63 GLOBAL RIGIDITY

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INTRODUCTION

Chapter 61 described the basic theory of infinitesimal rigidity of bar and joint structures and a number of related structures. In this chapter, we consider the stronger properties of:

(a) Global Rigidity: given a discrete configuration of points in Euclidean d-space, and a set of fixed pairwise distances, is the set of solutions unique, up to congruence in d-space?

(b) Universal Rigidity: given a discrete configuration of points in Euclidean d-space, and a set of fixed pairwise distances, is the set of solutions unique, up to congruence in all dimensions $d' \geq d$?

(c) How do global rigidity and universal rigidity depend on the combinatorial properties of the associated graph, in which the vertices and edges correspond to points and fixed pairwise distances, respectively, and how do they depend on the specific geometry of the initial configuration?

To study global rigidity, we use vocabulary and techniques drawn from (i) structural engineering: bars and joints, redundant first-order rigidity, static self-stresses (linear techniques); as well as from (ii) minima for energy functions with their companion stress matrices. There are both global rigidity theorems which hold for almost all realizations of a graph G based on combinatorial properties of the graph; and global rigidity theorems that hold for some specific realizations, and depend on the particular details of the geometry of (G, p).

Specifically, there are many globally rigid frameworks where the underlying graph is not generically globally rigid. It is computationally hard to test the global rigidity of such a particular framework.

For universal rigidity, we adapt the stress techniques from semidefinite programming. Even when universal rigidity occurs for some generic realizations of a graph, it may not occur for all such realizations, so it is not a generic property in the broad sense. However, it is weakly generic in the sense that a graph can have a full dimensional open subset of universally rigid realizations. Recent results show a strong connection: there exists a generic globally rigid realization of a graph if and only if there exists a generic universally rigid realization of that graph. So there is little difference in the combinatorics for generic realizations. However, a recent algorithm tests the universal rigidity of a given geometric framework (G, p), whereas no algorithm exists for testing global rigidity of a specific framework. Thus universal rigidity can be a valuable tool for confirming the global rigidity of a specific geometric framework (G, p).

Some results and techniques for universal rigidity were developed for tensegrity frameworks, where the bars with fixed lengths are replaced by cables (which can

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only become shorter) and struts (which can only become longer). These have a narrower set of stresses and relevant stress matrices, and provide some additional insights into the behaviour of real structures and the ways the techniques are applied.

Global rigidity has a significant range of applications, such as localization in sensor networks and molecular conformations. Some applications and extensions also involve variations of the structure. Work on global rigidity of symmetric structures (Chapter 62) is in its initial stages.

63.1 BASICS FOR GLOBAL AND UNIVERSAL RIGIDITY OF GRAPHS

Global rigidity results have both a combinatorial form, belonging to graphs, and a geometric form, depending on the special geometry of the realizations. The same two forms are found for universal rigidity.

63.1.1 BASICS FOR GLOBALLY RIGID GRAPHS

We begin with some basic results that apply to generic frameworks and therefore almost all frameworks on a given graph. As such, they can be presented in terms of the graphs.

GLOSSARY FOR GLOBAL AND UNIVERSAL RIGIDITY

Configuration of points in d-space: A map $p : V \to \mathbb{R}^d$ that assigns points $p_i \in \mathbb{R}^d$, $1 \le i \le n$, to an index set $V = \{1, 2, ..., n\}$.

Generic configuration: A configuration for which the set of the $d|V|$ coordinates of the points is algebraically independent over the rationals.

Congruent configurations: Two configurations p and q in d-space, on the same set V, related by an isometry T of $\mathbb{R}^d$ (with $T(p_i) = q_i$ for all $i \in V$).

Bar-and-joint framework in d-space (or ***framework:***) A pair (G, p) of a graph $G = (V, E)$ (no loops or multiple edges) and a configuration p in d-space for the vertex set V. We shall assume that there are no 0-length edges, that is, $p(u) \neq p(v)$ for all $uv \in E$.

Realization of graph G in d-space: A d-dimensional framework (G, p).

Globally rigid framework: A framework (G, p) in d-space for which every d-dimensional realization (G, q) of G with the same edge lengths as in (G, p) is congruent to (G, p).

Rigidity matrix: For a framework (G, p) in d-space, $R(G, p)$ is the $|E| \times d|V|$ matrix for the system of equations: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$ in the unknown velocities p'_i. The first-order flex equations are expressed as

$$R(G,p){p'}^T = \begin{bmatrix} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \end{bmatrix} \times {p'}^T = 0^T.$$

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Equilibrium stress: For a framework (G, p), an assignment of scalars ω_{ij} to the edges such that at each vertex i,

$$\sum_{\{j|\ \{i,j\}\in E\ \}} \omega_{ij}(p_i - p_j) = 0.$$

Equivalently, a row dependence for the rigidity matrix: $\omega R(G, p) = 0$.

Stress matrix: For a framework (G, p) in d-space and equilibrium stress ω the stress matrix Ω is the $|V| \times |V|$ symmetric matrix in which the entries are defined so that $\Omega[i, j] = -\omega_{ij}$ for the edges, $\Omega[i, j] = 0$ for nonadjacent vertex pairs, and $\Omega[i, i]$ is calculated so that each row and column sum is equal to zero.

Independent framework: A framework (G, p) for which the rigidity matrix has independent rows. Equivalently, there is only the zero equilibrium stress for (G, p).

Universally rigid framework: A framework (G, p) in d-space for which every other d'-dimensional realization (G, q) of G with the same edge lengths as in (G, p) is congruent to (G, p).

BASICS FOR GLOBAL RIGIDITY OF GENERIC FRAMEWORKS

It is a hard problem to decide if a given framework is globally rigid. Saxe [Sax79] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. See Example 63.1.5 below for some subtleties that may arise. The problem becomes tractable, however, if we consider generic frameworks, that is, frameworks (G, p) for which p is generic. For these frameworks we have the following fundamental necessary and sufficient conditions in terms of stresses and stress matrices due to Connelly [Con05] (sufficiency) and Gortler, Healy and Thurston [GHT10] (necessity), respectively.

THEOREM 63.1.1 Stress Matrix Condition for Global Rigidity

Let (G, p) be a generic framework in $\mathbb{R}^d$ on at least $d+2$ vertices. (G, p) is globally rigid in $\mathbb{R}^d$ if and only if (G, p) has an equilibrium stress ω for which the rank of the associated stress matrix Ω is $|V| - d - 1$.

Theorem 63.1.1 implies that global rigidity is a generic property in the following sense.

THEOREM 63.1.2 Generic Global Rigidity Theorem

For a graph G and a fixed dimension d the following are equivalent:

(a) *(G, p) is globally rigid for some generic configuration $p \in \mathbb{R}^d$;*

(b) *(G, p) is globally rigid for all generic configurations $p \in \mathbb{R}^d$.*

(c) *(G, p) is globally rigid for an open dense subset of configurations $p \in \mathbb{R}^d$.*

63.1.2 BASICS FOR UNIVERSAL RIGIDITY OF FRAMEWORKS

There are several key basic theorems for universally rigid generic frameworks. These depend on a stress matrix Ω being positive semi-definite (PSD) with maximal rank

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$n - d - 1$, where $n = |V|$. These properties can be broken down into two stages: dimensional rigidity, where the stress matrix has full rank $n-d-1$, leaving possible affine motions within the space, and super stability where the stress matrix is PSD and the directions of the framework do not lie on a conic (eliminating affine motions within the space).

BASIC RESULTS FOR UNIVERSAL RIGIDITY

For generic frameworks we have a more refined stress matrix condition, similar to that of Theorem 63.1.1, due to Gortler and Thurston [GT14a].

THEOREM 63.1.3 Stress Matrix Condition for Universal Rigidity

Let (G, p) be a generic framework in $\mathbb{R}^d$ on at least $d + 2$ vertices. Then (G, p) is universally rigid if and only if there exists an equilibrium stress ω for (G, p) for which the associated stress matrix Ω is positive semi-definite and has rank $|V|-d-1$.

Universal rigidity of frameworks is not a generic property. Example 63.1.5 below illustrates that the four-cycle $K_{2,2}$ has generic realizations which are universally rigid as well as generic realizations which are not universally rigid on the line. This leads to two different problems concerning the combinatorial aspects of universal rigidity. Given an integer $d \geq 1$, characterize the graphs G (i) for which *every* generic realization of G in $\mathbb{R}^d$ is universally rigid, (ii) for which *some* generic realization of G in $\mathbb{R}^d$ is universally rigid.

The graphs satisfying (i) will be called *generically universally rigid* in $\mathbb{R}^d$. The characterization of these graphs is an unsolved problem, even in $\mathbb{R}^1$.

For each universally rigid generic framework (G, q) in $\mathbb{R}^d$, there is an open neighbourhood $U(q)$ of q in $\mathbb{R}^{d|V|}$, such that for all $p \in U(q)$, the framework (G, p) is universally rigid [CGT16a]. For this reason, the graphs satisfying (ii) are called *openly universally rigid* or *weakly generically universally rigid* (*WGUR*, for short).

BASIC RESULTS CONNECTING GLOBAL AND UNIVERSAL RIGIDITY

It is clear that a framework that is universally rigid is also globally rigid and hence every weakly generically universally rigid graph is generically globally rigid. What may be surprising is that recent results [CGT16a] confirm that these two families of graphs are the same, for every fixed dimension. The following theorem summarizes this equivalence.

THEOREM 63.1.4 Equivalence of Generic Global Rigidity and Weak Generic Universal Rigidity

For a graph G on at least $d + 2$ vertices and a fixed dimension d, the following are equivalent:

(i) *a graph G is generically globally rigid in $\mathbb{R}^d$;*

(ii) *a graph is weakly generically universally rigid in $\mathbb{R}^d$;*

(iii) *there exists a generic framework (G, q) in $\mathbb{R}^d$ which is universally rigid;*

(iv) *there exists a generic framework (G, p) in $\mathbb{R}^d$ which is globally rigid;*

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(v) *there exists a generic framework* (G, p) *in* $\mathbb{R}^d$ *with a stress matrix* $\Omega(G, p)$ *which has rank* $n - d - 1$*;*

(vi) *there exists a generic framework* (G, q) *in* $\mathbb{R}^d$ *with a PSD stress matrix* Ω *which has rank* $n - d - 1$.

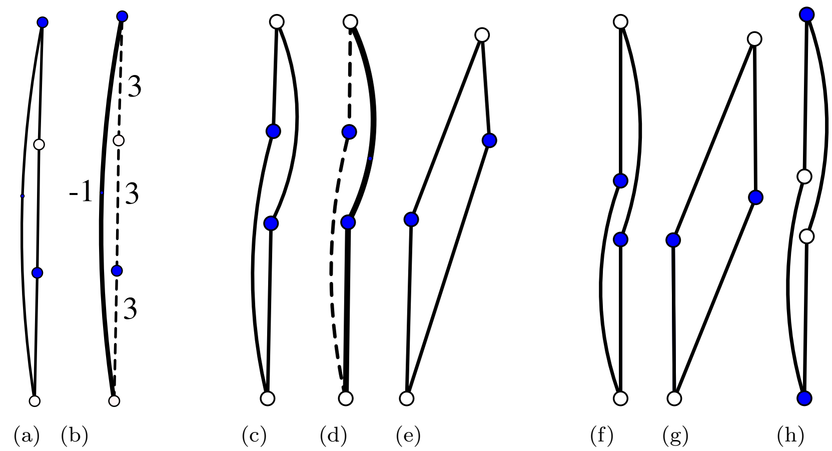

FIGURE 63.1.1
A range of frameworks on the graph $K_{2,2}$. *Figures (a,b) present a universally rigid framework, where (b) uses the convention of dashed lines for tension, and thick lines for compression. Figures (c,d,e) present a framework which is globally rigid but not universally rigid as (e) illustrates. Figures (f,g,h) present a framework which is not globally rigid as (f) and (h) have the same edge lengths on the line.*

We illustrate these definitions and connections with a collection of related simple frameworks on the line.

EXAMPLE 63.1.5 $K_{2,2}$ on the line

Consider the simple 'bow' framework of Figure 63.1.1(a). Intuitively, this is universally rigid (and therefore globally rigid). We can confirm this with a positive semi-definite (PSD) stress matrix, for the simple stress of Figure 63.1.1(b):

$$\Omega(G,p) = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

One can check that this is PSD by taking the $k \times k$ determinants from the top corner, down the diagonal $1 \leq k \leq 4$. Moreover, Ω has rank $2 = 4 - 1 - 1$, as required for universal (and global) rigidity in dimension 1. Although this specific framework is not generic, a small perturbation of the vertices will make it generic, and will preserve the PSD property. So all the theorems above apply to $K_{2,2}$. It is generically globally rigid and is weakly generically universally rigid.

We know that all generic frameworks on $K_{2,2}$ on the line will be globally rigid. Consider the framework in Figure 63.1.1(c) which is globally rigid. However, Figure 63.1.1(e) shows another realization with the same lengths, in the plane, so it is

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not universally rigid. It has a stress matrix Ω for the stress with signs illustrated in Figure 63.1.1(d) which has rank 2, but this cannot be PSD (by Figure 63.1.1(e)).

While the framework in Figure 63.1.1(c) is globally rigid, the framework in Figure 63.1.1(f) is in a special position with the same pattern and is not globally rigid. It can move through the plane as a parallelogram (Figure 63.1.1 (g) to Figure 63.1.1 (h)) to another framework, which is also on the line but not congruent. Figures 63.1.1 (f,h) have different stresses, each with a stress matrix of rank 2, but these frameworks are not globally rigid, reminding us that the theorems above are only guaranteed to be sufficient for generic frameworks.

63.2 COMBINATORICS FOR GENERIC GLOBAL RIGIDITY

The major goal in generic global rigidity is a combinatorial characterization of graphs with globally rigid generic realizations in d-space. The companion problem is to find efficient combinatorial algorithms to test graphs for generic global rigidity. For the plane (and the line), this is solved. Beyond the plane the results are incomplete, but some significant partial results are available.

Because of Theorem 63.1.4, all results for generic global rigidity are also results for weak generic universal rigidity. We will usually not mention this extension. For geometric—not generic—frameworks, there will be some key differences in the techniques and results.

GLOSSARY

Globally rigid graph in $\mathbb{R}^d$: A graph G for which some (or equivalently, all) generic configurations p produce globally rigid frameworks (G, p) in d-space.

Edge splitting operation (d-dimensional): Replaces an edge of graph G with a new vertex joined to the end vertices of the edge and to $d-1$ other vertices.

d-connected graph: A graph G such that removing any $d-1$ vertices (and all incident edges) leaves a connected graph. (Equivalently, a graph such that any two vertices can be connected by at least d paths that are vertex-disjoint except for their endpoints.)

k-tree-connected graph: A graph G which contains k edge-disjoint spanning trees.

Highly k-tree-connected graph: A graph G for which the removal of any edge leaves a k-tree-connected graph.

Rigid graph in $\mathbb{R}^d$: A graph G for which some (or equivalently, all) generic configurations p produce rigid frameworks (G, p) in d-space.

Redundantly rigid graph in $\mathbb{R}^d$: A graph $G = (V, E)$ for which $G - e$ is rigid in $\mathbb{R}^d$ for all $e \in E$.

M-circuit (or generic circuit) in $\mathbb{R}^d$: A graph $G = (V, E)$ for which a generic realization (G, p) in $\mathbb{R}^d$ is dependent, but $(G - e, p)$ is independent in d-space for all $e \in E$.

M-connected graph in $\mathbb{R}^d$: A graph $G = (V, E)$ for which every edge pair $e, f \in E$ belongs to a subgraph H of G which is an M-circuit.

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Cone graph: The graph $G * u$ obtained from $G = (V, E)$ by adding a new vertex u and the $|V|$ edges (u, i) for all vertices $i \in V$ (Figure 63.2.4.)

BASIC PROPERTIES FOR GLOBAL RIGIDITY OF GRAPHS IN ALL DIMENSIONS

The following necessary conditions, due to Hendrickson [Hen82], provide a basic link between local and global rigidity.

THEOREM 63.2.1 Hendrickson's Necessary Conditions

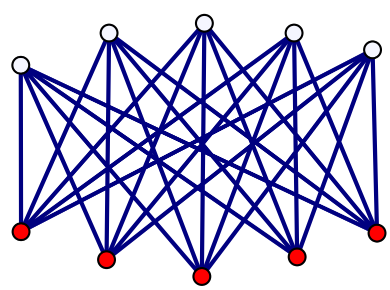
Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is

(a) $(d+1)$-connected; and

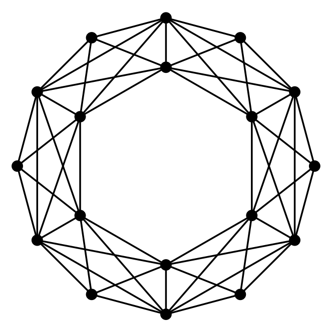
(b) redundantly rigid in $\mathbb{R}^d$.

It is clear that the $d+1$ connectivity condition is necessary for the broader class of general position frameworks which are globally rigid (or universally rigid). For a weaker converse see Theorem 63.5.4.

The necessity of redundant rigidity can also be observed from the Stress Matrix Condition Theorem 63.1.1. If an edge is not redundant the pair has a zero entry in the stress matrix. Removing the edge makes a smaller graph which is also globally rigid. On the other hand, removing the edge at a generic configuration, makes the graph flexible—a contradiction.



(a)



(b)

FIGURE 63.2.1
The two smallest known Hendrickson graphs in 3-space.

These necessary conditions together are also sufficient to imply the global rigidity of the graph in $\mathbb{R}^d$ for $d = 1, 2$, as we shall see below. This is not the case, however, for dimensions $d \geq 3$. We say that a graph G is a *Hendrickson graph* in $\mathbb{R}^d$ if it satisfies the necessary conditions (a) and (b) of Theorem 63.2.1 in $\mathbb{R}^d$ but it is not globally rigid in $\mathbb{R}^d$. For $d = 3$, Connelly [Con91] showed that the complete bipartite graph $K_{5,5}$ is a Hendrickson graph. He also constructed similar examples (specific complete bipartite graphs on $\binom{d+2}{2}$ vertices) for all $d \geq 3$. Frank and Jiang [FJ11] found two more (bipartite) Hendrickson graphs in $\mathbb{R}^4$ as well as infinite families in $\mathbb{R}^d$ for $d \geq 5$. Jordán, Király, and Tanigawa [JKT16] constructed infinite families of Hendrickson graphs for all $d \geq 3$ (see Figure 63.2.1 (b)). Further examples can be obtained by using the observation that the cone graph of a d-dimensional Hendrickson graph is a $d+1$-dimensional Hendrickson graph.

There is a generic global rigidity result for complete bipartite frameworks which combines results in [CG17, Con91].

THEOREM 63.2.2

A complete bipartite graph $K_{m,n}$ is generically globally rigid in $\mathbb{R}^d$ if and only if $m, n \geq d+1$ and $m+n \geq \binom{d+2}{2}+1$.

The examples with $m, n \geq d+1$ and $m+n = \binom{d+2}{2}$ include a number of Hendrickson graphs which are not globally rigid, such as $K_{5,5}$ in 3-space, and $K_{6,9}$, $K_{7,8}$ in 4-space.

INDUCTIVE CONSTRUCTIONS FOR GLOBAL RIGIDITY

Inductive constructions for graphs that preserve generic global rigidity are used both to prove theorems for general classes of frameworks and to analyze particular graphs.

Adding a new vertex of degree $d+1$ preserves global rigidity in $\mathbb{R}^d$. By applying a sequence of this operation to the base graph K_{d+1} we obtain the $(d+1)$-lateration graphs, which are thus sparse globally rigid graphs in $\mathbb{R}^d$. A finer and more useful operation that preserves global rigidity is edge splitting.

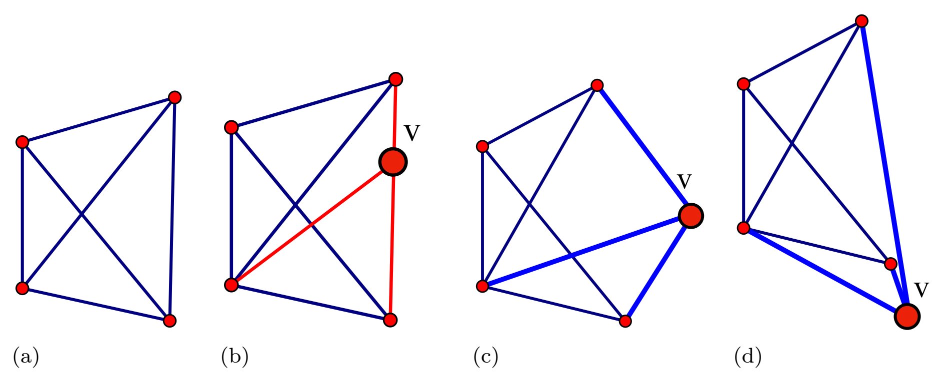


FIGURE 63.2.2
From these planar frameworks: (a) is universally rigid (it has a complete graph). The edge split (b) is globally rigid and (b,c) are universally rigid. Figure (d) is globally rigid, but not universally rigid (it lifts in $\mathbb{R}^3$).

THEOREM 63.2.3 Edge Split Theorem [Con05, JJS06]

Let $G=(V,E)$ be a graph obtained from a globally rigid graph H by a d-dimensional edge splitting operation. Then G is globally rigid in $\mathbb{R}^d$.

Another operation merges two graphs. For example, it is easy to see that the union of two graphs G_1, G_2 that are globally rigid in $\mathbb{R}^d$ is also globally rigid in

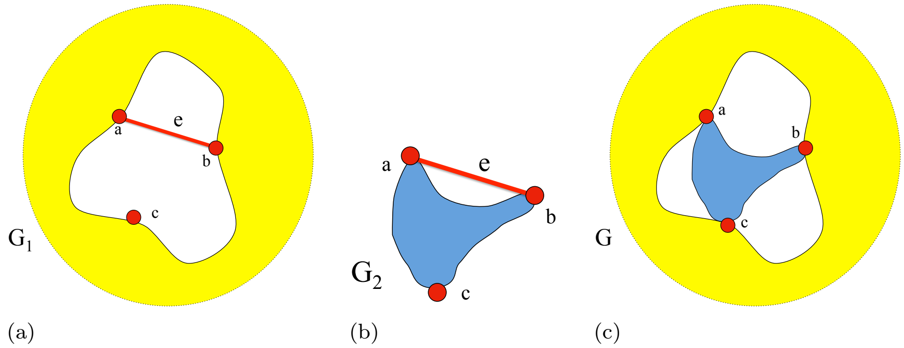


FIGURE 63.2.3
If G_1 (a) and G_2 (b) are globally rigid graphs sharing the edge e, then gluing on $d+1$ vertices creates a globally rigid graph without e.

$\mathbb{R}^d$, provided they share at least $d+1$ vertices. In fact it remains globally rigid even if we delete the edges of G_1 spanned by their common vertices before taking the union. An even stronger statement holds for graphs which have exactly $d+1$ vertices in common.

THEOREM 63.2.4 Gluing Theorem [Con11]

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are globally rigid graphs in $\mathbb{R}^d$ sharing at least $d+1$ vertices, then $G = (V_1 \cup V_2, E_1 \cup E_2 - G_1[V_1 \cap V_2])$ is globally rigid in $\mathbb{R}^d$.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are globally rigid graphs in $\mathbb{R}^d$ sharing exactly $d+1$ vertices and some edge e, then $G = (V_1 \cup V_2, E_1 \cup E_2 - e)$ is globally rigid in $\mathbb{R}^d$ (Figure 63.2.3).

The first statement of Theorem 63.2.4 extends to pairs of geometric globally rigid frameworks and to pairs of universally rigid frameworks, provided the overlapping vertices affinely span the space.

The following operation can also be used to construct globally rigid graphs from smaller graphs. Let G be a graph, X be a subset of $V(G)$, and let H be a graph on X (whose edges may not be in G). The pair (H, X) is called a *rooted minor* of (G, X) if H can be obtained from G by deleting and contracting edges of G, that is, if there is a partition $\{U_v | v \in X\}$ of $V(G)$ into $|X|$ subsets such that $v \in U_v$ and $G[U_v]$ are connected for all $v \in X$, and G has an edge between U_u and U_v for each $uv \in E(H)$. Let $K(X)$ denote the complete graph on the vertex set X.

THEOREM 63.2.5 Rooted Minor Theorem [Tan15]

Let G_1 and G_2 be graphs with $X = V(G_1) \cap V(G_2)$ and H be a graph on X. Suppose that $|X| \geq d+1$, G_1 is rigid in $\mathbb{R}^d$, (H, X) is a rooted minor of (G_2, X), $G_1 \cup H$ and $G_2 \cup K(X)$ are globally rigid or $G_1 \cup K(X)$ and $G_2 \cup H$ are globally rigid in $\mathbb{R}^d$. Then $G_1 \cup G_2$ is globally rigid in $\mathbb{R}^d$.

Note that the Rooted Minor Theorem implies the Gluing Theorem as well as the Edge Split Theorem.

GLOBALLY RIGID GRAPHS IN THE PLANE

Generic global rigidity has been completely characterized for dimensions up to 2. For all dimensions d it is easy to see that a graph G on at most $d+1$ vertices is globally rigid in $\mathbb{R}^d$ if and only if G is complete. Thus we shall formulate the results only for graphs with at least $d+2$ vertices.

The 1-dimensional result is folklore (see also Figure 63.1.1).

THEOREM 63.2.6 Global Rigidity on the Line

For a graph G with $|V| \geq 3$ the following are equivalent:

(a) *G is globally rigid in $\mathbb{R}^1$;*

(b) *G is 2-connected;*

(c) *there is a construction for G from K_3, using only edge splitting and edge addition.*

Note that 2-connected graphs are redundantly rigid in $\mathbb{R}^1$. The equivalence of (b) and (c) is clear by using the well-known ear-decompositions of 2-connected graphs.

The 2-dimensional result is based on the Edge Split Theorem (i.e., (c) implies (a) below), Hendrickson's Necessary Conditions (i.e., (a) implies (b)) and an inductive construction of 3-connected redundantly rigid graphs due to Jackson and Jordán [JJ05], which shows that (b) implies (c).

THEOREM 63.2.7 Global Rigidity in the Plane

For a graph G with $|V| \geq 4$ the following are equivalent:

(a) *G is globally rigid in $\mathbb{R}^2$;*

(b) *G is 3-connected and redundantly rigid in $\mathbb{R}^2$;*

(c) *there is a construction for G from K_4, using only edge splitting and edge addition.*

Note that if G has $|E| = 2|V| - 2$, that is, if G is a 3-connected M-circuit, then the edge splitting operation alone suffices in (c). This fact and an inductive construction for M-circuits can be found in Berg and Jordán [BJ03].

Since 6-connected graphs are redundantly rigid in $\mathbb{R}^2$, we obtain the following sufficient condition.

THEOREM 63.2.8 Sufficient Connectivity in the Plane

If a graph G is 6-connected, then G is globally rigid in $\mathbb{R}^2$.

The bound 6 above on vertex-connectivity is the best possible (see [LY82] for 5-connected graphs which are not even rigid). Jackson and Jordán [JJ09b] strengthened Theorem 63.2.8 by showing that 6-connectivity can be replaced by a weaker connectivity property, involving a mixture of vertex- and edge-connectivity parameters.

We remark that the characterization of globally rigid graphs in the plane has been used to deduce a number of variations (e.g., for zeolites [Jor10a], squares of graphs) and to solve related optimization problems (e.g., minimum cost anchor placement [Jor10b]).

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GENERICALLY GLOBALLY RIGID GRAPHS IN HIGHER DIMENSIONS

There are some additional generic results for higher dimensions. A useful observation, proved by Connelly and Whiteley [CW10], is that coning preserves global rigidity.

THEOREM 63.2.9 Generic Global Rigidity Coning

A graph G is globally rigid in $\mathbb{R}^d$ if and only if the cone $G * u$ is globally rigid in $\mathbb{R}^{d+1}$.

Tanigawa [Tan15] established another interesting connection between rigidity and global rigidity. We say that G is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$.

THEOREM 63.2.10 Vertex-redundant Rigidity Implies Global Rigidity

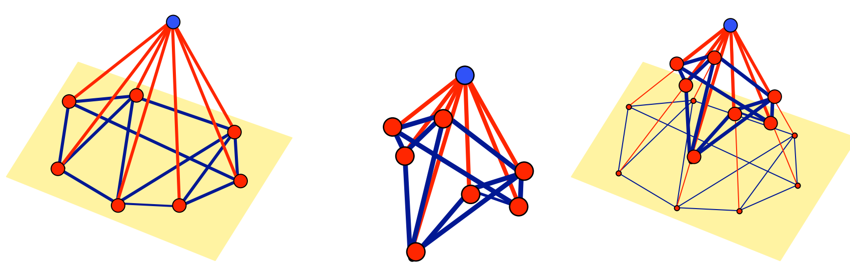
If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.

The concept of (vertex-)redundant rigidity can be generalized to d-dimensional *k-rigid* graphs, for all $k \geq 2$: these graphs remain rigid in $\mathbb{R}^d$ after removing any set of at most $k - 1$ vertices. Kaszanitzky and Király [KK16] solve a number of extremal questions related to these families. One can also consider higher degrees of redundant rigidity with respect to edge removal as well as similar notions for global rigidity. An interesting open problem is whether these graph properties can be tested in polynomial time for $d \geq 2$. Kobayashi et al. [KHKS16] show that a version of this problem for body-hinge graphs, in which sets of hinges are removed without losing global rigidity, is tractable.

Theorem 63.2.10 can be used to deduce a new sufficient condition for global rigidity as well as to solve an augmentation problem, see [Jor17].

THEOREM 63.2.11

If G is rigid in $\mathbb{R}^{d+1}$ then it is globally rigid in $\mathbb{R}^d$.



(a) (b) (c)

FIGURE 63.2.4

A redundantly rigid plane framework cones to a redundantly rigid framework in three-space (a). A redundantly rigid cone (b) slices back to a redundantly rigid framework in the plane (c).

THEOREM 63.2.12

Every rigid graph in $\mathbb{R}^d$ on $|V|$ vertices can be made globally rigid in $\mathbb{R}^d$ by adding at most $|V| - d - 1$ edges.

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The bound is tight for all $d \geq 1$: consider the complete bipartite graphs $K_{n-d,d}$.

A number of other related structures have also been investigated for generic global rigidity. These structures, which appear in engineering, robotics, and chemistry, can be defined in $\mathbb{R}^d$ for all $d \geq 2$. They consist of rigid bodies which are connected by disjoint bars (a body-bar framework) or by hinges (a body-hinge structure) that constrain the relative motion of the corresponding pairs of bodies to a rotation about an affine subspace of dimension $d-2$. For example in 3-space a hinge is a line (segment) which restricts the relative motion of the two bodies connected by the hinge to a rotation about the shared hinge line.

The *underlying multigraph* H of such a framework has one vertex for each body and one edge for each bar (resp. hinge), connecting the vertices of the bodies it connects. (See Section 61.2 for definitions and infinitesimal rigidity results.)

These frameworks can be modeled as bar-and-joint frameworks, with each body replaced by a large enough rigid framework on a complete graph. In the case of a body-bar framework these complete graphs are disjoint and the connecting bars correspond to vertex-disjoint bars between the complete subgraphs. For body-hinge frameworks the complete graphs corresponding to the bodies are constructed so that if two bodies are connected by a hinge then the complete graphs share $d-1$ vertices. The underlying graphs of these special bar-and-joint frameworks are called (d-dimensional) *body-bar graphs* and *body-hinge graphs*, respectively.

We shall formulate the results concerning the global rigidity of generic body-bar and body-hinge frameworks in terms of body-bar and body-hinge graphs, respectively. For these graphs, global rigidity has been fully characterized in $\mathbb{R}^d$ for all $d \geq 1$ in terms of their underlying multigraphs. The body-bar result is due to Connelly, Jordán, and Whiteley [CJW13].

THEOREM 63.2.13 Body-Bar Global Rigidity

Let G be a d-dimensional body-bar graph on at least $d+2$ vertices with underlying multigraph H and let $d \geq 1$ be an integer. Then the following are equivalent:

(a) *G is globally rigid in $\mathbb{R}^d$;*

(b) *G is redundantly rigid in $\mathbb{R}^d$;*

(c) *H is highly $\binom{d+1}{2}$-tree-connected.*

For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e. The body-hinge version was solved by Jordán, Király, and Tanigawa [JKT16].

THEOREM 63.2.14 Body-Hinge Global Rigidity

Let G be a d-dimensional body-hinge graph on at least $d+2$ vertices with underlying multigraph H and let $d \geq 3$ be an integer. Then the following are equivalent:

(a) *G is globally rigid in $\mathbb{R}^d$;*

(b) *$(\binom{d+1}{2} - 1)H$ is highly $\binom{d+1}{2}$-tree-connected.*

The two-dimensional characterization is slightly different.

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THEOREM 63.2.15 Body-Hinge Global Rigidity in the Plane

A 2-dimensional body-hinge graph G on at least 4 vertices with underlying graph H is globally rigid in $\mathbb{R}^2$ if and only if H is 3-edge-connected.

A recent result, due to Jordán and Tanigawa [JT17], characterizes globally rigid braced triangulations in $\mathbb{R}^3$.

GLOBALLY LINKED PAIRS AND THE NUMBER OF REALIZATIONS

Even if a graph is not globally rigid, some parts of it may have a unique realization with the given edge lengths. One notion that can be used to analyse these parts is as follows. We say that a pair $\{u, v\}$ of vertices of G is *globally linked* in G in $\mathbb{R}^d$ if for all generic d-dimensional realizations (G, p) we have that the distance between $q(u)$ and $q(v)$ is the same in all realizations (G, q) equivalent with (G, p). Thus a graph G is globally rigid in $\mathbb{R}^d$ if and only if all pairs of vertices in G are globally linked in $\mathbb{R}^d$. It is not hard to see that a pair $\{u, v\}$ is globally linked in G in $\mathbb{R}^1$ if and only if there exist two openly vertex-disjoint paths from u to v in G, which is equivalent to $\{u, v\}$ sharing a globally rigid subgraph. It follows that global linkedness is a generic property on the line.

In higher dimensions global linkedness is not a generic property and it remains an open problem to find a combinatorial characterization of globally linked pairs even in $\mathbb{R}^2$ (see Figure 63.2.5). This section lists some of the partial results that have been proven in the planar case by Jackson, Jordán, and Szabadka [JJS06, JJS14].

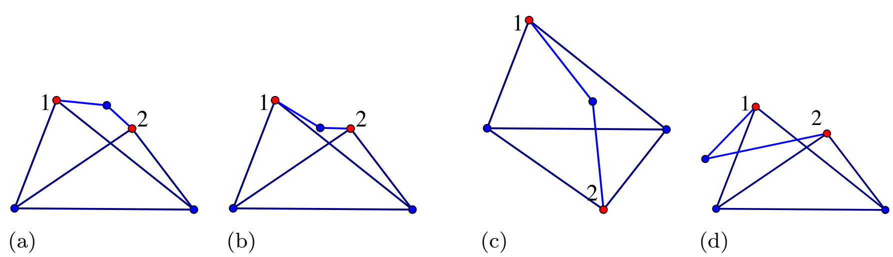

FIGURE 63.2.5
The pair $\{1, 2\}$ is globally linked in (a,b) although the framework is not globally rigid. In (c,d), with the same graph, $\{1, 2\}$ are not globally linked, so being globally linked is not a generic property of the graph.

THEOREM 63.2.16

Let G and H be graphs such that G is obtained from H by a two-dimensional edge splitting operation on edge xy and vertex w. If $H - xy$ is rigid in $\mathbb{R}^2$ and that $\{u, v\}$ is globally linked in H in $\mathbb{R}^2$, then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$.

For the family of M-connected graphs, global linkedness has been characterized as follows, see [JJS06]. Note that globally rigid graphs are M-connected and M-connected graphs are redundantly rigid in $\mathbb{R}^2$, see [JJ05].

THEOREM 63.2.17

Let $G = (V, E)$ be an M-connected graph in $\mathbb{R}^2$ and let $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if there exist three openly vertex-disjoint paths from u to v in G (Figure 63.2.6).

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In minimally rigid graphs, or more broadly, in all independent graphs, there are no globally linked pairs, other than the adjacent pairs of vertices [JJS14].

THEOREM 63.2.18

Let $G = (V, E)$ be an independent graph in $\mathbb{R}^2$ and $u, v \in V$. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $uv \in E$.

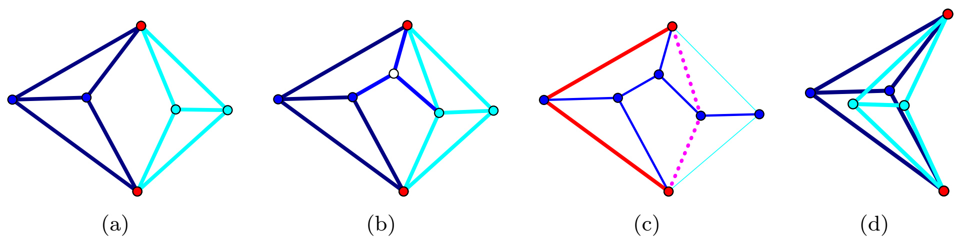

FIGURE 63.2.6

Edge splitting preserves globally linked pairs, as well as M-circuits (a,b). (c) Three openly vertex disjoin paths. (d) A framework with $h = 2$ and its two noncongruent realizations.

A related observation is that if a pair $\{u, v\}$ is globally linked in G in $\mathbb{R}^d$, then either the edge uv is present in G, or it is in an M-circuit in $G + uv$ in $\mathbb{R}^d$.

One can also determine the maximum number of pairwise equivalent but non-congruent generic realizations of an M-connected graph. Given a rigid generic framework (G, p), let $h(G, p)$ denote the number of distinct congruence classes of frameworks which are equivalent to (G, p). Given a rigid graph G, let $h(G) = \max\{h(G, p)\}$, where the maximum is taken over all generic frameworks (G, p).

Borcea and Streinu [BS04] investigated the number of realizations of minimally rigid generic frameworks (G, p) in the plane. Their results imply that $h(G) \leq 4^n$ for all rigid graphs G. They also construct an infinite family of generic minimally rigid frameworks (G, p) for which $h(G, p)$ has order $12^{\frac{n}{3}}$ which is approximately $(2.28)^n$. One can determine the exact value of $h(G, p)$ for all generic realizations (G, p) of an M-connected graph $G = (V, E)$. For $u, v \in V$, let $b(u, v)$ denote the number of connected components of $G - \{u, v\}$ and put $c(G) = \sum_{u,v \in V}(b(u, v) - 1)$. The next equality was verified by Jackson, Jordán, and Szabadka [JJS06].

THEOREM 63.2.19

Let G be an M-connected graph in $\mathbb{R}^2$. Then $h(G, p) = 2^{c(G)}$ for all generic realizations (G, p) of G.

It follows that $h(G) \leq 2^{\frac{n-2}{2}-1}$ for all M-connected graphs G. A family of graphs attaining this bound is a collection of K_4's joined along a common edge.

OPEN PROBLEMS ON GENERIC GLOBAL RIGIDITY

As noted earlier, the major open problem in this area is to find a combinatorial characterization of (and an efficient deterministic algorithm for testing) globally rigid graphs in $\mathbb{R}^d$ for $d \geq 3$. There are several related conjectures.

The next conjecture is open for $d \geq 3$.

CONJECTURE 63.2.20 Sufficient Connectivity Conjecture

For every $d \geq 1$ there is a (smallest) integer $f(d)$ such that every $f(d)$-connected graph G is globally rigid in $\mathbb{R}^d$.

Some of the results mentioned earlier imply that $f(1) = 2$ and $f(2) = 6$. By replacing globally rigid by rigid in Conjecture 63.2.20 we obtain a conjecture of Lovász and Yemini [LY82] from 1982. By Theorem 63.2.10 the two conjectures are equivalent.

Note that no degree of vertex-connectivity suffices to imply generic universal rigidity (c.f. Theorem 63.5.2).

A molecular model in three-space treats each atom and its set of neighbours as pairwise adjacent joints (due to dihedral constraints). Thus bonds may be interpreted as hinges between such complete subframeworks or simply as rigid bars between the corresponding atoms. In the former case we obtain molecular-hinge frameworks (which are geometrically singular body-hinge frameworks since the lines of all bonds of an atom are concurrent in the center of the atom) while in the latter case (where all pairs of neighbours of the atom are also connected by rigid bars) we obtain bar-and-joint frameworks whose underlying graph is a square of another graph, that is, a *molecular graph.* (See Section 61.2 for related results.) These models are central to applications of (global) rigidity to protein structures with thousands of atoms [Whi99].

CONJECTURE 63.2.21 Molecular Graph Global Rigidity Conjecture

Let G be a graph with no cycles of length at most four. Then G^2 is generically globally rigid in $\mathbb{R}^3$ if and only if G^2 is 4-connected and the multigraph $5G$ is highly 6-tree connected.

A graph G is *minimally globally rigid* in $\mathbb{R}^d$ if it is globally rigid but removing any edge from G leaves a graph which is not globally rigid.

CONJECTURE 63.2.22 Minimally Globally Rigid Conjecture

Let G be minimally globally rigid in $\mathbb{R}^d$. Then

(a) $|E| \leq (d+1)|V| - \binom{d+2}{2}$ *and*

(b) *the minimum degree of G is at most $2d + 1$.*

This conjecture has been verified for $d = 1, 2$ (with slightly better bounds); see [Jor17]. The bounds on the edge number and the minimum degree would be close to being tight for all d, see the complete bipartite graph $K_{d+1,n-d-1}$.

Let v be a vertex of degree at least $d-1$ in G with neighbour set $N(v)$. The d-dimensional *vertex splitting* operation at vertex v in G partitions $N(v)$ into three (possibly empty) sets A_1, C, A_2 with $|C| = d-1$, deletes vertex v and its incident edges, and adds two new vertices v_1, v_2 and new edges from v_i to all vertices in $A_i \cup C$, for $i = 1, 2$, as well as the edge v_1v_2 (the so-called *bridging edge*). The operation is *nontrivial* if A_1, A_2 are both nonempty. The vertex splitting operation preserves rigidity. It is conjectured to preserve global rigidity, provided that it is nontrivial. The 2-dimensional version of this conjecture was verified in [JS09]. For higher dimensions this conjecture is open even in the following weaker form.

CONJECTURE 63.2.23 Vertex Split Conjecture

Let H be globally rigid in $\mathbb{R}^d$ and let G be obtained from H by a nontrivial vertex

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splitting operation with bridging edge e. If $G - e$ is rigid in $\mathbb{R}^d$ then G is globally rigid in $\mathbb{R}^d$.

A recent result of [JT17] shows that if G is obtained from a globally rigid graph H with maximum degree at most $d+2$ by a sequence of nontrivial vertex splitting operations, then G is globally rigid in $\mathbb{R}^d$.

There is a conjectured characterization of globally linked pairs in $\mathbb{R}^2$. Sufficiency follows from Theorem 63.2.17.

CONJECTURE 63.2.24 Globally Linked Pairs Conjecture

Let $G = (V, E)$ be a graph and let $u, v \in V$. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $uv \in E$ or there is an M-connected subgraph H of G with $\{u, v\} \subseteq V(H)$ for which there exist three openly vertex-disjoint paths from u to v in H.

ALGORITHMS FOR GENERIC GLOBAL RIGIDITY

The Stress Matrix Condition for Generic Global Rigidity (Theorem 63.1.1) implies that there is a randomized polynomial time algorithm for testing whether a given graph G is globally rigid in $\mathbb{R}^d$, for any fixed d. See [CW10, GHT10] for the details.

We have deterministic polynomial-time algorithms for the cases in which global rigidity is well characterized: low dimensions ($d = 1, 2$) and special classes of graphs (body-bar, body-hinge). These algorithms are based on testing low dimensional (redundant) rigidity (see Chapter 61), checking whether a graph is k-vertex-connected (for which there exist well-known efficient network flow-based algorithms in general, and linear time algorithms up to $k = 3$) and testing (high) k-tree-connectivity.

For the latter problem, the tree partitions of Theorems 63.2.13 and 63.2.14 can be computed by matroidal algorithms of order $O(|V|^3)$ time.

We also have algorithms for identifying finer substructures (rigid, redundantly rigid or M-connected components), see e.g., [Jor16].

63.3 VARIATIONS AND CONNECTIONS TO OTHER FIELDS

GLOSSARY

Direction-length framework: A pair (G, p) of a loopless edge-labeled graph $G = (V; D, L)$ and a configuration p in d-space for the vertex set V. The edge set of G consists of direction edges D and length edges L that represent direction and length constraints, respectively. A direction edge uv fixes the gradient of the line through $p(u)$ and $p(v)$, whereas a length edge uv specifies the distance between the points $p(u)$ and $p(v)$.

Mixed graph: A graph $G = (V; D, L)$ whose edge set consists of direction edges D and length edges L.

Pure graph: A (mixed) graph $G = (V; D, L)$, in which $L = \emptyset$ (a direction-pure graph) or $D = \emptyset$ (a length-pure graph), respectively.

Congruent realizations: Two realizations (G, p) and (G, q) of G for which (G, q) can be obtained from (G, p) by a translation and, possibly, a dilation by -1.

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Globally rigid direction-length framework: A direction-length framework (G,p) in d-space for which every other d-dimensional realization (G,p) with the same edge lengths (for $e \in L$) and edge directions (for $e \in D$) as in (G,p) is congruent to (G,p).

Direction balanced graph: A 2-connected mixed graph in which both sides of any 2-vertex-separation contain a direction edge.

63.3.1 OTHER CONSTRAINTS

In computer-aided design, many different patterns of constraints (lengths, angles, incidences of points and lines, etc.) are used to design or describe configurations of points and lines, up to congruence. With distances between points, the geometry becomes that of global rigidity, with several positive results on the generic behaviour. For some other constraints the corresponding combinatorial problems (concerning "generic" configurations) are unsolved, and perhaps not solvable in polynomial time.

However, for special designs, e.g., for direction constraints in $\mathbb{R}^d$ and even for the combination of distance and direction constraints in $\mathbb{R}^2$, these problems have been solved, using extensions of the techniques and results for bar-and-joint frameworks.

DIRECTION-LENGTH FRAMEWORKS

By the Generic Global Rigidity Theorem (Theorem 63.1.2) global rigidity is a generic property for length-pure frameworks (with respect to the global rigidity definition for bar-and-joint frameworks). Whiteley [Whi96] showed that global rigidity is equivalent to first-order rigidity for direction-pure frameworks in $\mathbb{R}^d$. This implies that global rigidity is a generic property for direction-pure frameworks in $\mathbb{R}^d$ for all $d \geq 1$ (with respect to the adapted global rigidity definition allowing translations and dilations).

It is not known whether global rigidity is a generic property for direction-length frameworks, even in $\mathbb{R}^2$. In the plane we do have a number of partial results. In particular, we have a list of necessary conditions [JJ10a, JK11]. Let (G,p) be a generic direction-length framework with at least three vertices and $G = (V; D, L)$. If (G,p) is globally rigid in $\mathbb{R}^d$ then (i) G is not pure, (ii) G is rigid, (iii) G is 2-connected, (iv) G is direction balanced, (v) the only 2-edge-cuts in G consist of incident direction edges, and (vi) if $|L| \geq 2$ then $G - e$ is rigid for all $e \in L$.

The characterization of generically rigid mixed graphs gives rise to a rigidity matroid defined on the edge set of a mixed graph $G = (V; D, L)$ in which M-circuits and M-connected graphs are well-characterized.

Jackson and Jordán [JJ10a] characterized global rigidity for mixed graphs whose edge set is a circuit in the two-dimensional (mixed) rigidity matroid.

THEOREM 63.3.1 Mixed Global Rigidity for Plane Circuits

Let (G,p) be a generic realization of a mixed graph whose rigidity matroid is a circuit. Then (G,p) is globally rigid in $\mathbb{R}^2$ if and only if G is direction balanced.

Recently Clinch [Cli16] extended Theorem 63.3.1 to M-connected mixed graphs.

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THEOREM 63.3.2 Mixed Global Rigidity for Plane Connected Matroids

Let (G, p) be a generic realization of a mixed graph whose rigidity matroid is connected. Then (G, p) is globally rigid in $\mathbb{R}^2$ if and only if G is direction balanced.

The proofs of the previous theorems rely on inductive constructions of the corresponding families of mixed graphs, using the mixed versions of the two-dimensional extension operations [JJ10b]. The d-dimensional versions of these operations are treated in Nguyen [Ngu12].

A recent result by Clinch, Jackson, and Keevash [CJK16] characterizes mixed graphs with the property that *all* generic realizations in $\mathbb{R}^2$ are globally rigid. There is also a special case of d-dimensional direction-length frameworks where global rigidity is fully understood.

THEOREM 63.3.3

Let G be a mixed graph in which all pairs of adjacent vertices are connected by both a length and a direction edge, and let (G, p) be a d-dimensional generic realization of G. Then (G, p) is globally rigid if and only if G is 2-connected.

GLOBAL RIGIDITY OF FRAMEWORKS ON SURFACES

Another direction of research investigates frameworks in $\mathbb{R}^3$ whose vertices are constrained to lie on 2-dimensional surfaces. (See Section 61.2 for related results.) Generic rigidity has been characterized for various surfaces and there are some partial results concerning generic global rigidity on the sphere, cylinder, cone, and ellipsoid. In fact the spherical case has been settled by a result of Connelly and Whiteley, who proved that a generic framework on the sphere is globally rigid if and only if the corresponding generic framework is globally rigid in the plane [CW10].

Hendrickson's necessary conditions have been extended to the cylinder and the cone: a generic globally rigid framework with at least five vertices must be redundantly rigid and 2-connected. A sufficient condition, in terms of the corresponding stress matrix has also been found by Jackson and Nixon [JN15b]. Jackson, McCourt and Nixon [JMN14] conjecture that the necessary conditions above are also sufficient to guarantee global rigidity on the cylinder and the cone. The missing piece is an inductive construction for the graph family in question using operations which are known to preserve global rigidity.

63.3.2 CONNECTIONS TO OTHER FIELDS

There are a number of situations where global rigidity (or universal rigidity) is a key part of the computational process. These problems are often expressed in terms of "distance geometry." Dependence among the distances being used is a synonym for having a self-stress in the corresponding framework.

LOCALIZATION

An important problem is computing the location of sensors, scattered randomly in the plane, using pairwise distances. This problem is a setting for applying global rigidity [JJ09a, AEG$^+$06, SY07]. The computation needs access to a set of distance

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measurements which form a globally rigid graph, in order to have a unique solution. Typically, a few locations are known; these are called anchors and they function like pinned vertices. We are seeking a globally rigid pinned framework [Jor10b]. Given the expected errors in the measurements, the calculations reasonably assume the vertices are generic.

There are several simple approaches which have been used to make the calculations manageable. One is to find a trilateral subgraph: a graph which is formed inductively from the anchors by adding 3-valent vertices. Each step of the calculation has modest overhead and the locations are built up, one at a time. If the sensors have many neighbours with measured distances, then we can select such a trilateration graph with high probability [AEG$^+$06].

MOLECULAR CONFORMATIONS

Another significant computation problem is finding the shape (conformation) of a molecule from NMR (nuclear magnetic resonance) data. NMR measures (approximately) a set of pairwise distances, typically between pairs of hydrogen atoms. Again one needs to calculate the (relative) locations with data that is redundant at least in some parts of most molecules. Some parts will be underdetermined (e.g., tails of a protein may not be fixed). Some of the programs for optimizing these calculations use "smoothing" through small universally rigid subgraphs to refine the errors in the measurements (e.g., [CH88]).

63.4 GEOMETRY OF GLOBAL RIGIDITY

Unlike infinitesimal rigidity, graphs which are not generically globally rigid can have special position realizations which are globally rigid and even universally rigid, provided that they have a nontrivial stress with an appropriate stress matrix and some additional features. There are layers of analysis here that we shall return to in the following sections, including the study of super stability and tensegrity frameworks.

GLOSSARY

Special position of a graph G in d-space: Any configuration $p \in \mathbb{R}^{d|V|}$ such that the rigidity matrix $R(G,p)$, or any submatrix, has rank smaller than the maximum rank (the rank at a configuration with algebraically independent coordinates). These form an algebraic variety (see Chapter 61).

Affine spanning set for d-space: A configuration p of points such that every point $q \in \mathbb{R}^d$ can be expressed as an affine combination of the p_i: $q = \sum_i \lambda_i p_i$, with $\sum_i \lambda_i = 1$. (Equivalently, the affine coordinates $(p_i, 1)$ span the vector space $\mathbb{R}^{d+1}$.)

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BASIC GEOMETRIC RESULTS

There are some basic results for specific geometric frameworks. The stronger geometric results will appear for Universal Rigidity. The following three theorems are due to Connelly and Whiteley [CW10].

THEOREM 63.4.1 Stress Matrix and Infinitesimal Rigidity Certificate

If there is a realization (G, p) of G in $\mathbb{R}^d$ which is infinitesimally rigid with an equilibrium stress ω for which the associated stress matrix Ω has rank $|V| - d - 1$ then G is generically globally rigid in $\mathbb{R}^d$.

This theorem does not guarantee that (G, p) is itself globally rigid.

THEOREM 63.4.2 Stability Lemma

Given a framework (G, p) which is globally rigid and infinitesimally rigid in $\mathbb{R}^d$, there is an open neighborhood U of p in $\mathbb{R}^{d|V|}$ such that for all $q \in U$ the framework (G, q) is globally rigid and infinitesimally rigid.

The 2-dimensional case of the Stability Lemma has been generalized in [JJS14]: if (G, p) is infinitesimally rigid in $\mathbb{R}^2$, then there is an open neighbourhood U of p in $\mathbb{R}^{d|V|}$ for which $h(G, q) \leq h(G, p)$ for all $q \in U$.

An immediate corollary of the Stability Lemma is a different kind of global rigidity certificate.

THEOREM 63.4.3 Global Rigidity and Infinitesimal Rigidity Certificate

If there is a realization (G, p) of G which is globally rigid and infinitesimally rigid in $\mathbb{R}^d$ then G is generically globally rigid in $\mathbb{R}^d$.

63.5 UNIVERSALLY RIGID FRAMEWORKS

We return to universal rigidity of a framework (G, p), working in several layers, which turn out to be connected by important theorems and classes of PSD stress matrices. Here are several basic theorems for generic universally rigid frameworks. These depend on a stress matrix Ω being positive semi-definite (PSD) with maximal rank $n - d - 1$, where there are n vertices.

We begin with a few results for particular classes of graphs. We then follow two related but narrower concepts: dimensional rigidity and super stability. While not all universally rigid frameworks are super stable, a recent result of Connelly and Gortler [CG15] shows that every universally rigid framework can be iteratively built by a nested sequence of super stable structures.

We first present the results for bar frameworks. In the next section, we attend to the signs of the coefficients in the stress, and present the results for tensegrity frameworks, which is the form in which many of them appear in engineering and tensegrity installations.

The complexity of testing the universal rigidity of frameworks is open.

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GLOSSARY

Dimensionally rigid framework: A framework (G, p) in d-space with an affine span of dimension d, is dimensionally rigid in $\mathbb{R}^d$ if every framework (G, q) equivalent to (G, p) in any dimension has an affine span of dimension at most d.

Edge directions lie on a conic at infinity: The edge directions of a framework (G, p) lie on a conic at infinity if there is a symmetric matrix Q such that $m_{i,j} Q m_{i,j}^T = 0$ for all $m_{i,j} = (p_i - p_j)$ with $\{i, j\} \in E$.

Super stable framework: A framework (G, p) in d-space for which there is a stress matrix which is positive semi-definite with rank $|V| - d - 1$ and the edge directions do not lie on a conic at infinity.

Strictly separated by a quadric: Two sets of points P, Q in $\mathbb{R}^d$, with affine coordinates, for which there exists a quadric represented by a symmetric $(d+1) \times (d+1)$ matrix Q such that $p_i^T Q p_i > 0$ for all $p_i \in P$ and $q_j^T Q q_j < 0$ for all $q_j \in Q$.

BASIC RESULTS FOR UNIVERSAL RIGIDITY

We have described some basic results in Section 63.1.1. Several specific classes of graphs have more complete answers.

Connelly and Gortler characterized universally rigid complete bipartite frameworks for all $d \geq 1$ [CG17], extending the 1-dimensional result from [JN15a]. These geometric results lie behind some results for global rigidity cited earlier in Theorem 63.2.2.

THEOREM 63.5.1 Complete Bipartite Frameworks

Let G be a complete bipartite graph and let (G, p) be a d-dimensional realization of G in general position, with $m + n > d + 1$. Then (G, p) is not universally rigid if and only if the two vertex classes are strictly separated by a quadric.

THEOREM 63.5.2 Bipartite Universal Rigidity [JN15a]

The only generically universally rigid bipartite graph in $\mathbb{R}^d$ is K_2, for all $d \geq 1$.

There is also a sufficient condition which works for squares of graphs.

THEOREM 63.5.3 Universally Rigid Squares [GGLT13]

Let G be a $(d+1)$-connected graph. Then every generic d-dimensional realization of G^2 is universally rigid.

THEOREM 63.5.4 General Position Realizations [Alf17]

If a graph G is $(d+1)$-connected, then there exists a general position configuration p in $\mathbb{R}^d$ such that the framework (G, p) is universally rigid (therefore globally rigid).

DIMENSIONAL RIGIDITY

Dimensional rigidity, introduced by Alfakih [Alf07], is a concept that says a framework, with the given edge lengths, always lives in a restricted dimension. Again,

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beyond the small complete graphs, this is linked to properties of stress matrices. This is weaker than universal rigidity, as there may be noncongruent affinely equivalent frameworks. However, the concepts are clearly linked, and dimensional rigidity can be a key step towards proving universal rigidity, see e.g., [CG15].

THEOREM 63.5.5 Stress matrices for dimensional rigidity [Alf11]

1. *A framework (G, p) with n vertices whose affine span is d-dimensional with $n \geq d+2$, is dimensionally rigid if and only if (G, p) has a nonzero PSD stress matrix.*
2. *If a framework (G, p) with n vertices in $\mathbb{R}^d$ has an equilibrium stress with a PSD stress matrix of rank $n-d-1$, then (G, p) is dimensionally rigid in $\mathbb{R}^d$.*

THEOREM 63.5.6 [AY13, CGT16b]

A framework (G, p) in $\mathbb{R}^d$, with d-dimensional affine span, has a nontrivial affine flex if and only if it has an equivalent noncongruent affine image in $\mathbb{R}^d$ if and only if the edge directions lie on a conic at infinity.

THEOREM 63.5.7 Dimensional Rigidity

If a framework (G, p) with n vertices in $\mathbb{R}^d$ is dimensionally rigid in $\mathbb{R}^d$, and (G, q) is equivalent to (G, p), then q is an affine image of p.

As a result, we have the following strong connection to universal rigidity.

THEOREM 63.5.8 Dimensional Rigidity and Universal Rigidity

A framework (G, p) with n vertices in $\mathbb{R}^d$ is universally rigid if and only if it is dimensionally rigid and the edge directions do not lie on a conic at infinity.

SUPER STABILITY

The focus of the stress matrix approach remains finding a positive semi-definite stress matrix of appropriate rank. This is captured by the notion of super stability, defined by the following result of Connelly.

THEOREM 63.5.9 Super Stability

Let (G, p) be a framework whose affine span is all of $\mathbb{R}^d$, with an equilibrium stress ω and stress matrix Ω. Suppose further that

1. *Ω is positive semi-definite (PSD);*
2. *the rank of Ω is $n-d-1$;*
3. *the member directions of (G, p) do not lie on a conic at infinity.*

Then (G, p) is universally rigid.

A framework (G, p) satisfying conditions (1-3) is called *super stable.*

THEOREM 63.5.10 Generic Global Rigidity Implies Super Stability [CGT16a]

If G is generically globally rigid in $\mathbb{R}^d$, then there exists a framework (G, p) in $\mathbb{R}^d$ that is infinitesimally rigid in $\mathbb{R}^d$ and super stable. Moreover, every framework within a small enough open neighborhood of (G, p) will be infinitesimally rigid in $\mathbb{R}^d$ and super stable, including some generic framework.

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Not all universally rigid frameworks are super stable. However [CG15] shows that every universally rigid framework can be constructed through a nested sequence of affine spaces, each with an associated super stable framework. Such a nested sequence with associated super stable frameworks is a certificate for universal (global) rigidity of the framework and it is algorithmically efficient to confirm that the certificate is correct. What is not algorithmically efficient, is finding the sequence. Failure to find this sequence also generates a certificate that the framework is not universally rigid—by finding a higher dimensional set which includes equivalent, but not congruent, frameworks. Again, this is not efficient.

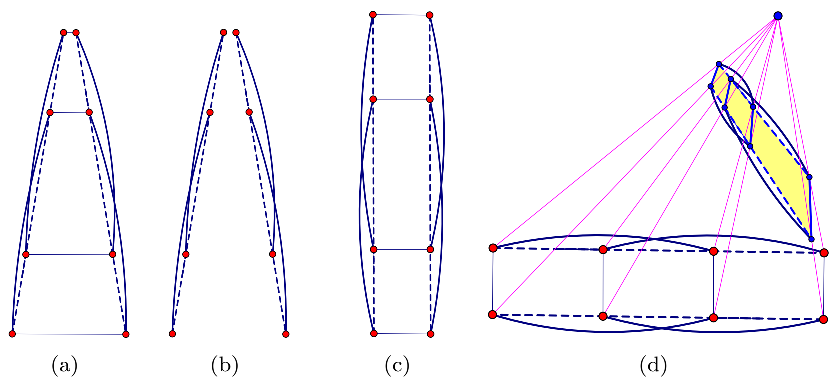

FIGURE 63.5.1
A universally rigid framework (a) will have super stable components (b), and members that have no self-stress at the first level of iteration. (c) shows a projectively equivalent framework which is not even rigid, though it is dimensionally rigid. (d) shows a cone of both (a) and (c) which is globally rigid, with a nonglobally rigid base.

Universal rigidity of a framework (G, p) is not preserved by projective transformations, because the property that the edge directions lie on a conic at infinity may change when a new "infinity" appears under the projection. That this is the only failure was confirmed by [Alf14, AN13]. Moreover, when the edge vectors at d vertices linearly span the space, the edge directions automatically avoid conics at infinity [Alf14].

63.5.1 PROJECTIVE TRANSFORMATIONS, CONING AND CHANGE OF METRIC

Given rigidity properties of a framework (G, p) in dimension d, we can often extend these properties to a coned framework in $(d+1)$-space, and then project to a different hyperplane to obtain another d-dimensional realization (G, q) as a slice of the cone, preserving the properties. This is a concrete form of a projective transformation from (G, p) to (G, q). Alternatively, there can be a direct analysis of the impact of a projective transformation from p to $q = f(p)$ on the various properties [SW07]. As Chapter 61 describes, these projective transformations preserve the first-order rigidity of (G, p), and have a clearly described impact on the coefficients of any equilibrium stress of (G, p). These processes also provide the tools to confirm the transfer of these basic properties among Euclidean, spherical, Minkowskian, and hyperbolic metrics, all of which live in a common projective space. The geometric transfer, for a specific projective configuration $\tilde{p}$ in any of the metrics, then gives corresponding combinatorial transfers of generic properties.

Here we look at the extensions of these techniques and results to global rigidity, dimensional rigidity, super stability, and universal rigidity for bar-and-joint frameworks.

GLOSSARY

Projective transform of a configuration p of points in d-space: A configuration q on the same vertices in d-space such that there is an invertible $(d+1)\times(d+1)$ matrix T for which $T(p_i, 1) = \lambda_i(q_i, 1)$ for all $1 \le i \le n$ (where λ_i is a scalar and $(p_i, 1)$ is the vector p_i extended with an additional 1—the affine coordinates of p_i).

Cone slice from v_0: For a $(d{+}1)$-dimensional configuration p of the vertices $\{v_0, v_1, \ldots, v_n\}$, a configuration q in d-space (placed as a hyperplane in $(d{+}1)$-space) of the vertices $\{v_1, .., v_n\}$, such that q_i is on the line through p_0, p_i for all $1 \le i \le n$. We use $q = \Pi_0(p)$ to denote that q is obtained from p by a cone slice from v_0 (Figure 63.2.4(b,c)).

CONING

The Generic Global Rigidity Coning Theorem (Theorem 63.2.9) fails as a statement for specific geometric frameworks, as there are both examples where the plane framework is globally rigid and the cone is not [CW10], and examples where the cone framework is globally rigid and the projection back to the plane is not globally rigid (or even rigid) [CG15]. See Figure 63.5.1 for illustrative examples.

On the other hand there are strong geometric results for dimensional rigidity and super stability. In the following statements on coning (due to Connelly and Gortler [CG17]) we shall use (G, p) and $(G * u, q)$ to denote a framework in d-space and a corresponding cone framework in $(d + 1)$-space. Note this includes all cone frameworks with the same cone slice.

THEOREM 63.5.11 Dimensional Rigidity Coning

The framework (G, p) is dimensionally rigid in $\mathbb{R}^d$ if and only if the cone framework $(G * u, q)$ is dimensionally rigid in $\mathbb{R}^{d+1}$.

THEOREM 63.5.12 Super Stability Coning

The framework (G, p) is super stable in $\mathbb{R}^d$ if and only if the cone framework $(G * u, q)$ is super stable in $\mathbb{R}^{d+1}$.

There is a standard *sliding* operation on cone frameworks, where vertices p_i slide along the line through p_0 and p_i, perhaps even through p_0 to the other side along the same line, so that they remain distinct from the cone vertex p_0. The following result from [CG17] follows from the fact that sliding does not change the projected framework of the cone to $\mathbb{R}^d$.

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COROLLARY 63.5.13 Sliding in Cones

Let $(G * u, p)$ be a cone framework in $\mathbb{R}^{d+1}$ and let $(G * u, q)$ be obtained from it by sliding. Then $(G * u, p)$ is super stable (resp. dimensionally rigid) if and only if $(G * u, q)$ is super stable (resp. dimensionally rigid).

THEOREM 63.5.14 Universal Rigidity Coning

If the framework (G, p) is super stable in $\mathbb{R}^d$ then the cone framework $(G * u, q)$ is universally rigid in $\mathbb{R}^{d+1}$.

If the cone framework $(G * u, q)$ is universally rigid in $\mathbb{R}^{d+1}$ then the framework (G, p) is dimensionally rigid in $\mathbb{R}^d$.

The only failure for projecting a universally rigid cone framework to a universally rigid framework can come from the appearance of affine motions due to the projection having edge directions on a conic at infinity. If, for example, the framework (G, p) is in general position, this cannot happen, as was observed by Alfakih [AY13, Alf17].

PROJECTIVE TRANSFORMATIONS

A general projection in $\mathbb{R}^d$ can be formed by coning to $\mathbb{R}^{d+1}$, rotating, and reprojecting, perhaps several times. As a result, the coning results guarantee the projective invariance of key properties.

THEOREM 63.5.15 Projective Invariance of Dimensional Rigidity and Super Stability

A framework (G, p) is super stable (resp. dimensionally rigid) in $\mathbb{R}^d$ if and only if every invertible projective transformation which keeps vertices finite is super stable (resp. dimensionally rigid) in $\mathbb{R}^d$.

This projective invariance is almost always true for universal rigidity. The key is whether the directions of the edges meet the set X which is going to infinity, in a conic.

THEOREM 63.5.16 Projection and Universal Rigidity

If a framework (G, p) is universally rigid in $\mathbb{R}^d$, and X is a hyperplane avoiding all vertices, such that the edge directions do not meet it in a conic, then any invertible projective transformation T which takes X to infinity makes $(G, T(p))$ universally rigid.

When a nested sequence of affine spaces, with a sequence of PSD matrices is used to iteratively demonstrate the dimensional rigidity (universal rigidity) of a given framework (G, p), the projected framework $(G, T(p))$ has the projected affine sequence to demonstrate its dimensional rigidity (universal rigidity).

It is well known that global rigidity is not projectively invariant for specific frameworks (G, p). However, the rank of any stress matrix Ω on the framework is invariant under projective transformations [CW10]. We might move to a configuration where the rank of the stress matrix is not sufficient to guarantee global rigidity.

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THEOREM 63.5.17

If a framework (G,p) is globally rigid in $\mathbb{R}^d$ then there is an open neighborhood $O(p)$ among projective images of p such that (G,q) is globally rigid in $\mathbb{R}^d$ for all $q \in O(p)$.

CHANGE OF METRIC

It is a classical result that infinitesimal rigidity, and associated properties (even with symmetry) transfer from Euclidean metric to all the other metric spaces built by Cayley-Klein metrics on the shared projective space [SW07, CW10]. See also Chapters 61 and 62. It is natural to ask how the properties of global rigidity and universal rigidity transfer. Coning, along with sliding, takes frameworks from the Euclidean metric in $\mathbb{R}^d$ to the Spherical metric in $\mathbb{S}^d$.

Given a projective configuration p in d-dimensional projective space, a corresponding configuration in $\mathbb{R}^d$ (all vertices finite) is called $R(p)$ and a corresponding configuration in $\mathbb{S}^d$ is called $S(p)$.

THEOREM 63.5.18 Super Stability Transfer to the Spherical Metric

For a given graph G and a fixed configuration p in projective space of dimension d, the framework $(G, R(p))$ is super stable in Euclidean metric space $\mathbb{R}^d$ if only if $(G, S(p))$ is super stable in the spherical metric $\mathbb{S}^d$.

THEOREM 63.5.19 Global and Universal Rigidity Transfer to the Spherical Metric

A given graph G is generically globally rigid in Euclidean metric space if and only if G is generically globally rigid in the spherical metric.

For a given graph G and a fixed configuration p in projective space of dimension d, (G,p) is universally rigid in the spherical metric if and only if the framework (G,p) is universally rigid in Euclidean metric space, for almost all projections.

Basically, all results and methods transfer from Euclidean metric space to the spherical metric.

Gortler and Thurston [GT14b] considered generic global rigidity in complex space and pseudo-Euclidean metrics (metrics with more general signatures). Here are a few results for generic configurations.

THEOREM 63.5.20 Transfer of Metrics: Complex Space

A graph G is globally rigid at some (all) generic configurations in Euclidean metric space if and only if G is globally rigid at some (all) generic configurations in complex metric space of the same dimension.

THEOREM 63.5.21 Transfer of Metrics: Pseudo-Euclidean

If a graph G is globally rigid at generic configurations in Euclidean metric space, then G is globally rigid at all *generic configurations in every alternate Cayley-Klein metric.*

If a graph G contains a d-simplex, then G is globally rigid at some generic configurations in Euclidean d-space if and only if G is globally rigid at every generic configuration in every d-dimensional alternate Cayley-Klein metric.

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The gap in Theorem 63.5.21 is that there may be additional graphs that are globally rigid for *some* generic configurations in a pseudo-Euclidean space but not globally rigid at other generic configurations in the same space, and are therefore *never* globally rigid at a generic configuration in Euclidean metric space. Alternatively, it is an *open problem* in the pseudo-Euclidean metric whether global rigidity is "generic": *does one generic globally rigid framework imply that all generic frameworks are globally rigid?*

One technique for these transfers among metrics is the *Pogorelov map*, which takes any pair of equivalent frameworks (G, p), (G, q) in one metric M to another pair (G, p'), (G, q') in another metric M' through the process of averaging to an infinitesimally flexible framework $(G, p + q)$ in M. By first-order principles, this transfers to confirm that $(G, p+q)$ is infinitesimally flexible in M', and de-averages to an equivalent pair (G, p'), (G, q') in M' [GT14b, CW10].

63.6 TENSEGRITY FRAMEWORKS

In a tensegrity framework, we replace some (or all) of the equalities for bars with inequalities for the distances, corresponding to *cables* (the distance can shrink but not expand) and *struts* (the distance can expand but not shrink). See Chapter 61 for basic results on the infinitesimal rigidity of tensegrity frameworks. Many results on universal rigidity transfer directly from results for bar frameworks, as soon as we align the cables and struts with the sign pattern of the self-stress with a PSD stress matrix.

GLOSSARY

Tensegrity graph: A graph $T = (V; B, C, S)$ with a partition (or labelling) of the edges into three classes, called *bars, cables*, and *struts*. The edges of T may be called *members*. In figures, cables are indicated by dashed lines, struts by thick lines, and bars by single thin lines.

Tensegrity framework in $\mathbb{R}^d$***:*** A pair (T, p), where T is a tensegrity graph and p is a configuration p of the vertex set of T in $\mathbb{R}^d$.

(T, p) ***dominates*** (T, q)***:*** For each member of T, the appropriate condition holds:

$$\begin{aligned} |p_i - p_j| &= |q_i - q_j| \quad \text{when} \quad \{i, j\} \in B \\ |p_i - p_j| &\geq |q_i - q_j| \quad \text{when} \quad \{i, j\} \in C \\ |p_i - p_j| &\leq |q_i - q_j| \quad \text{when} \quad \{i, j\} \in S. \end{aligned}$$

Globally rigid tensegrity framework: A d-dimensional tensegrity framework (T, p) for which any other realization (T, q) in $\mathbb{R}^d$, dominated by (T, p) is congruent to p.

Universally globally rigid tensegrity framework: A d-dimensional tensegrity framework (T, p) for which any other realization (T, q) in any dimension, dominated by (T, p) is congruent to p.

Proper equilibrium stress on a tensegrity framework (T, p): An assignment ω of scalars to the members of T such that:

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(a) $\omega_{ij} \geq 0$ for cables $\{i,j\} \in C$;
(b) $\omega_{ij} \leq 0$ for struts $\{i,j\} \in S$; and
(c) for each vertex i, $\sum_{\{j \mid \{i,j\} \in E\}} \omega_{ij}(p_j - p_i) = 0$.

Strict proper equilibrium stress: A proper equilibrium stress ω with the inequalities in (a) and (b) strict.

Underlying bar framework: For a tensegrity framework (T,p), where $T = (V;B,C,S)$, the bar framework $(\bar{G},p)$ on the graph $\bar{G} = (V,E)$ with $E = B \cup C \cup S$.

Spiderweb: A labelled graph $G_- = (V_0,V_1;C)$, with pinned vertices V_0 and cable members only, with $C \subset V_1 \times [V_0 \cup V_1]$, and a configuration p for $V_0 \cup V_1$.

Spiderweb self-stress on (G_-,p)***:*** An assignment ω of nonnegative scalars to C such that for each unpinned vertex $i \in V_1$, $\sum_{\{j|\{i,j\}\in C\}} \omega_{ij}(p_j - p_i) = 0$.

Spiderweb flex: A flex $p(t)$ for (G_-,p) with all pinned vertices fixed.

63.6.1 BASIC RESULTS FOR TENSEGRITIES

All the definitions and results for dimensional rigidity and superstability of frameworks extend to proper equilibrium stresses of tensegrity frameworks. The following result confirms that super stability transfers from bar frameworks to tensegrity frameworks. These transfers flow from [CG17, CGT16a, CGT16b].

THEOREM 63.6.1 Dimensional Rigidity of Tensegrity Frameworks

Let (T,p) be a d-dimensional tensegrity framework, where the affine span of the joint positions is all of $\mathbb{R}^d$, with a proper equilibrium stress ω and stress matrix Ω. Suppose further that

(i) *Ω is positive semi-definite,*

(ii) *the rank of Ω is $n-d-1$.*

Then (T,p) is dimensionally rigid.

THEOREM 63.6.2 The Fundamental Theorem of Tensegrity Frameworks

Let (T,p) be a d-dimensional tensegrity framework, where the affine span of the joint positions is all of $\mathbb{R}^d$, with a proper equilibrium stress ω and stress matrix Ω. Suppose further that

(i) *Ω is positive semi-definite,*

(ii) *the rank of Ω is $n-d-1$,*

(iii) *the stressed member directions of (G,p) do not lie on a conic at infinity.*

Then (T,p) is universally rigid.

For example, Connelly [Con82] proved that a tensegrity framework obtained from a planar polygon by putting a joint at each vertex, a cable along each edge, and struts connecting other vertices (a *tensegrity polygon*) such that the resulting tensegrity has some nonzero proper equilibrium stress satisfies conditions (i)-(iii) and hence it is universally globally rigid. Geleji and Jordán [GJ13] characterized

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the tensegrity polygons for which all convex realizations in $\mathbb{R}^2$ possess a nonzero proper equilibrium stress.

The following result of Bezdek and Connelly [Con06] is an initial analogous result in 3-space.

THEOREM 63.6.3

If a tensegrity framework in $\mathbb{R}^3$ has cables along the edges of a convex centrally symmetric polyhedron, and struts connecting antipodal vertices, then it is super stable.

Note that within these theorems are many examples of universally rigid tensegrity frameworks which are *not infinitesimally rigid*. There is a version of the stability theorem that extends even to these special position frameworks.

COROLLARY 63.6.4 Tensegrity Stability Theorem

If we take a globally rigid framework where Ω is positive semi-definite of the required rank, then within the variety of projectively equivalent frameworks there is an open neighborhood $O(p)$ within which all frameworks are universally rigid.

A special result for the modified spiderwebs further illustrates the role of tensegrity frameworks.

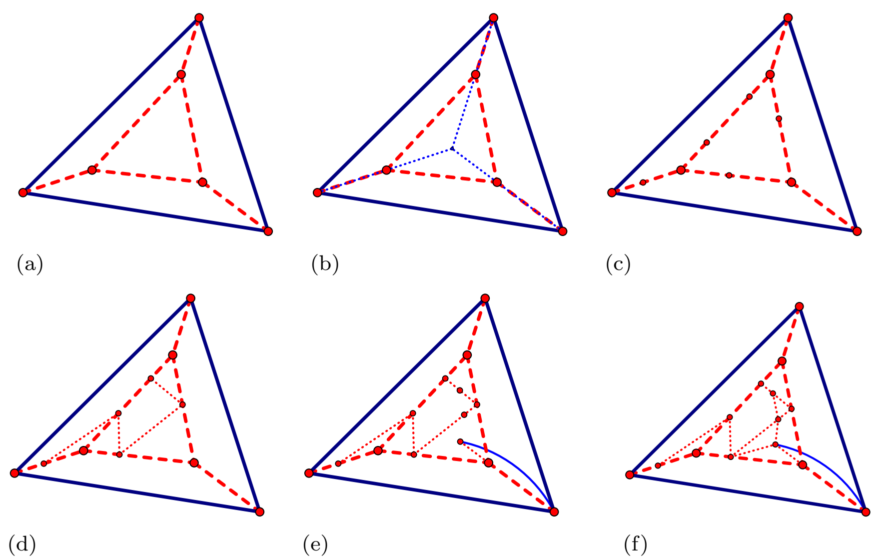

FIGURE 63.6.1
The development of super stable tensegrity frameworks. Figure (a) has a spiderweb self-stress, due to the geometry of the three concurrent edges (b). It is super stable but not infinitesimally rigid. Figure (c,d) shows ways of extending the framework that preserve universal rigidity, although the added edges in (d) will not have a nonzero self-stress, following the inductions below. Moreover, we can add vertices outside a cable edge (Figure (e) and continue to add edges and new vertices, preserving the universal rigidity.

THEOREM 63.6.5 Spiderweb Rigidity

Any spiderweb (G_-, p) in d-space with a spiderweb self-stress, positive on all cables, is super stable in d-space.

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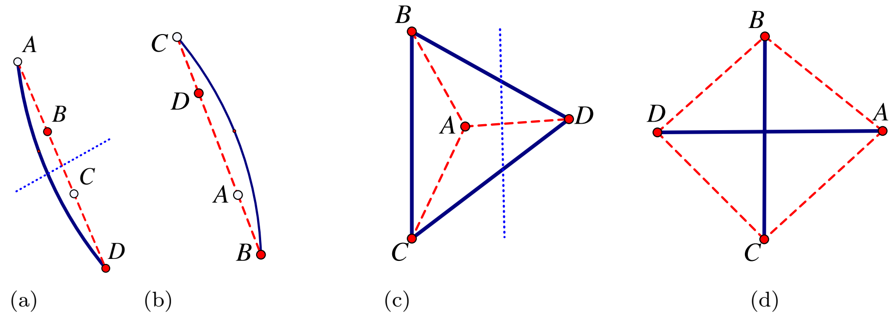

FIGURE 63.6.2
If a universally rigid tensegrity framework is projected, the edges cut by the line going to infinity change class and their stress changes sign (a,b), (c,d)

THEOREM 63.6.6 Projective Invariance of Super Stability of Tensegrity Frameworks

Let $f : X \to \mathbb{R}^d$ be an invertible projective transformation, where X is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$, and suppose that for each i, $p_i \notin X$. Further assume that the member directions of (G, p) do not lie on a conic at infinity, and that the member directions of $(G, f(p))$ do not lie on a conic on X for a projective transformation f. Then the tensegrity framework (G, p) is super stable if and only if $(G, f(p))$ is super stable, where the strut/cable designation for $\{i, j\}$ changes only when the line segment $[p_i, p_j]$ intersects X and bars go to bars.

Coning also extends to tensegrity frameworks. Make all the edges incident to the cone vertex bars, and initially preserve the designation of cables and struts, with configurations near the original $\mathbb{R}^d$ (relative to the cone vertex p_0). Then switch the cable and strut designation under sliding only when a vertex slides across the cone vertex (one vertex at a time). This process will transfer all the relevant tensegrity results from the Euclidean metric to tensegrities in the spherical metric, with the same caveats as for bar frameworks.

INDUCTIVE CONSTRUCTIONS FOR TENSEGRITIES

The *tensegrity gluing* of two tensegrity graphs $T_1 = (V_1; B_1, C_1, S_1)$ and $T_2 = (V_2; B_2, C_2, S_2)$ is the tensegrity graph T with vertices $V = V_1 \cup V_2$, $B = B_1 \cup B_2 \cup (C_1 \cap S_2) \cup (S_1 \cap C_2)$, $C = C_1 \cup C_2 - B$, and $S = S_1 \cup S_2 - B$.

The following is a corollary of the previous gluing results.

COROLLARY 63.6.7 Tensegrity Gluing

If (T_1, p_1) and (T_2, p_2) are universally rigid (resp. dimensionally rigid) tensegrity frameworks in $\mathbb{R}^d$ sharing at least $d+1$ points affinely spanning $\mathbb{R}^d$ then the tensegrity glued framework $(T, p_1 \cup p_2)$ is universally rigid (resp. dimensionally rigid) in $\mathbb{R}^d$.

Appendix A of [CW96] includes a process of: (i) adding a new vertex p_c anywhere along the interior of a cable $[p_i p_j]$, splitting the cable into two cables $\{i, c\}$, $\{c, j\}$ and (ii) inserting a new vertex p_s along the exterior of a strut $[p_i p_j]$ (say

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along the p_j side) splitting the strut into a strut $\{i, s\}$ and a cable $\{s, j\}$. We note that the strut insertion is the same as a cable insertion on a projective image of the framework. See Figure 63.6.1(c,d,e).

These insertions provide inductive steps that preserve dimensional rigidity, super stability, and universal rigidity of specific geometric tensegrity frameworks. This provides a geometric inductive process for building up universally rigid frameworks, onto which one can then hang further—even nongeneric—frameworks by geometric gluing.

Applied just to cables, the insertions take a spiderweb to a spiderweb. Again, one can insert vertices and then glue in another spiderweb, iterating to create larger spiderwebs.

63.7 SOURCES AND RELATED MATERIALS

SURVEYS AND BASIC SOURCES

We refer the reader to the following books, book chapters, survey articles and recent substantial articles for a more detailed overview of this field.

[CG16]: A forthcoming book on frameworks and tensegrities with a number of connections to global rigidity.

[Alf14]: Basic results for universal rigidity and dimensional rigidity.

[CG15]: Key recent article for super stability and the iterative constructions, with extensions to tensegrity frameworks.

[CG17]: Basic results for complete bipartite frameworks and current results in universal rigidity, including coning and projection.

[JJ09a]: A survey chapter on graph theoretic techniques in sensor network localization.

[AEG$^+$06]: An introduction to the links between sensor network localization and combinatorial rigidity.

[Jor10b]: A survey chapter on rigid and globally rigid pinned frameworks.

[Jor16]: A short monograph on generic rigidity and global rigidity in the plane.

[Whi96]: An expository article presenting matroidal aspects of first-order rigidity, redundant rigidity, scene analysis, and multivariate splines.

[Wiki]: A wiki site with a number of preprints, including original papers on global rigidity.

RELATED CHAPTERS

Chapter 9: Geometry and topology of polygonal linkages
Chapter 57: Solid modeling
Chapter 61: Rigidity and scene analysis
Chapter 62: Symmetry and rigidity

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