60 GEOMETRIC APPLICATIONS OF THE GRASSMANN-CAYLEY ALGEBRA

Neil L. White

INTRODUCTION

Grassmann-Cayley algebra is first and foremost a means of translating synthetic projective geometric statements into invariant algebraic statements in the bracket ring, which is the ring of projective invariants. A general philosophical principle of invariant theory, sometimes referred to as ***Gram's theorem***, says that any projectively invariant geometric statement has an equivalent expression in the bracket ring; thus we are providing here the practical means to carry this out. We give an introduction to the basic concepts, and illustrate the method with several examples from projective geometry, rigidity theory, and robotics.

60.1 BASIC CONCEPTS

Let P be a $(d-1)$-dimensional projective space over the field F, and V the canonically associated d-dimensional vector space over F. Let S be a finite set of n points in P and, for each point, fix a homogeneous coordinate vector in V. We assume that S spans V, hence also that $n \geq d$. Initially, we choose all of the coordinates to be distinct, algebraically independent indeterminates in F, although we can always specialize to the actual coordinates we want in applications. For $p_i \in S$, let the coordinate vector be $(x_{1,i}, \ldots, x_{d,i})$.

GLOSSARY

Bracket: A $d \times d$ determinant of the homogeneous coordinate vectors of d points in S. Brackets are relative projective invariants, meaning that under projective transformations their value changes only in a very predictable way (in fact, under a basis change of determinant 1, they are literally invariant). Hence brackets may also be thought of as coordinate-free symbolic expressions. The bracket of $u_1, \ldots, u_d$ is denoted by $[u_1, \ldots, u_d]$.

Bracket ring: The ring B generated by the set of all brackets of d-tuples of points in S, where $n = |S| \geq d$. It is a subring of the ring $F[x_{1,1}, x_{1,2}, \ldots, x_{d,n}]$ of polynomials in the coordinates of points in S.

Straightening algorithm: A normal form algorithm in the bracket ring.

Join of points: An exterior product of k points, $k \leq d$, computed in the exterior algebra of V. We denote such a product by $a_1 \vee a_2 \vee \cdots \vee a_k$, or simply $a_1 a_2 \cdots a_k$, rather than $a_1 \wedge a_2 \wedge \cdots \wedge a_k$, which is commonly used in exterior algebra. A

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

concrete version of this operation is to compute the Plücker coordinate vector of (the subspace spanned by) the k points, that is, the vector whose components are all $k \times k$ minors (in some prespecified order) of the $d \times k$ matrix whose columns are the homogeneous coordinates of the k points.

Extensor of step k, or ***decomposable k-tensor:*** A join of k points. Extensors of step k span a vector space $V^{(k)}$ of dimension $\binom{d}{k}$. (Note that not every element of $V^{(k)}$ is an extensor.)

Antisymmetric tensor: Any element of the direct sum $\Lambda V = \oplus_k V^{(k)}$.

Copoint: Any antisymmetric tensor of step $d-1$. A copoint is always an extensor.

Join: The exterior product operation on ΛV. The join of two tensors can always be reduced by distributivity to a linear combination of joins of points.

Integral: $E = u_1 u_2 \cdots u_d$, for any vectors $u_1, u_2, \ldots, u_d$ such that $[u_1, u_2, \ldots, u_d] = 1$. Every extensor of step d is a scalar multiple of the integral E.

Meet: If $A = a_1 a_2 \cdots a_j$ and $B = b_1 b_2 \cdots b_k$, with $j + k \geq d$, then

$$A \wedge B = \sum_{\sigma} \operatorname{sgn}(\sigma)[a_{\sigma(1)}, \ldots, a_{\sigma(d-k)}, b_1, \ldots, b_k] a_{\sigma(d-k+1)} \cdots a_{\sigma(j)}$$
$$\equiv [\dot{a}_1, \ldots, \dot{a}_{d-k}, b_1, \ldots, b_k]\, \dot{a}_{d-k+1} \cdots \dot{a}_j \ .$$

The sum is taken over all permutations σ of $\{1, 2, \ldots, j\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(d-k)$ and $\sigma(d-k+1) < \sigma(d-k+2) < \cdots < \sigma(j)$. Each such permutation is called a ***shuffle*** of the $(d-k, j-(d-k))$ split of A, and the dots represent such a signed sum over all the shuffles of the dotted symbols.

Grassmann-Cayley algebra: The vector space $\Lambda(V)$ together with the operations $\vee$ and $\wedge$.

PROPERTIES OF GRASSMANN-CAYLEY ALGEBRA

(i) $A \vee B = (-1)^{jk} B \vee A$ and $A \wedge B = (-1)^{(d-k)(d-j)} B \wedge A$, if A and B are extensors of steps j and k.

(ii) $\vee$ and $\wedge$ are associative and distributive over addition and scalar multiplication.

(iii) $A \vee B = (A \wedge B) \vee E$ if $\operatorname{step}(A) + \operatorname{step}(B) = d$.

(iv) A meet of two extensors is again an extensor.

(v) The meet is dual to the join, where duality exchanges points and copoints.

(vi) ***Alternative Law:*** Let $a_1, a_2, \ldots, a_k$ be points and $\gamma_1, \gamma_2, \ldots, \gamma_s$ copoints. Then if $k \geq s$,

$$(a_1 a_2 \cdots a_k) \wedge (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_s) = [\dot{a}_1, \gamma_1][\dot{a}_2, \gamma_2] \cdots [\dot{a}_s, \gamma_s]\, \dot{a}_{s+1} \vee \cdots \vee \dot{a}_k \ .$$

Here the dots refer to all shuffles over the $(1, 1, \ldots, 1, k-s)$ split of $a_1 \cdots a_k$, that is, a signed sum over all permutations of the a's such that the last $k-s$ of them are in increasing order.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

60.2 GEOMETRY ↔ G.-C. ALGEBRA → BRACKET ALGEBRA

If X is a projective subspace of dimension $k-1$, pick a basis $a_1, a_2, \ldots, a_k$ and let $A = a_1 a_2 \cdots a_k$ be an extensor. We call $X = \overline{A}$ the ***support*** of A.

(i) If $A \neq 0$ is an extensor, then A determines $\overline{A}$ uniquely.

(ii) If $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A \vee B} = \overline{A} + \overline{B}$.

(iii) If $\overline{A} \cup \overline{B}$ spans V, then $\overline{A \wedge B} = \overline{A} \cap \overline{B}$.

TABLE 60.2.1 Examples of geometric conditions and corresponding Grassmann-Cayley algebra statements.

GEOMETRIC CONDITION	DIM	G.-C. ALGEBRA STATEMENT	BRACKET STATEMENT
Point $\overline{a}$ is on the line $\overline{bc}$ (or $\overline{b}$ is on $\overline{ac}$, etc.)	2	$a \wedge bc = 0$	$[abc] = 0$
Lines $\overline{ab}$ and $\overline{cd}$ intersect	3	$ab \wedge cd = 0$	$[abcd] = 0$
Lines $\overline{ab}$, $\overline{cd}$, $\overline{ef}$ concur	2	$ab \wedge cd \wedge ef = 0$	$[\overset{\bullet}{a}cd][\overset{\bullet}{b}ef] = 0$
Planes $\overline{abc}$, $\overline{def}$, and $\overline{ghi}$ have a line in common	3	$abc \wedge def \wedge ghi = 0$	$[\overset{\bullet}{a}def][\overset{\bullet}{b}ghi][\overset{\bullet}{c}xyz] = 0 \ \forall x, y, z$
The intersections of $\overline{ab}$ with $\overline{cd}$ and of $\overline{ef}$ with $\overline{gh}$ are collinear with $\overline{i}$	2	$(ab \wedge cd) \vee (ef \wedge gh) \vee i = 0$	$[\overset{\bullet}{a}cd][\overset{\diamond}{e}gh][\overset{\bullet}{b}\overset{\diamond}{f}i] = 0$

The geometric conditions in Table 60.2.1 should be interpreted projectively. For example, the concurrency of three lines includes as a special case that the three lines are mutually parallel, if one prefers to interpret the conditions in affine space. Degenerate cases are always included, so that the concurrency of three lines includes as a special case the equality of two or even all three of the lines, for example.

Most of the interesting geometric conditions translate into Grassmann-Cayley conditions of step 0 (or, equivalently, step d), and therefore expand into bracket conditions directly. When the Grassmann-Cayley condition is not of step 0, as in the example in Table 60.2.1 of three planes in three-space containing a common line, then the Grassmann-Cayley condition may be joined with an appropriate number of universally quantified points to get a conjunction of bracket conditions. The joined points may also be required to come from a specified basis to make this a conjunction of a finite number of bracket conditions.

In this fashion, any incidence relation in projective geometry may be translated into a conjunction of Grassmann-Cayley statements, and, conversely, Grassmann-Cayley statements may be translated back to projective geometry just as easily, provided they involve only join and meet, not addition.

Many identities in the Grassmann-Cayley algebra yield algebraic, coordinate-free proofs of important geometric theorems. These proofs typically take the form "the left-hand side of the identity is 0 if and only if the right-hand side of the identity is 0," and the resulting equivalent Grassmann-Cayley conditions translate to interesting geometric conditions as above.

TABLE 60.2.2 Examples of Grassmann-Cayley identities and corresponding geometric theorems, in dimension 2.

GEOMETRIC THEOREM	G.-C. ALGEBRA IDENTITY
Desargues's theorem: Derived points $ab \wedge a'b'$, $ac \wedge a'c'$, and $bc \wedge b'c'$ are collinear if and only if abc or $a'b'c'$ are collinear or aa', bb', and cc' concur.	$(ab \wedge a'b') \vee (ac \wedge a'c') \vee (bc \wedge b'c') = [abc][a'b'c'](aa' \wedge bb' \wedge cc')$
Pappus's theorem and Pascal's theorem: If abc and $a'b'c'$ are both collinear sets, then $(bc' \wedge b'c)$, $(ca' \wedge c'a)$, and $(ab' \wedge a'b)$ are collinear.	$[ab'c'][a'bc'][a'b'c]\underline{[abc]} - [abc'][ab'c][a'bc]\underline{[a'b'c']} = (bc' \wedge b'c) \vee (ca' \wedge c'a) \vee (ab' \wedge a'b)$
Pappus's theorem (alternate version): If $aa'x$, $bb'x$, $cc'x$, $ab'y$, $bc'y$, and $ca'y$ are collinear, then ac', ba', cb' concur.	$aa' \wedge bb' \wedge cc' + ab' \wedge bc' \wedge ca' + ac' \wedge ba' \wedge cb' = 0$
Fano's theorem: If no three of a, b, c, d are collinear, then $(ab \wedge cd)$, $(bc \wedge ad)$, and $(ca \wedge bd)$ are collinear if and only if char $F = 2$.	$(ab \wedge cd) \vee (bc \wedge ad) \vee (ca \wedge bd) = 2\,[abc][abd][acd][bcd]$

The identities in Table 60.2.2 are proved by expanding both sides, using the rules for join and meet, and then verifying the equality of the resulting expressions by using the straightening algorithm of bracket algebra (see [Stu93]).

The right-hand side of the identity for the first version of Pappus's theorem is also the Grassmann-Cayley form of the geometric construction used in Pascal's theorem, and hence is 0 if and only if the six points lie on a common conic (Pappus's theorem being the degenerate case of Pascal's theorem in which the conic consists of two lines). Hence the left-hand side of the same identity is the bracket expression that is 0 if and only if the six points lie on a common conic. In particular, if abc and $a'b'c'$ are both collinear, we see immediately from the underlined brackets that the left-hand side is 0.

Numerous other projective geometry incidence theorems may be proved using the Grassmann-Cayley algebra. We illustrate this with an example modified from [RS76]. Other examples may be found in the same reference.

THEOREM 60.2.1

In 3-space, if triangles abc and $a'b'c'$ are in perspective from the point d, then the lines $a'bc \wedge ab'c'$, $b'ca \wedge bc'a'$, $c'ab \wedge ca'b'$, and $a'b'c' \wedge abc$ are all coplanar.

Proof. We prove the general case, where a, b, c, d, a', b', c' are all distinct, triangles abc and $a'b'c'$ are nondegenerate, and d is in neither the plane abc nor the plane $a'b'c'$. Then, since a, a', d are collinear, we may write $\alpha a' = \beta a + d$ for nonzero scalars α and β. Since we are using homogeneous coordinates for points, a, and similarly a', may be replaced by nonzero scalar multiples of themselves without changing the geometry. Thus, without loss of generality, we may write $a' = a + d$. Similarly, $b' = b + d$ and $c' = c + d$. Now

$$L_1 := a'bc \wedge ab'c' = [a'ab'c']bc - [bab'c']a'c + [cab'c']a'b$$
$$= [dabc]bc + [badc]ca + [cabd]ab + [badc]cd + [cabd]db$$
$$= [abcd](-bc - ac + ab + cd - bd).$$

Similarly,

$$L_2 := b'ca \wedge bc'a' = [abcd](ac + ab + bc + ad - cd),$$
$$L_3 := c'ab \wedge ca'b' = [abcd](-ab + bc - ac + bd - ad),$$
$$L_4 := a'b'c' \wedge abc = [abcd](bc - ac + ab).$$

Now we check that any two of these lines intersect. For example,

$$L_1 \wedge L_2 = [abcd]^2(-bc - ac + ab + cd - bd) \wedge (ac + ab + bc + ad - cd) = 0.$$

However, this shows only that either all four lines are coplanar or all four lines concur. To prove the former, it suffices to check that the intersection of $\overline{L_1}$ and $\overline{L_4}$ is distinct from that of $\overline{L_2}$ and $\overline{L_4}$. Notice that $L_1 \wedge L_4$ does not tell us the point of intersection, because $\overline{L_1}$ and $\overline{L_4}$ do not jointly span V, by our previous computation. But if we choose a generic vector x representing a point in general position, it follows from $\overline{L_1} \neq \overline{L_4}$, which must hold in our general case, that $(L_1 \vee x) \wedge L_4$ is nonzero and does represent the desired point of intersection. Then we compute

$$\begin{aligned}(L_1 \vee x) \wedge L_4 &= [abcd]^2(-bcx - acx + abx + cdx - bdx) \wedge (bc - ac + ab)\\ &= [abcd]^2(2[abcx] - [bcdx] - [acdx])(c - b)\\ &= \alpha(c - b)\end{aligned}$$

for some nonzero scalar α. Similarly, $(L_2 \vee x) \wedge L_4 = \beta(c - a)$ for some nonzero scalar β. By the nondegeneracy of the triangle abc, these two points of intersection are distinct. □

60.3 CAYLEY FACTORIZATION: BRACKET ALGEBRA → GEOMETRY

(1)	Projective geometry		
	↕		
(2)	Grassmann-Cayley algebra		
	↓	↑	Cayley factorization
(3)	Bracket algebra		
	↓		
(4)	Coordinate algebra		

(1)↔(2)→(3) in the chart above is explained in Section 60.2 above, with (2)→(1) being straightforward only in the case of a Grassmann-Cayley expression involving only joins and meets. (3)→(4) is the trivial expansion of a determinant into a polynomial in its d^2 entries. (4)→(3) is possible only for invariant polynomials (under the special linear group); see [Stu93] for an algorithm.

PHILOSOPHY OF INVARIANT THEORY: It is best for many purposes to avoid level (4), and to work instead with the symbolic coordinate-free expressions on levels (2) and (3).

Cayley factorization, (3)→(2), refers to the translation of a bracket polynomial into an equivalent Grassmann-Cayley expression involving only joins and meets. The input polynomial must be homogeneous (i.e., each point must occur the same number of times in the brackets of each bracket monomial of the polynomial), and Cayley factorization is not always possible. No practical algorithm is known in general, but an algorithm [Whi91] is known that finds such a factorization—or else announces its impossibility—in the multilinear case (each point occurs exactly once in each monomial). This algorithm is practical up to about 20 points.

MULTILINEAR CAYLEY FACTORIZATION

The multilinear Cayley factorization (MCF) algorithm is too complex to present here in detail; instead, we give an example and indicate roughly how the algorithm proceeds on the example.

Let

$$\begin{aligned} P \;=\; & -[acj][deh][bfg] - [cdj][aeh][bfg] - [cdj][abe][fgh] \\ & + [acj][bdf][egh] - [acj][bdg][efh] + [acj][bdh][efg]. \end{aligned}$$

Note that P is multilinear in the 9 points. The MCF algorithm now looks for sets of points $x, y, \ldots, z$ such that the extensor $xy \cdots z$ could be part of a Cayley factorization of P. For this choice of P, it turns out that no such set larger than a pair of elements occurs. An example of such a pair is a, d; in fact, if d is replaced by a in P, leaving two a's in each term of P, although in different brackets, the resulting bracket polynomial is equal to 0, as can be verified using the straightening algorithm. The MCF algorithm, using the straightening algorithm as a subroutine, finds that $(a, d), (b, h), (c, j), (f, g)$ are all the pairs with this property.

The algorithm now looks for combinations of these extensors that could appear as a meet in a Cayley factorization of P. (For details, see [Whi91].) It finds in our example that $ad \wedge cj$ is such a combination. As soon as a single such combination is found, an algebraic substitution involving a new variable, $z = ad \wedge cj$, is performed, and a new bracket polynomial of smaller degree involving this new variable is derived; the algorithm then begins anew on this polynomial. If no such combination is found, the input bracket polynomial is then known to have no Cayley factorization. In our example, this derived polynomial turns out to be $P = [zef][gbh] - [zeg][fbh]$, which of necessity is still multilinear. The MCF algorithm proceeds to find (and we can directly see by consulting Table 60.2.1) that $P = ze \wedge fg \wedge bh$. Thus, our final Cayley factorization is output as

$$P = ((ad \wedge cj) \vee e) \wedge fg \wedge bh.$$

It is significant that this algorithm requires no backtracking. For example, once $ad \wedge cj$ is found as a possible meet in a Cayley factorization of P, it is known that if P has a Cayley factorization at all, then it must also have one using the factor $ad \wedge cj$; hence we are justified in factoring it out, i.e., substituting a new variable for it. Other Cayley factorizations may be possible, for example,

$$P = ((fg \wedge bh) \vee (ad \wedge cj)) \wedge e.$$

Note that these two factorizations have the same geometric meaning.

60.4 APPLICATIONS

60.4.1 ROBOTICS

GLOSSARY

Robot arm: A set of rigid bodies, or links, connected in series by joints that allow relative movement of the successive links, as described below. The first link is regarded as fixed in position, or tied to the ground, while the last link, called the ***end-effector***, is the one that grasps objects or performs tasks.

Revolute joint: A joint between two successive links of a robot arm that allows only a rotation between them. In simpler terms, a revolute joint is a hinge connecting two links.

Prismatic joint: A joint between two successive links of a robot arm that allows only a translational movement between the two links.

Screw joint: A joint between two successive links of a robot arm that allows only a screw movement between the two links.

TABLE 60.4.1 Modeling instantaneous robotics.

ROBOTICS CONCEPT	GRASSMANN-CAYLEY EQUIVALENT
Revolute joint on axis $\overline{ab}$	$\alpha(a \vee b)$, a 2-extensor
Rotation about line $\overline{ab}$	$\beta(a \vee b)$
Motion of point p in rotation about line $\overline{ab}$	$\beta(a \vee b) \vee p$
Screw joint	indecomposable 2-tensor
Prismatic joint	2-extensor at infinity
Motion space of the end-effector, where $j_1, j_2, \ldots, j_k$ are joints in series	span of the extensors $< j_1, j_2, \ldots, j_k >$

We are considering here only the instantaneous kinematics or statics of robot arms, that is, positions and motions at a given instant in time. A robot arm has a ***critical configuration*** if the joint extensors become linearly dependent. If the arm has six joints in three-space, a critical configuration means a loss of full mobility. If the arm has a larger number of joints, criticality is defined as any six of the joint extensors becoming linearly dependent. This can mean severe problems with the driving program in real-life robots, even when the motion space retains full dimensionality.

In one sense, criticality is trivial to determine, since we need only compute a determinant function, called the ***superbracket***, on the six-dimensional space $\Lambda^2(V)$. However, if we want to know all the critical configurations of a given robot arm, this becomes a nontrivial question, that of determining all of the zeroes of the superbracket. To answer it, we need to express the superbracket in terms of ordinary brackets. This has been done in [MW91], where the superbracket of the six 2-extensors $a_1a_2, b_1b_2, \ldots, f_1f_2$ is given by

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

$$\begin{aligned}
[[a_1a_2, b_1b_2, c_1c_2, d_1d_2, e_1e_2, f_1f_2]] \;=\; \\
- \, [a_1a_2b_1b_2][c_1c_2\overset{\bullet}{d_1}\overset{\diamond}{e_1}][\overset{\bullet}{d_2}\overset{\diamond}{e_2}f_1f_2] \\
+ \, [a_1a_2\overset{\bullet}{b_1}\overset{\diamond}{c_1}][\overset{\bullet}{b_2}\overset{\diamond}{c_2}d_1d_2][e_1e_2f_1f_2] \\
- \, [a_1a_2\overset{\bullet}{b_1}\overset{\diamond}{c_1}][\overset{\bullet}{b_2}d_1d_2\overset{\triangleleft}{e_1}][\overset{\diamond}{c_2}\overset{\triangleleft}{e_2}f_1f_2] \\
+ \, [a_1a_2\overset{\bullet}{b_1}\overset{\diamond}{d_1}][\overset{\bullet}{b_2}c_1c_2\overset{\triangleleft}{e_1}][\overset{\diamond}{d_2}\overset{\triangleleft}{e_2}f_1f_2].
\end{aligned}$$

(Here the dots, diamonds, and triangles have the same meaning as the dots in Section 60.1.)

Consider the particular example of the six-revolute-joint robot arm illustrated in Figure 60.4.1, whose first two joints lie on intersecting lines, whose third and fourth joints are parallel, and whose last two joints also lie on intersecting lines. The larger cylinders in the figure represent the revolute joints. To express the superbracket, we must choose two points on each joint axis. We may choose $b_1 = a_2$, $d_1 = c_2$ (where this point is at infinity), and $f_1 = e_2$, as shown by the black dots. The thin cylinders represent the links; for example, the first link, between a_2 and b_2, is connected to the ground (not shown) by joint a_1a_2, and can therefore only rotate around the axis $\overline{a_1a_2}$.

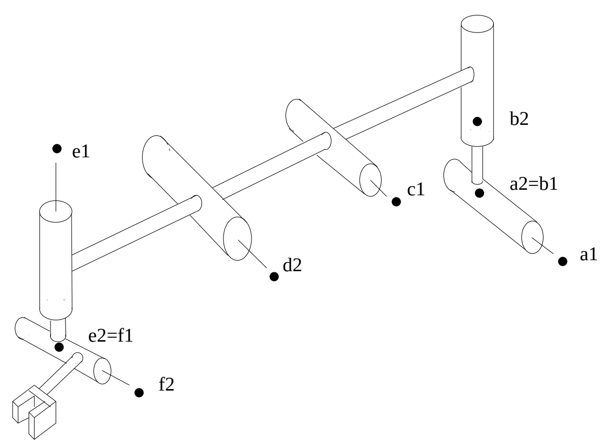

FIGURE 60.4.1
Six-revolute-joint robot arm.

Plugging in and deleting terms with a repeated point inside a bracket, we get

$$- \;\; [a_1a_2b_2\overset{\bullet}{c_1}][a_2c_2d_2e_2][\overset{\bullet}{c_2}e_1e_2f_2] \tag{60.4.1}$$

$$+ \;\; [a_1a_2b_2\overset{\bullet}{c_2}][a_2c_1c_2e_2][\overset{\bullet}{d_2}e_1e_2f_2] \tag{60.4.2}$$

$$= \;\; [\overset{\bullet}{c_1}a_1a_2b_2][\overset{\bullet}{d_2}a_2c_2e_2][\overset{\bullet}{c_2}e_1e_2f_2]\,, \tag{60.4.3}$$

where each of (60.4.1) and (60.4.2) has two terms because of the dotting, and the same four terms constitute (60.4.3), since two of the six terms generated by the dotting are zero because of the repetition of c_2 in the second bracket.

Finally, we recognize (60.4.3) as the bracket expansion of

$$(c_1d_2c_2) \wedge (a_1a_2b_2) \wedge (a_2c_2e_2) \wedge (e_1e_2f_2).$$

We then recognize that the geometric conditions for criticality are any positions that make this Grassmann-Cayley expression 0, namely

(i) one or more of the planes $\overline{c_1c_2d_2}, \overline{a_1a_2b_2}, \overline{a_2c_2e_2}, \overline{e_1e_2f_2}$ is degenerate, or

(ii) the four planes have nonempty intersection.

Notice that in an actual robot arm of the type we are considering, none of the degeneracies in (i) can actually occur.

See Section 51.1 for more information.

60.4.2 BAR FRAMEWORKS

Consider a generically $(d-1)$-isostatic graph G (see Section 61.1 of this Handbook), that is, a graph for which almost all realizations in $(d-1)$-space as a bar framework are minimally first-order rigid. Since first-order rigidity is a projective invariant (see Theorem 61.1.25), we would like to know the projective geometric conditions that characterize all of its nonrigid (first-order flexible) realizations. By Gram's theorem, these conditions must be expressible in terms of bracket conditions, and [WW83] shows that the first-order flexible realizations are characterized by the zeroes of a single bracket polynomial C_G, called the *pure condition* (see Theorem 61.1.27). Furthermore, [WW83] gives an algorithm to construct the pure condition C_G directly from the graph G. Then we require Cayley factorization to recover the geometric incidence condition, if it is not already known. Consider the following examples, illustrated in Figure 60.4.2.

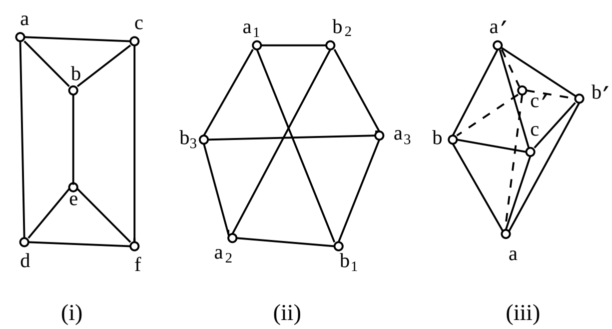

FIGURE 60.4.2
Three examples of bar frameworks.

(i) The graph G is the edge skeleton of a triangular prism, realized in the plane. We have $C_G = [abc][def]([abe][dfc] - [dbe][afc])$, and we may recognize the factor in parentheses as the third example in Table 60.2.1. Thus $C_G = 0$, and the framework is first-order flexible, if and only if one of the triangles $\overline{abc}$ or $\overline{def}$ is degenerate, or the three lines $\overline{ad}$, $\overline{be}$, $\overline{cf}$ are concurrent, or one or more of these lines is degenerate.

(ii) The graph G is $K_{3,3}$, a complete bipartite graph, realized in the plane. Then $C_G = [a_1a_2a_3][a_1b_2b_3][b_1a_2b_3][b_1b_2a_3] - [b_1b_2b_3][b_1a_2a_3][a_1b_2a_3][a_1a_2b_3]$, and this is the second example in Table 60.2.2. Thus $C_G = 0$, and the framework is first-order flexible, if and only if the six points lie on a common conic or, equivalently by Pascal's theorem, the three points $\overline{a_1b_2} \wedge \overline{a_2b_1}$, $\overline{a_1b_3} \wedge \overline{a_3b_1}$, $\overline{a_2b_3} \wedge \overline{a_3b_2}$ are collinear.

(iii) The graph G is the edge skeleton of an octahedron, realized in Euclidean 3-space. Then $C_G = [abc'a'][bca'b'][cab'c'] + [abc'b'][bca'c'][cab'a']$, and this can be recognized directly as the expansion of the Grassmann-Cayley expression $abc \wedge a'bc' \wedge a'b'c \wedge ab'c'$. Thus $C_G = 0$, and the framework is first-order flexible, if and only if the four alternating octahedral face planes $\overline{abc}$, $\overline{a'bc'}$, $\overline{a'b'c}$, and $\overline{ab'c'}$ concur, or any one or more of these planes is degenerate. This, in turn, is equivalent to the same condition on the other four face planes, $\overline{abc'}$, $\overline{ab'c}$, $\overline{a'bc}$, $\overline{a'b'c'}$.

60.4.3 BAR-AND-BODY FRAMEWORKS

A ***bar-and-body framework*** consists of a finite number of $(d-1)$-dimensional rigid bodies, free to move in Euclidean $(d-1)$-space, and connected by rigid bars, with the connections at the ends of each bar allowing free rotation of the bar relative to the rigid body; i.e., the connections are "universal joints." Each rigid body may be replaced by a first-order rigid bar framework in such a way that the result is one large bar framework, thus in one sense reducing the study of bar-and-body frameworks to that of bar frameworks. Nevertheless, the combinatorics of bar-and-body frameworks is quite different from that of bar frameworks, since the original rigid bodies are not allowed to become first-order flexible in any realization, contrary to the case with bar frameworks.

A generically isostatic bar-and-body framework has a pure condition, just as a bar framework has, whose zeroes are precisely the special positions in which the framework has a first-order flex. However, this pure condition is a bracket polynomial in the *bars* of the framework, as opposed to a bracket polynomial in the vertices, as was the case with bar frameworks. An algorithm to directly compute the pure condition for a bar-and-body framework, somewhat similar to that for bar frameworks, is given in [WW87]. We illustrate with the example in Figure 60.4.3, consisting of three rigid bodies and six bars in the plane. We may interpret the word "plane" here as "real projective plane."

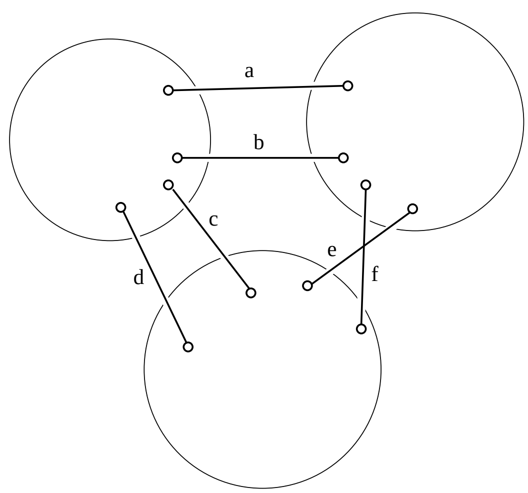

FIGURE 60.4.3
A bar-and-body framework.

Hence $V = \mathbb{R}^3$, and we let $W = \Lambda^2(V) \cong V^* \cong \mathbb{R}^3$. We think of the endpoints of the bars as elements of V, and hence the lines determined by the bars are two-extensors of these points, or elements of W. The algorithm produces the pure condition $[abc][def] - [abd][cef]$. This bracket polynomial may be Cayley factored as $ab \wedge cd \wedge ef$, as seen in Table 60.2.1. Now we switch to thinking of $a, b, \ldots, f$ as 2-extensors in V rather than elements of W, and recall that there is a duality between V and W, hence between $\Lambda(V)$ and $\Lambda(W)$. Thus, the framework has a first-order flex if and only if $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) = 0$ in $\Lambda(V)$. Hence the desired geometric condition for the existence of a first-order flex is that the three points $\overline{a \wedge b}$, $\overline{c \wedge d}$, and $\overline{e \wedge f}$ are collinear. Now $\overline{a \wedge b}$ is just the center of relative (instantaneous) motion for the two bodies connected by those two bars: think of fixing one of the bodies and then rotating the other body about this center; the lengths of the two bars are instantaneously preserved. The geometric result we have obtained is just a restatement of the classical theorem of Arnhold-Kempe that in any flex of three rigid bodies, the centers of relative motion of the three pairs of bodies must be collinear.

60.4.4 AUTOMATED GEOMETRIC THEOREM-PROVING

J. Richter-Gebert [RG95] uses Grassmann-Cayley algebra to derive bracket conditions for projective geometric incidences in order to produce coordinate-free automatic proofs of theorems in projective geometry. By introducing two circular points at infinity, the same can be done for theorems in Euclidean geometry [CRG95].

Richter-Gebert's technique is to reduce each hypothesis to a *binomial* equation, that is, an equation with a single product of brackets on each side. For example, as we have seen, the concurrence of three lines $\overline{ab}, \overline{cd}, \overline{ef}$ may be rewritten as $[acd][bef] = [bcd][aef]$. Similarly, the collinearity of three points $\overline{a}, \overline{b}, \overline{c}$ may be expressed as $[abd][bce] = [abe][bcd]$, avoiding the much more obvious expression $[abc] = 0$ since it is not of the required form. If all binomial equations are now multiplied together, and provided they were appropriately chosen in the first place, common factors may be canceled (which involves nondegeneracy assumptions, so that the common factors are nonzero), resulting in the desired conclusion. A surprising array of theorems may be cast in this format, and this approach has been successfully implemented.

More recent work along similar lines, extending it especially to conic geometry, is by H. Li and Y. Wu [LW03a, LW03b].

60.4.5 COMPUTER VISION

Much of computer vision study involves projective geometry, and hence is very amenable to the techniques of the Grassmann-Cayley algebra. One reference that explicitly applies these techniques to a system of up to three pinhole cameras is Faugeras and Papadopoulo [FP98].

60.5 SOURCES AND RELATED MATERIAL

SURVEYS

[DRS74] and [BBR85]: These two papers survey the properties of the Grassmann-Cayley algebra (called the "double algebra" in [BBR85]).

[Whi95]: A more elementary survey than the two above.

[Whi94]: Emphasizes the concrete approach via Plücker coordinates, and gives more detail on the connections to robotics.

RELATED CHAPTERS

Chapter 9: Geometry and topology of polygonal linkages
Chapter 51: Robotics
Chapter 61: Rigidity and scene analysis
Chapter 62: Rigidity of symmetric frameworks

REFERENCES

[BBR85] M. Barnabei, A. Brini, and G.-C. Rota. On the exterior calculus of invariant theory. *J. Algebra*, 96:120–160, 1985.

[CRG95] H. Crapo and J. Richter-Gebert. Automatic proving of geometric theorems. In N. White, editor, *Invariant Methods in Discrete and Computational Geometry*, pages 167–196. Kluwer, Dordrecht, 1995.

[DRS74] P. Doubilet, G.-C. Rota, and J. Stein. On the foundations of combinatorial theory: IX, combinatorial methods in invariant theory. *Stud. Appl. Math.*, 53:185–216, 1974.

[FP98] O. Faugeras and T. Papadopoulo. Grassmann-Cayley algebra for modelling systems of cameras and the algebraic equations of the manifold of trifocal tensors. *Philos. Trans. Roy. Soc. London*, Ser. A, 356:1123–1152, 1998.

[LW03a] H. Li and Y. Wu Automated short proof generation for projective geometric theorems with Cayley and bracket algebras, I. Incidence geometry. *J. Symbolic Comput.*, 36:717–762, 2003.

[LW03b] H. Li and Y. Wu Automated short proof generation for projective geometric theorems with Cayley and bracket algebras, II. Conic geometry. *J. Symbolic Comput.*, 36:763–809, 2003.

[MW91] T. McMillan and N.L. White. The dotted straightening algorithm. *J. Symbolic Comput.*, 11:471–482, 1991.

[RG95] J. Richter-Gebert. Mechanical theorem proving in projective geometry. *Ann. Math. Artif. Intell.*, 13:139–172, 1995.

[RS76] G.-C. Rota and J. Stein. Applications of Cayley algebras. In *Colloquio Internazionale sulle Teorie Combinatorie*, pages 71–97, Accademia Nazionale dei Lincei, 1976.

[Stu93] B. Sturmfels. *Algorithms in Invariant Theory*. Springer-Verlag, New York, 1993.

[Whi91] N. White. Multilinear Cayley factorization. *J. Symbolic Comput.*, 11:421–438, 1991.

[Whi94] N. White. Grassmann-Cayley algebra and robotics. *J. Intell. Robot. Syst.*, 11:91–107, 1994.

[Whi95] N. White. A tutorial on Grassmann-Cayley algebra. In N. White, editor, *Invariant Methods in Discrete and Computational Geometry*, pages 93–106. Kluwer, Dordrecht, 1995.

[WW83] N. White and W. Whiteley. The algebraic geometry of stresses in frameworks. *SIAM J. Algebraic Discrete Methods*, 4:481–511, 1983.

[WW87] N. White and W. Whiteley. The algebraic geometry of motions in bar-and-body frameworks. *SIAM J. Algebraic Discrete Methods*, 8:1–32, 1987.