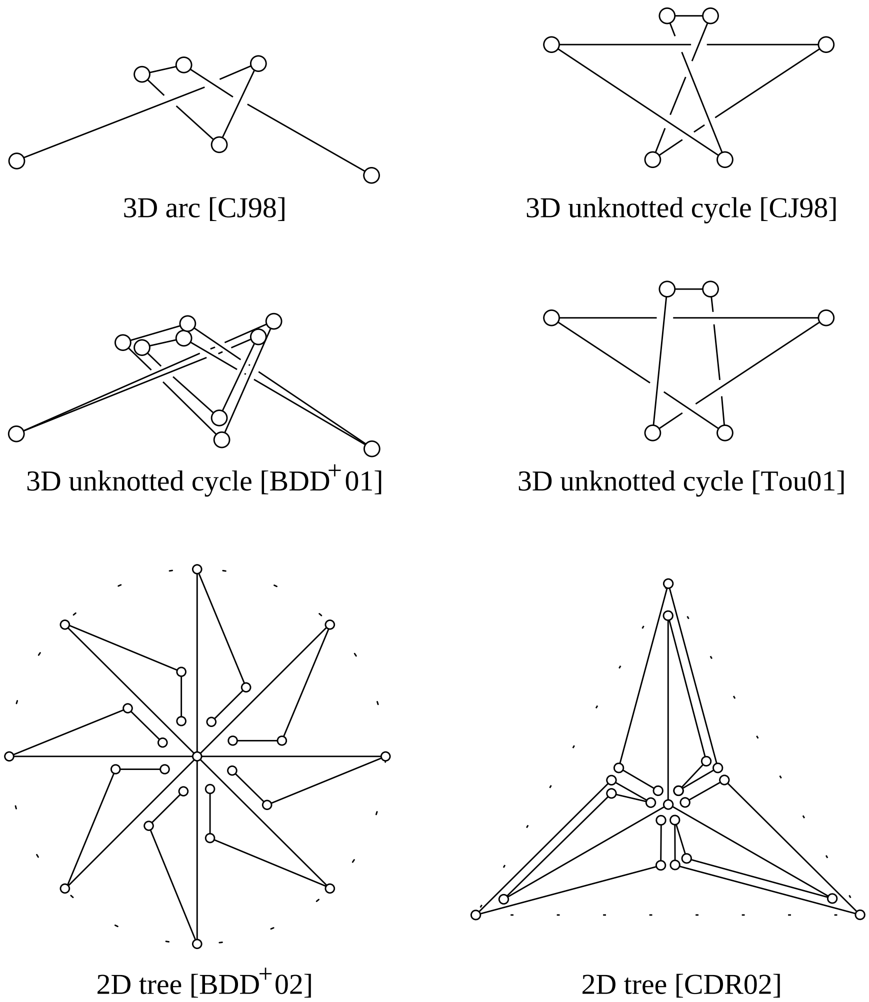
FIGURE 9.3.2
Known examples of locked linkages.



The last cell of Table 9.3.1 to be filled was that 2D arcs and cycles never to lock [CDR03]. Indeed, the following more general theorem holds:

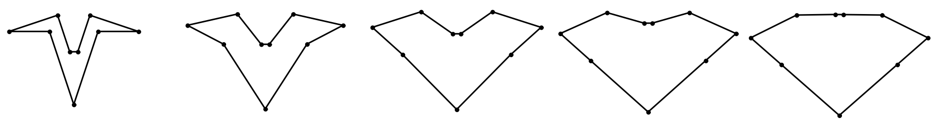
THEOREM 9.3.2 *Straightening 2D arcs and convexifying 2D cycles* [CDR03]

Given a disjoint collection of polygonal arcs and polygonal cycles in the plane, there is a motion that avoids self-intersection and, after finite time, straightens every outermost arc and convexifies every outermost cycle. (An arc or cycle is ***outermost*** *if it is not contained within another cycle.)*

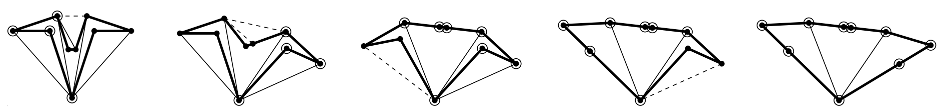
In this theorem, arcs and cycles contained within other cycles may not straighten or convexify—they simply "come along for the ride"—but this is the best we could hope for in general.

There are now three methods for solving this problem. See Figure 9.3.3 for a visual comparison on a simple example. The first method is based on flow through an ordinary differential equation defined implicitly by a convex optimization problem [CDR03]. The second method is more combinatorial and is based on algebraic motions defined by single-degree-of-freedom mechanisms given by pseudotriangulations [Str00]. The third method is based on energy minimization via gradient descent [CDIO04], a technique also adapted to polygon morphing [IOD09].

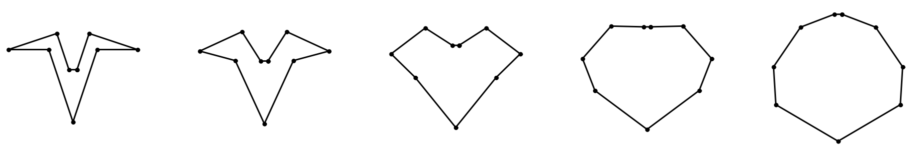
FIGURE 9.3.3
Convexifying a common polygon via all three convexification methods.



(a) Via convex programming [CDR03].



(b) Via pseudotriangulations [Str00]. Pinned vertices are circled.



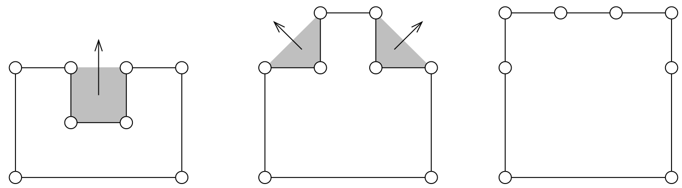
(c) Via energy minimization [CDIO04].

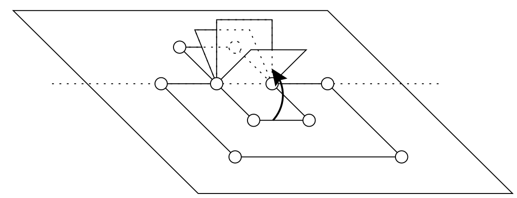
The first two motions have the additional property of being ***expansive***—the distance between every pair of vertices never decreases over time—while the third motion only relies on the existence of such a motion. The first and last motions, being flow-based, preserve any initial symmetries of the linkage. Characterizing by continuity, the three motions are respectively piecewise-C^1, piecewise-C^∞, and C^∞. Only the last motion has a corresponding finite-time algorithm to compute a motion that is *piecewise-linear* through configuration space, i.e., the motion can be decomposed into steps where each angle in each step changes at a constant rate. The number of steps has a pseudopolynomial bound, that is, a bound polynomial in the number n of bars and the ratio between the largest and smallest distances among nonincident edges. This algorithm is also easy to implement.

SPECIAL CLASSES OF LINKAGES

In addition to these results for general classes of linkages, various special classes have been shown to have different properties. Polygonal arcs in 3D that lie on

FIGURE 9.3.4
Flipping a polygon until it is convex.





the surface of a convex polyhedron, or having a non-self-intersecting orthogonal projection, are never locked [BDD+01]. Polygonal cycles in 3D having a non-self-intersecting orthogonal projection are also never locked [CKM+01].

Connelly et al. [CDD+10] introduced the idea of *adorned* arc linkages, where each bar has a planar shape (Jordan region) rigidly attached to it. Provided each shape is *slender*—meaning that the distance along the shape boundary to each hinge is monotone—any expansive motion of the underlying arc avoids introducing collisions among the shapes too. This result was later used to prove that hinged dissections can be made continuously foldable without collisions [AAC+12].

FLIPS, FLIPTURNS, DEFLATIONS, POPS, POPTURNS

One of the first papers essentially about unlocking linkages is by Erdős [Erd35], who asked whether a particular "flipping" algorithm always convexifies a planar polygon by motions through 3D in a finite number of steps. A ***flip*** rotates by 180° a subchain of the polygon, called a ***pocket***, whose endpoints are consecutive vertices along the convex hull of the polygon. Each such flip never causes the polygon to self-intersect.[1] Nagy [SN39] was the first to claim a proof that a polygon admits only finitely many flips before convexifying. This result was subsequently rediscovered several times; see [Tou05, Grü95]. Unfortunately, a detailed analysis of these proofs [DGOT08] shows that many of the claimed proofs, including Nagy's original, are incorrect or incomplete. Fortunately, multiple correct proofs remain, including the first correct published proof by Reshetnyak [Res57] and a newer, simplified proof [DGOT08]. Thus, pocket flipping is one suitable strategy for convexifying a 2D polygon by motions in 3D.

Joss and Shannon (1973) first proved that the number of flips required to

[1] Erdős [Erd35] originally proposed flipping multiple pockets at once, but such an operation can lead to self-intersection; Nagy [SN39] fixed this problem by proposing flipping only one pocket at once.

convexify a polygon cannot be bounded in terms of the number of vertices, but this work remains unpublished; see [Grü95, Tou05]. However, it may still be possible to bound the number of flips using other metrics:

PROBLEM 9.3.3 *Bounding the number of flips* [M. Overmars, Feb. 1998]

Bound the maximum number of flips a polygon admits in terms of natural measures of geometric closeness such as the sharpest angle, the diameter, and the minimum distance between two nonincident edges.

A related computational problem is to compute the extreme numbers of flips:

PROBLEM 9.3.4 *Maximizing or minimizing flips* [Dem02]

What is the complexity of minimizing or maximizing the length of a convexifying sequence of flips for a given polygon?

Several variations on flips have also been considered. Grünbaum and Zaks [GZ01] generalized Nagy's results to polygons with self-intersections—still they can be convexified by finitely many flips—with further consideration in [DGOT08]. Wegner [Weg93] introduced the notion of ***deflations***, which are the exact reverse of flips, and Fevens et al. [FHM$^+$01] showed that some quadrilaterals admit infinitely many deflations. On the other hand, every pentagon will stop deflating after finitely many carefully chosen deflations, and any infinite deflation sequence of a pentagon results from deflating an induced quadrilateral on four of the vertices [DDF$^+$07].

Flipturns are similar to flips, except that the pocket is temporarily severed from the rest of the linkage and rotated 180° in the plane around the midpoint of the hull edge. Such an operation is not a valid linkage motion, but it has the advantage that the number of flipturns that a polygon admits before convexification is $O(n^2)$ [ACD$^+$02, ABC$^+$00]. This bound is tight up to a constant factor [Bie06], and there is extensive work on finding the precise constants [ACD$^+$02], though some gaps remain to be closed. Also, related to Problem 9.3.4, it is known that maximizing the length of a convexifying flipturn sequence is weakly NP-hard [ACD$^+$02]. Minimizing the number of flipturns leads to the following interesting problem:

PROBLEM 9.3.5 *Number of required flipturns* [Bie06]

Is there a polygon that requires $\Omega(n^2)$ flipturns to convexify, or can all polygons be convexified by $o(n^2)$ carefully chosen flipturns?

The best known lower bound is $\Omega(n)$.

Pops and ***popturns*** are variations on flips and flipturns, respectively, where the "pocket" being reflected or rotated consists of exactly two incident edges of the polygon, not necessarily related to the convex hull. Allowing the polygon to intersect itself, popturns can convexify any polygon [ABB$^+$07], but pops cannot convexify some polygons (even initially non-self-intersecting) [DH10]. Forbidding self-intersection, there is a simple characterization of which polygons can be convexified by popturns [ABB$^+$07].

INTERLOCKED LINKAGES

Combinations of polygonal arcs and cycles in 3D that can or cannot be locked (or more accurately, "interlocked") are studied in [DLOS03, DLOS02]. More precisely, this work studies the shortest (fewest-bar) 3D arcs and cycles that can interlock with each other. For example, three 3-arcs (arcs with three bars each) can interlock, as can a 3-arc and a 4-arc, or a 3-cycle and a 4-arc, or a 3-arc and a 4-cycle. However, two 3-arcs and arbitrarily many 2-arcs never interlock, and nor can a 3-cycle and a 3-arc. Also considered in [DLOS02] is the case that some of the pieces have restricted motion, e.g., all angles being fixed, or just rigid motions being allowed. Glass et al. [GLOZ06] proved that a 2-arc and an 11-arc can interlock.

9.4 UNIVERSALITY RESULTS

TRACING CURVES

The classic motivation of building linkages is to design a planar linkage in which one of the vertices traces a portion of a desired curve given by some polynomial function. In particular, Watt posed the problem of finding a linkage with some vertices pinned so that one vertex would trace out a line (segment). Watt's problem, at first thought to be impossible, was finally solved by Peaucellier in [Pea73], as well as by Lipkin in [Lip71]. See also [Kem77] and [Har74].

Later, Kempe [Kem76] described a linkage that would trace out a portion of any algebraic curve in the plane. However, his description is very brief and it leaves unspecified what portion of the algebraic curve is actually traced out, and whether there are other, possibly unwanted components or pieces of other algebraic curves that can also be traced out. This question also arises for the linkages that trace a line segment.

GLOSSARY

Real algebraic set: A subset of $\mathbb{R}^N$ given by a finite number of polynomial equations with real coefficients.

Real semi-algebraic set: A subset of $\mathbb{R}^N$ given by a finite number of polynomial equations and inequalities with real coefficients.

It is important to realize the distinction between an algebraic set and a semi-algebraic set. For example, a circle (excluding its interior) is an algebraic set, while a (closed) line segment is a semi-algebraic set but not an algebraic set. The linear projection of an algebraic set is always a semi-algebraic set, but it may not be an algebraic set. The configuration space of a linkage is an algebraic set, but the locus of possible positions of one of its vertices is only guaranteed to be a semi-algebraic set, because it represents the projection onto the coordinates corresponding to one of the vertices of the linkage.

ARBITRARY CONFIGURATION SPACES

One of the more precise results related to Kempe's result is the following:

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THEOREM 9.4.1 *Creating linkage configuration spaces* [KM95]

Let M be any compact smooth manifold. Then there is a planar linkage whose configuration space is diffeomorphic to a disjoint union of some number of copies of M.

This result was also claimed by Thurston, but there does not seem to be a written proof by him. As a consequence of this result, we obtain the following precise version of what Kempe was trying to claim. This consequence is proved by King [Kin99] using the techniques of Kapovich-Millson [KM02] and Thurston.

THEOREM 9.4.2 *Tracing out an algebraic curve* [Kin99]

Let X be any set in the plane that is the polynomial image of a closed interval. Then there is a linkage in the plane with some pinned vertices such that one of the vertices traces out X exactly.

See [JS99, BM56] for other discussions of how to create linkages to trace out at least a portion of a given algebraic curve. King [Kin98] also generalizes this result to higher dimensions and to the semialgebraic sets arising from projecting the configuration space down to consider some subset of the vertices. See also [KM02] for connections to universality theorems concerning configuration spaces of lines in the plane, for example, as in the work of [Mnë88]. The complexity results of [HJW85] described in Section 9.5 build off a universality construction similar to those mentioned above.

Abbott, Barton, and Demaine [Abb08] characterized the number of bars required to draw a polynomial curve of degree n in d dimensions, as $\Theta(n^d)$. Their constructions also have stronger continuity properties than King's, so that a single motion can construct the entire algebraic set. By giving a polynomial-time construction of a configuration of the linkage, they also prove coNP-hardness of testing rigidity.

Abel et al. [ADD$^+$16] prove that these universality results hold when linkages either do not or are required not to have crossing bars in their configurations. They further prove $\forall\mathbb{R}$-completeness of testing rigidity and global rigidity, and $\exists\mathbb{R}$-completeness of graph realization, even in the noncrossing scenario.

9.5 COMPUTATIONAL COMPLEXITY

There are a variety of algorithmic questions that can be asked about a given linkage. Most of these questions are computationally difficult to answer, either NP-hard or PSPACE-hard. Nonetheless, given the importance of these problems, there is work on developing (exponential-time) algorithms.

GLOSSARY

Ruler folding problem: Given a polygonal arc (i.e., a sequence of bar lengths) and a desired length L, is there a configuration of the arc (ruler) in which the bars lie along a common line segment of length L? If so, find such a configuration. (The problem can also be phrased as reconfiguration, provided the linkage is permitted to self-intersect.)

Reachability problem: Given a configuration of a linkage, a distinguished vertex, and a point in the plane, is it possible to reconfigure the linkage so that the distinguished vertex touches the given point? If so, find such a reconfiguration. In this problem, the linkage has one or more vertices pinned to particular locations in the plane.

Reconfiguration problem: Given two configurations of a linkage, is it possible to reconfigure one into the other? If so, find such a reconfiguration.

Locked decision problem: Given a linkage, is it locked?

HARDNESS RESULTS

One of the simplest complexity results is about the ruler folding problem, obtained via a reduction from set partition:

THEOREM 9.5.1 *Complexity of ruler folding* [HJW85]

The ruler folding problem is NP-complete.

Building on this result, the same authors establish

THEOREM 9.5.2 *Complexity of arc reachability* [HJW85]

The reachability problem is NP-hard for a planar polygonal arc in the presence of four line-segment obstacles and permitting the arc to self-intersect.

For general linkages instead of arcs, stronger complexity results exist:

THEOREM 9.5.3 *Complexity of reachability* [HJW84]

The reachability problem is PSPACE-hard for a planar linkage without obstacles and permitting the linkage to self-intersect.

On the other hand, a similar result holds for a polygonal arc among obstacles:

THEOREM 9.5.4 *Complexity of arc reachability among obstacles* [JP85]

The reachability problem is PSPACE-hard for a planar polygonal arc in the presence of polygonal obstacles and permitting the arc to self-intersect.

Finally, when the linkage is not permitted to self-intersect, and there are no obstacles, hardness is known in cases when the linkage can be locked; see Section 9.3.

THEOREM 9.5.5 *Complexity of non-self-intersecting arc reconfiguration* [AKRW03]

The reconfiguration problem is PSPACE-hard for a 3D polygonal arc or a 2D polygonal tree when the linkage is not permitted to self-intersect.

ALGORITHMS

Algorithms for linkage reconfiguration problems can be obtained from general motion-planning results in Chapter 50 (Section 50.1.1). This connection seems to have first been made explicit in [AKRW03]. To apply the roadmap algorithm of Canny [Can87] (Theorem 50.1.2), we first phrase the algorithmic linkage problems into the motion-planning framework; see also [DO07, ch. 2].

The configuration space of a given linkage is the subset of $\mathbb{R}^{vc}$ in which every point satisfies certain bar-length constraints and, if desired, non-intersection constraints between all pairs of bars. Both types of constraints can be phrased using constant-degree polynomial equations and inequalities, e.g., the former by setting the squared length of each bar to the desired value. (There are also embeddings of the configuration space into Euclidean space with fewer than vc dimensions, dependent on the number of degrees of freedom in the linkage, but the vc-dimensional parameterization is most naturally semi-algebraic.)

Returning to the motion-planning framework, the polynomial equations and inequalities are precisely the obstacle surfaces. The configuration space has dimension $k = vc$, and there are $n \leq b^2$ obstacle surfaces where b is the number of bars, each with degree $d = O(1)$. We can factor out the trivial rigid motions by supposing that one bar of the linkage is pinned, reducing k to $(v-2)c$. Now running the roadmap algorithm produces a representation of the entire configuration space. By path planning within this space, we can solve the reconfiguration problem. By a simple pass through the representation, we can tell whether the space is connected, solving the locked decision problem. By slicing the space with a polynomial specifying that a particular vertex is at a particular point in the plane, we can solve the reachability problem.

Plugging $k \leq vc$, $n \leq b^2$, and $d = O(1)$ into the roadmap algorithm with deterministic running time $O(n^k(\log n)d^{O(k^4)})$ and randomized expected running time $O(n^k(\log n)d^{O(k^2)})$, we obtain

COROLLARY 9.5.6 *Roadmap algorithm applied to linkages* [AKRW03]
The reachability, reconfiguration, and locked decision problems can be solved for an arbitrary linkage with v vertices and b bars in $\mathbb{R}^c$ using $O(b^{2vc}(\log b)2^{O(vc)^4})$ deterministic time or $O(b^{2vc}(\log b)2^{O(vc)^2})$ expected randomized time.

9.6 KINEMATICS

According to Bottema and Roth [BR79b], "kinematics is that branch of mechanics which treats the phenomenon of motion without regard to the cause of the motion. In kinematics there is no reference to mass or force; the concern is only with relative positions and their changes." Kinematics is a subject with a long history and which has had, at various times, notable influence on and has to some extent has been partially identified with such areas as algebraic geometry, differential geometry, mechanics, singularity theory, and Lie theory. It has often been a subject studied from an engineering point of view, and there are many detailed calculations with respect to particular mechanisms of interest. As a representative example, we consider four-bar mechanisms (Figure 9.6.1):

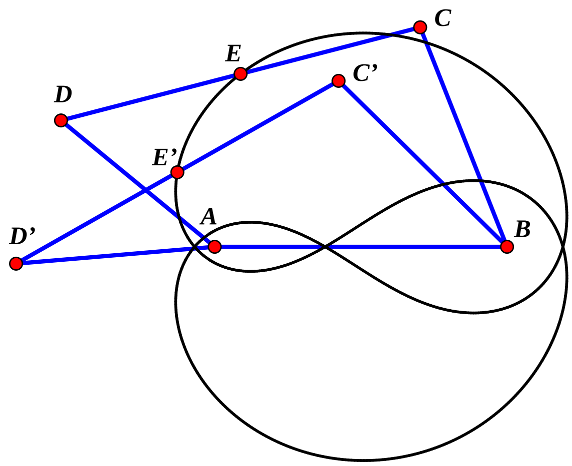

FIGURE 9.6.1
The coupler curve of the midpoint E of the coupler CD as it moves relative to the frame AB in a four-bar mechanism.

GLOSSARY

Mechanism: A linkage with one degree of freedom, modulo global translation and rotation.

Four-bar mechanism: A four-bar polygonal cycle; see Figure 9.6.1 for an example. Sometimes called a ***three-bar mechanism***.

Frame: We generally fix a frame of reference for a mechanism by pinning one bar, fixing its position in the plane. This bar is called the ***frame***. In Figure 9.6.1, bar AB is pinned.

Coupler: A distinguished bar other than the frame. In Figure 9.6.1, we consider the coupler CD.

Coupler motion: The motion of the entire plane induced by the relative motion of the coupler with respect to the frame.

Coupler curve: The path traced during the coupler motion by any point rigidly attached to the coupler (e.g., via two additional bars). Figure 9.6.1 shows the coupler curve of the midpoint E of the coupler bar CD.

FOUR-BAR MECHANISM

Coupler curves can be surprisingly complex. In the generic case, a coupler curve of a four-bar mechanism is an algebraic curve of degree 6. Substantial effort has been put into cataloging the different shapes of coupler curves that can arise from four-bar and other mechanisms. A sample theorem in this context is the following:

THEOREM 9.6.1 *Multiplicity of coupler curves* [Rob75]

Any coupler curve of a four-bar mechanism can be generated by two other four-bar mechanisms.

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GLOSSARY

Infinitesimal motion or ***first-order flex:*** The first derivative of a motion at an instantaneous moment in time, assigning a velocity vector to each point involved in the motion. (See Chapter 61 for a more thorough explanation in the context of rigidity.)

Pole or ***instantaneous pole:*** The instantaneous fixed point of a first-order motion of the plane. For a rotation, the pole is the center of rotation. For a translation, the pole is a point at infinity in the projective plane. A combination of rotation and translation can be rewritten as a pure rotation.

Polode: The locus of poles over time during a motion of the plane.

POLES

Some of the central theorems in kinematics treat the instantaneous case. Poles characterize the first-order action of a motion at each moment in time. Together, the polode can be viewed relative to either the fixed plane of the frame (the ***fixed polode***) or the moving plane of the coupler (***moving polode***). Apart from degenerate cases, a planar motion can be described by the moving polode rolling along the fixed polode. A basic theorem in the context of poles is the following:

THEOREM 9.6.2 *Three-Pole Theorem*

For any three motions of the plane, the instantaneous poles of the three mutual relative motions are collinear at any moment in time.

FURTHER READING

For a general introduction and sampling of the field of kinematics, see [Hun78, BR79b, Sta97, McC90, Pot94, McC00]. For relations to singularity theory, see e.g., [GHM97]. For examples, analysis, and synthesis of specific mechanisms such as the four-bar mechanism, see [GN86, Mik01, Sta99, Ale95, BS90, Leb67, Con79, Con78]. For some typical examples from an engineering viewpoint, see e.g., [CP91, Che02, Ler00]. See also Section 60.4 of this Handbook.

9.7 APPLICATIONS

Applications of linkages arise throughout science and engineering. We highlight three modern applications: robotics, manufacturing, and protein folding.

APPLICATIONS IN ENGINEERING

The study of linkages in fact originated in the context of mechanical engineering, e.g., for the purpose of converting circular motion into linear motion. Today, one of the driving applications for linkages is *robotics*, in particular *robotic arms*.

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A robotic arm can be modeled as a linkage, typically a polygonal chain. Some robotic arms have hinges that force the bars to remain coplanar, modeled by 2D chains; other arms have universal joints, modeled by 3D chains; other arms pose additional different constraints (such as incident bars being coplanar, yet the whole linkage need not be coplanar), leading to other models of linkage folding. Some planar robotic arms reserve slightly offset planar planes for the bars, modeled by a planar polygonal chain that permits self-intersection. Most other robotic arms are modeled by disallowing self-intersection.

The reachability problem is largely motivated by robotic arms, where the "hand" at one end of the arm must be placed at a particular location, e.g., to pick up an object, but the rest of the configuration is secondary. In other contexts, the entire configuration of the arm is important, and we need to plan a motion to a target configuration, leading to the reconfiguration problem. The locked decision problem is the first question one might ask about the simplicity/complexity of motion planning for a particular type of linkage. However, all of these problems are typically studied in the context of linkages without obstacles, but in robotics there are almost always obstacles. Some obstacles, such as a halfplane representing the floor, can often be avoided; but more generally the problems become much more complicated. See Chapter 51.

Another area with linkage applications is *manufacturing*. Given a straight hydraulic tube or piece of wire, a typical goal is to produce a desired folded configuration. In these contexts, we want to bend the wire as little as possible. In particular, a typical constraint is to bend the wire only monotonically: once it is bent one way, it cannot be bent the other way. This constraint forces straight segments of the target shape to remain straight throughout the motion. Thus, the problem can be modeled as straightening a polygonal chain, either in 2D or 3D depending on the application, with additional constraints. For example, the expansive motions described in Section 9.3 fold all joints monotonically; however, their reliance on bending most joints simultaneously may be undesirable. Arkin et al. [AFMS01] consider the restriction in which only a single joint can be rotated at once, together with additional realistic constraints arising in wire bending.

APPLICATIONS IN BIOLOGY

A crude model of a protein backbone is a polygonal chain in 3D, and a similarly crude model of an entire protein is a polygonal tree in 3D. In both cases, the vertices represent atoms, and the bars represent bonds between atoms (which in reality stay roughly the same length). In proteins, these bar/bond lengths are typically all within a factor of 2 of each other. Two atoms cannot occupy the same space, which can be roughly modeled by disallowing self-intersection. One interesting open problem in this context is the following:

PROBLEM 9.7.1 *Equilateral or near-equilateral locked linkages* [BDD+01]

Is there a locked equilateral arc, cycle, or tree in 3D? More generally, what is the smallest value of $\alpha \geq 1$ for which there is a locked arc/cycle/tree in 3D with all edge lengths between 1 and α?

These crude models may lead to some biological insight, but they do not capture several aspects of real protein folding.

One aspect that can easily be incorporated into linkage folding is that the angles between incident bars is typically fixed. This *fixed-angle constraint* can alternatively be viewed as adding bars between vertices originally at distance two from each another. Soss et al. [Sos01, SEO03, ST00] initiated study of such fixed-angle linkages in computational geometry, in particular establishing NP-hardness of deciding reconfigurability or flattenability. Aloupis et al. [ADD+02, ADM+02] consider when fixed-angle linkages are not locked in the sense that all flat states are reachable from each other by motions avoiding self-intersection. Borcea and Streinu [BS11b, BS11a] gave a polynomial-time algorithm for computing the maximum or minimum attainable distance between the endpoints of a fixed-angle arc; the maximum problem was also solved by Benbernou and O'Rourke [Ben11].

A more challenging aspect of protein folding is the *thermodynamic hypothesis* [Anf72]: that folding is encouraged to follow energy-minimizing pathways. Indeed, the bars are not strictly binding, nor are they completely fixed in length; they are merely encouraged to do so, and sometimes violate these constraints. Unfortunately, these properties are difficult to model, and energy functions defined so far are either incomplete or difficult to manipulate. Also, the implications on linkage-folding problems remain unclear.

One particularly simple energy-based model of protein folding studied in both computer science and biology is the HP (Hydrophilic-Hydrophobic) model; see e.g., [ABD+03, CD93, Dil90, Hay98, ZKL07]. This model is particularly discrete, modeling a protein as an equilateral chain on a lattice, typically square or cubic grid, but possibly also a triangular or tetrahedral lattice. The model captures only hydrophobic bonds and forces, clustering to avoid external water. Finding the optimal folding even in this simple model is NP-complete [BL98, CGP+98], though there are several constant-factor approximation algorithms [HI96, New02, ABD+97] and some practical heuristics [ZKL07]. One interesting open problem is whether designing a protein to fold into particular shape is easier than finding the shape to which a particular protein folds [ABD+03]:

PROBLEM 9.7.2 *HP protein design* [ABD+03]

What is the complexity of deciding whether a given subset of the lattice is an optimal folding of some HP protein, and if so finding such a protein? What if it must be the unique optimal folding of the HP protein?

A result related to the second half of this problem is that arbitrarily long HP proteins with unique optimal foldings exist, at least for open and closed chains in a 2D square grid [ABD+03].

9.8 SOURCES AND RELATED MATERIAL

FURTHER READING

[DO07]: The main book on geometric folding in general, Part I of which focuses on linkage folding. Parts II and III consider folding (reconfiguration) of objects of larger intrinsic dimension, in particular 2 (pieces of paper) and 3 (polyhedra).

[Dem12]: An online class about geometric folding in general, and linkage folding in particular.

[O'R98, Dem00, Dem02]: Older surveys on geometric folding in general.

RELATED CHAPTERS

Chapter 37: Computational and quantitative real algebraic geometry
Chapter 50: Algorithmic motion planning
Chapter 51: Robotics
Chapter 52: Computer graphics
Chapter 53: Modeling motion
Chapter 60: Geometric applications of the Grassmann-Cayley algebra
Chapter 61: Rigidity and scene analysis
Chapter 65: Applications to structural molecular biology

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