# 6 ORIENTED MATROIDS

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## INTRODUCTION

The theory of *oriented matroids* provides a broad setting in which to model, describe, and analyze combinatorial properties of geometric configurations. Mathematical objects of study that appear to be disjoint and independent, such as *point and vector configurations*, *arrangements of hyperplanes*, *convex polytopes*, *directed graphs*, and *linear programs* find a common generalization in the language of oriented matroids.

The oriented matroid of a finite set of points $P$ extracts relative position and orientation information from the configuration; for example, it can be given by a list of signs that encodes the orientations of all the bases of $P$. In the passage from a concrete point configuration to its oriented matroid, metrical information is lost, but many structural properties of $P$ have their counterparts at the—purely combinatorial—level of the oriented matroid. (In computational geometry, the oriented matroid data of an unlabelled point configuration are sometimes called the *order type*.) From the oriented matroid of a configuration of points, one can compute not only that face lattice of the convex hull, but also the set of all its triangulations and subdivisions (cf. Chapter 16).

We first introduce oriented matroids in the context of several models and motivations (Section 6.1). Then we present some equivalent axiomatizations (Section 6.2). Finally, we discuss concepts that play central roles in the theory of oriented matroids (Section 6.3), among them *duality*, *realizability*, the study of *simplicial cells*, and the treatment of *convexity*.

## 6.1 MODELS AND MOTIVATIONS

This section discusses geometric examples that are usually treated on the level of concrete coordinates, but where an "oriented matroid point of view" gives deeper insight. We also present these examples as standard models that provide intuition for the behavior of general oriented matroids.

### 6.1.1 ORIENTED BASES OF VECTOR CONFIGURATIONS

#### GLOSSARY

***Vector configuration X:*** A matrix $X = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, usually assumed to have full rank $d$.



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***Matroid of X:*** The pair $M_X = (E, \mathcal{B}_X)$, where $E := \{1, 2, \ldots, n\}$ and $\mathcal{B}_X$ is the set of all subsets of $E$ that correspond to bases of (the column space of) $X$.

***Matroid:*** A pair $M = (E, \mathcal{B})$, where $E$ is a finite set, and $\mathcal{B} \subset 2^E$ is a nonempty collection of subsets of $E$ (the ***bases*** of $M$) that satisfies the ***Steinitz exchange axiom***: For all $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \backslash B_2$, there exists an $f \in B_2 \backslash B_1$ such that $(B_1 \backslash e) \cup f \in \mathcal{B}$.

***Chirotope of X:*** The map

$$\chi_X : \begin{array}{rcl} E^d & \to & \{-1, 0, +1\} \\ (\lambda_1, \ldots, \lambda_d) & \mapsto & \operatorname{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})). \end{array}$$

***Signs:*** The elements of the set $\{-, 0, +\}$, used as a shorthand for the corresponding elements of $\{-1, 0, +1\}$.

Ordinary (unoriented) *matroids*, as introduced in 1935 by Whitney (see Kung [Kun86], Oxley [Oxl92]), can be considered as an abstraction of vector configurations in finite dimensional vector spaces over arbitrary fields. All the bases of a matroid $M$ have the same cardinality $d$, which is called the ***rank*** of the matroid. Equivalently, we can identify $M$ with the characteristic function of the bases $B_M : E^d \to \{0, 1\}$, where $B_M(\lambda) = 1$ if and only if $\{\lambda_1, \ldots, \lambda_d\} \in \mathcal{B}$.

One can obtain examples of matroids as follows: Take a finite set of vectors

$$X = \{x_1, x_2, \ldots, x_n\} \subset V$$

of rank $d$ in a $d$-dimensional vector space $V$ over an arbitrary field $K$ and consider the set of bases of $V$ formed by subsets of the points in $X$. In other words, the pair

$$M_X = (E, \mathcal{B}_X) = \big(\, \{1, \ldots, n\},\ \big\{\{\lambda_1, \ldots, \lambda_d\} \mid \det(x_{\lambda_1}, \ldots, x_{\lambda_d}) \neq 0\big\} \,\big)$$

forms a matroid.

The basic information about the incidence structure of the points in $X$ is contained in the underlying matroid $M_X$. However, the matroid alone without additional orientation information contains only very restricted information about a geometric configuration. For example, any configuration of $n$ points in the plane in ***general position*** (i.e., no three points on a line) after ***homogenization*** (that is, appending a coordinate 1 to the each point) yields a vector configuration in $\mathbb{R}^3$ such that any three vectors are linearly independent. Thus all such point configurations yield the same matroid $M = U_{3,n}$: Here the matroid retains no information beyond the dimension and size of the configuration, and the fact that it is in general position.

In contrast to matroids, the theory of ***oriented matroids*** considers the structure of dependencies in vector spaces over *ordered* fields. Roughly speaking, an oriented matroid is a matroid where in addition every basis is equipped with an orientation. These oriented bases have to satisfy an oriented version of the Steinitz exchange axiom (to be described later). For the affine setting, oriented matroids not only describe the incidence structure between the points of $X$ and the hyperplanes spanned by points of $X$ (this is the matroid information); they also encode the positions of the points relative to the hyperplanes: "Which points lie on the positive side of a hyperplane, which points lie on the negative side, and which lie on the hyperplane?" If $X \in V^n$ is a configuration of $n$ points in a $d$-dimensional vector space $V$ over an ordered field $K$, we can describe the corresponding oriented

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matroid $\chi_X$ by the chirotope, which encodes the orientation of the $(d+1)$-tuples of points in $X$. The chirotope is very closely related to the oriented matroid of $X$, but it encodes much more information than the corresponding matroid, including orientation and convexity information about the underlying configuration.

## 6.1.2 CONFIGURATIONS OF POINTS

### GLOSSARY

***Affine point configuration:*** A matrix $X' = (x'_1, \ldots, x'_n) \in (\mathbb{R}^{d-1})^n$, usually with the assumption that $x'_1, \ldots, x'_n$ affinely span $\mathbb{R}^{d-1}$.

***Associated vector configuration:*** The matrix $X \in (\mathbb{R}^d)^n$ of full rank $d$ obtained from an ordered/labeled point configuration $X' = (x'_1, \ldots, x'_n)$ by adding a row of ones. This corresponds to the embedding of the affine space $\mathbb{R}^{d-1}$ into the linear vector space $\mathbb{R}^d$ via $p \longmapsto x = \binom{p}{1}$.

***Homogenization:*** The step from a point configuration to the associated vector configuration.

***Oriented matroid of an affine point configuration:*** The oriented matroid of the associated vector configuration.

***Covector of a vector configuration $X$:*** Any partition of $X = (x_1, \ldots, x_n)$ induced by an oriented linear hyperplane into the points on the hyperplane, on its positive side, and on its negative side. The partition is denoted by a sign vector $C \in \{-, 0, +\}^n$.

***Oriented matroid of $X$:*** The collection $\mathcal{L}_X \subseteq \{-, 0, +\}^n$ of all covectors of $X$.

Let $X := (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ be an $n \times d$ matrix and let $E := \{1, \ldots, n\}$. We interpret the columns of $X$ as $n$ vectors in the $d$-dimensional real vector space $\mathbb{R}^d$. For a linear functional $y^T \in (\mathbb{R}^d)^*$ we set

$$C_X(y) = (\mathrm{sign}(y^T x_1), \ldots, \mathrm{sign}(y^T x_n)).$$

Such a sign vector is called a ***covector*** of $X$. We denote the collection of all covectors of $X$ by

$$\mathcal{L}_X := \{C_X(y) \mid y \in \mathbb{R}^d\}.$$

The pair $\mathcal{M}_X = (E, \mathcal{L}_X)$ is called the ***oriented matroid*** of $X$. Here each sign vector $C_X(y) \in \mathcal{L}_X$ describes the positions of the vectors $x_1, \ldots, x_n$ relative to the linear hyperplane $H_y = \{x \in \mathbb{R}^d \mid y^T x = 0\}$: the sets

$$\begin{aligned} C_X(y)^0 &:= \{e \in E \mid C_X(y)_e = 0\} \\ C_X(y)^+ &:= \{e \in E \mid C_X(y)_e > 0\} \\ C_X(y)^- &:= \{e \in E \mid C_X(y)_e < 0\} \end{aligned}$$

describe how $H_y$ partitions the set of points $X$. Thus $C_X(y)^0$ contains the points on $H_y$, while $C_X(y)^+$ and $C_X(y)^-$ contain the points on the positive and on the negative side of $H_y$, respectively. In particular, if $C_X(y)^- = \emptyset$, then all points not on $H_y$ lie on the positive side of $H_y$. In other words, in this case $H_y$ determines a face of the positive cone

$$\mathrm{pos}(x_1, \ldots, x_n) := \{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \mid 0 \leq \lambda_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

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of all points of $X$. The face lattice of the cone $\text{pos}(X)$ can be recovered from $\mathcal{L}_X$. It is simply the set $\mathcal{L}_X \cap \{+,0\}^E$, partially ordered by the order induced from the relation "$0 < +$."

If, in the configuration $X$, we have $x_{i,d} = 1$ for all $1 \leq i \leq n$, then we can consider $X$ as representing homogeneous coordinates of an *affine* point set $X'$ in $\mathbb{R}^{d-1}$. Here the affine points correspond to the original points $x_i$ after removal of the $d$th coordinate. The face lattice of the convex polytope $\text{conv}(X') \subset \mathbb{R}^{d-1}$ is then isomorphic to the face lattice of $\text{pos}(X)$. Hence, $\mathcal{M}_X$ can be used to recover the *convex hull* of $X'$.

Thus oriented matroids are generalizations of point configurations in linear or affine spaces. For general oriented matroids we weaken the assumption that the hyperplanes spanned by points of the configuration are flat to the assumption that they only satisfy certain topological incidence properties. Nonetheless, this kind of picture is sometimes misleading since not all oriented matroids have this type of representation (compare the "Type II representations" of [BLS$^+$93, Sect. 5.3]).

## 6.1.3 ARRANGEMENTS OF HYPERPLANES AND OF HYPERSPHERES

### GLOSSARY

***Hyperplane arrangement $\mathcal{H}$:*** Collection of (oriented) linear hyperplanes in $\mathbb{R}^d$, given by normal vectors $x_1, \ldots, x_n$.

***Hypersphere arrangement induced by $\mathcal{H}$:*** Intersection of $\mathcal{H}$ with the unit sphere $S^{d-1}$.

***Covectors of $\mathcal{H}$:*** Sign vectors of the cells in $\mathcal{H}$; equivalently, $\mathbf{0}$ together with the sign vectors of the cells in $\mathcal{H} \cap S^{d-1}$.

We obtain a different picture if we polarize the situation and consider *hyperplane arrangements* rather than configurations of points. For a real matrix $X := (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ consider the system of hyperplanes $\mathcal{H}_X := (H_1, \ldots, H_n)$ with

$$H_i := \{y \in \mathbb{R}^d \mid y^T x_i = 0\}.$$

Each vector $x_i$ induces an orientation on $H_i$ by defining

$$H_i^+ := \{y \in \mathbb{R}^d \mid y^T x_i > 0\}$$

to be the ***positive side*** of $H_i$. We define $H_i^-$ analogously to be the ***negative side*** of $H_i$. To avoid degenerate cases we assume that $X$ contains at least one proper basis (i.e., the matrix $X$ has rank $d$). The hyperplane arrangement $\mathcal{H}_X$ subdivides $\mathbb{R}^d$ into polyhedral cones. Without loss of information we can intersect with the unit sphere $S^{d-1}$ and consider the sphere system

$$\mathcal{S}_X \ := \ \left(H_1 \cap S^{d-1}, \ldots, H_n \cap S^{d-1}\right) \ = \ \mathcal{H}_X \cap S^{d-1}.$$

Our assumption that $X$ contains at least one proper basis translates to the fact that the intersection of all $H_1 \cap \ldots \cap H_n \cap S^{d-1}$ is empty. $\mathcal{H}_X$ induces a cell decomposition $\Gamma(\mathcal{S}_X)$ on $S^{d-1}$. Each face of $\Gamma(\mathcal{S}_X)$ corresponds to a sign vector in $\{-,0,+\}^E$ that indicates the position of the cell with respect to the $(d-2)$-spheres

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$H_i \cap S^{d-1}$ (and therefore with respect to the hyperplanes $H_i$) of the arrangement. The list of all these sign vectors is exactly the set $\mathcal{L}_X$ of covectors of $\mathcal{H}_X$.

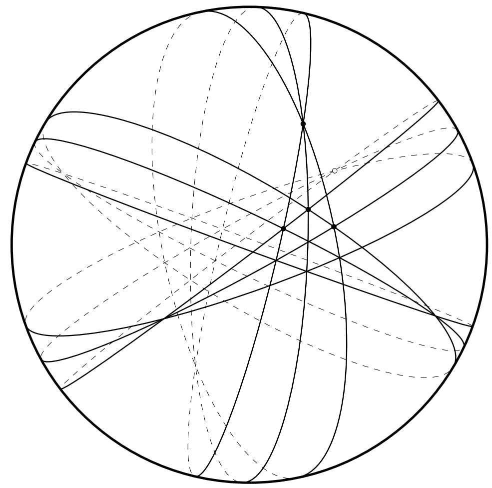

FIGURE 6.1.1
*An arrangement of nine great circles on $S^2$. The arrangement corresponds to a Pappus configuration.*

While the visualization of oriented matroids by sets of points in $\mathbb{R}^n$ does not fully generalize to the case of nonrepresentable oriented matroids, the picture of hyperplane arrangements has a well-defined extension that also covers all the nonrealizable cases. We will see that as a consequence of the topological representation theorem of Folkman and Lawrence (Section 6.2.4) every rank $d$ oriented matroid can be represented as an arrangement of oriented *pseudospheres* (or pseudohyperplanes) embedded in the $S^{d-1}$ (or in $\mathbb{R}^d$, respectively). Arrangements of pseudospheres are systems of topological $(d-2)$-spheres embedded in $S^{d-1}$ that satisfy certain intersection properties that clearly hold in the case of "straight" arrangements.

## 6.1.4 ARRANGEMENTS OF PSEUDOLINES

## GLOSSARY

***Pseudoline:*** Simple closed curve $p$ in the projective plane $\mathbb{R}\mathrm{P}^2$ that is topologically equivalent to a line (i.e., there is a self-homeomorphism of $\mathbb{R}\mathrm{P}^2$ mapping $p$ to a straight line).

***Arrangement of pseudolines:*** Collection of pseudolines $\mathcal{P} := (p_1, \ldots, p_n)$ in the projective plane, any two of them intersecting exactly once.

***Simple arrangement:*** No three pseudolines meet in a common point. (Equivalently, the associated oriented matroid is ***uniform***.)

***Equivalent arrangements:*** Arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ that generate isomorphic cell decompositions of $\mathbb{R}\mathrm{P}^2$. (In this case there exists a self-homeomorphism of $\mathbb{R}\mathrm{P}^2$ mapping $\mathcal{P}_1$ to $\mathcal{P}_2$.)

***Stretchable arrangement of pseudolines:*** An arrangement that is equivalent to an arrangement of projective lines.

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An *arrangement of pseudolines* in the projective plane is a collection of pseudolines such that any two pseudolines intersect in exactly one point, where they cross. (See Grünbaum [Grü72] and Richter [Ric89].) We will always assume that $\mathcal{P}$ is ***essential***, i.e., that the intersection of all the pseudolines $p_i$ is empty.

An arrangement of pseudolines behaves in many respects just like an arrangement of $n$ lines in the projective plane. (In fact, there are only very few combinatorial theorems known that are true for straight arrangements, but not true in general for pseudoarrangements.) Figure 6.1.1 shows a small example of a nonstretchable arrangement of pseudolines. (It is left as a challenging exercise to the reader to prove the nonstretchability.) Up to isomorphism this is the only simple nonstretchable arrangement of 9 pseudolines [Ric89] [Knu92]; every arrangement of 8 (or fewer) pseudolines is stretchable [GP80].

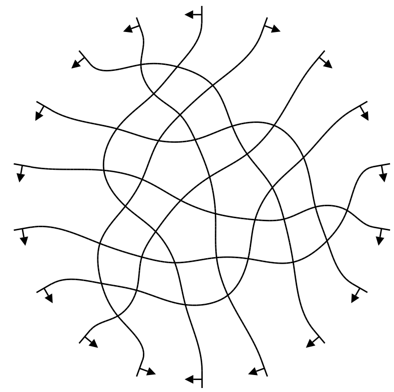

FIGURE 6.1.2
*A nonstretchable arrangement of nine pseudolines. It was obtained by Ringel* [Rin56] *as a perturbation of the Pappus configuration.*

To associate with a projective arrangement $\mathcal{P}$ an oriented matroid we represent the projective plane (as customary) by the 2-sphere with antipodal points identified. With this, every arrangement of pseudolines gives rise to an arrangement of *great pseudocircles* on $S^2$. For each great pseudocircle on $S^2$ we choose a positive side. Each cell induced by $\mathcal{P}$ on $S^2$ now corresponds to a unique sign vector. The collection of all these sign vectors again forms a set of covectors $\mathcal{L}_{\mathcal{P}} \backslash \{\mathbf{0}\}$ of an oriented matroid of rank 3. Conversely, as a special case of the topological representation theorem (see Theorem 6.2.4 below), *every* oriented matroid of rank 3 has a representation by an *oriented* pseudoline arrangement.

Thus we can use pseudoline arrangements as a standard picture to represent rank 3 oriented matroids. The easiest picture is obtained when we restrict ourselves to the upper hemisphere of $S^2$ and assume w.l.o.g. that each pseudoline crosses the equator exactly once, and that the crossings are distinct (i.e., no intersection of the great pseudocircles lies on the equator). Then we can represent this upper hemisphere by an arrangement of mutually crossing, oriented affine pseudolines in the plane $\mathbb{R}^2$. (We did this implicitly while drawing Figure 6.1.2.) For a reasonably elementary proof of the fact that rank 3 oriented matroids are equivalent to arrangements of pseudolines see Bokowski, Mock, and Streinu [BMS01].

By means of this equivalence, all problems concerning pseudoline arrangements can be translated to the language of oriented matroids. For instance, the problem of stretchability is equivalent to the realizability problem for oriented matroids.

## 6.2 AXIOMS AND REPRESENTATIONS

In this section we define oriented matroids formally. It is one of the main features of oriented matroid theory that the same object can be viewed under quite different aspects. This results in the fact that there are many different equivalent axiomatizations, and it is sometimes very useful to "jump" from one point of view to another. Statements that are difficult to prove in one language may be easy in another. For this reason we present here several different axiomatizations. We also give a (partial) dictionary that indicates how to translate among them. For a complete version of the basic equivalence proofs—which are highly nontrivial—see [BLS$^+$93, Chapters 3 and 5].

We will give axiomatizations of oriented matroids for the following four types of representations:

- collections of covectors,
- collections of cocircuits,
- signed bases, and
- arrangements of pseudospheres.

In the last part of this section these concepts are illustrated by an example.

### GLOSSARY

***Sign vector:*** Vector $C$ in $\{-,0,+\}^E$, where $E$ is a finite index set, usually $\{1,\ldots,n\}$. For $e\in E$, the $e$-component of $C$ is denoted by $C_e$.

***Positive, negative, and zero part of C:***

$$\begin{aligned} C^+ &:= \{e\in E \mid C_e = +\},\\ C^- &:= \{e\in E \mid C_e = -\},\\ C^0 &:= \{e\in E \mid C_e = 0\}. \end{aligned}$$

***Support of C:*** $\underline{C} := \{e\in E \mid C_e \neq 0\}$.

***Zero vector:*** $\mathbf{0} := (0,\ldots,0)\in\{-,0,+\}^E$.

***Negative of a sign vector:*** $-C$, defined by $(-C)^+ := C^-$, $(-C)^- := C^+$ and $(-C)^0 = C^0$.

***Composition of C and D:*** $(C\circ D)_e := \begin{cases} C_e & \text{if } C_e\neq 0,\\ D_e & \text{otherwise.}\end{cases}$

***Separation set of C and D:*** $S(C,D) := \{e\in E \mid C_e = -D_e \neq 0\}$.

We partially order the set of sign vectors by "$0 < +$" and "$0 < -$". The partial order on sign vectors, denoted by $C\le D$, is understood componentwise; equivalently, we have

$$C\le D \iff \left[\, C^+\subset D^+ \text{ and } C^-\subset D^- \,\right].$$

For instance, if $C := (+,+,-,0,-,+,0,0)$ and $D := (0,0,-,+,+,-,0,-)$, then we have:

$$C^+ = \{1,2,6\},\quad C^- = \{3,5\},\quad C^0 = \{4,7,8\},\quad \underline{C} = \{1,2,3,5,6\},$$

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$$C \circ D = (+,+,-,+,-,+,0,-), \quad C \circ D \geq C, \quad S(C,D) = \{5,6\}.$$

Furthermore, for $x \in \mathbb{R}^n$, we denote by $\sigma(x) \in \{-,0,+\}^E$ the image of $x$ under the componentwise sign function $\sigma$ that maps $\mathbb{R}^n$ to $\{-,0,+\}^E$.

## 6.2.1 COVECTOR AXIOMS

**Definition:** An ***oriented matroid*** given in terms of its covectors is a pair $\mathcal{M} := (E,\mathcal{L})$, where $\mathcal{L} \subseteq \{-,0,+\}^E$ satisfies

(CV0) $\mathbf{0} \in \mathcal{L}$

(CV1) $C \in \mathcal{L} \implies -C \in \mathcal{L}$

(CV2) $C, D \in \mathcal{L} \Longrightarrow C \circ D \in \mathcal{L}$

(CV3) $C, D \in \mathcal{L},\ e \in S(C,D) \Longrightarrow$
there is a $Z \in \mathcal{L}$ with $Z_e = 0$ and with $Z_f = (C \circ D)_f$ for $f \in E \backslash S(C,D)$.

It is not difficult to check that these covector axioms are satisfied by the sign vector system $\mathcal{L}_X$ of the cells in a hyperplane arrangement $\mathcal{H}_X$, as defined in the previous section. The first two axioms are satisfied trivially. For (CV2) assume that $x_C$ and $x_D$ are points in $\mathbb{R}^d$ with $\sigma(x_C^T \cdot X) = C \in \mathcal{L}_X$ and $\sigma(x_D^T \cdot X) = D \in \mathcal{L}_X$. Then (CV2) is implied by the fact that for sufficiently small $\varepsilon > 0$ we have $\sigma((x_C + \varepsilon x_D)^T \cdot X) = C \circ D$. The geometric content of (CV3) is that if $H_e := \{y \in \mathbb{R}^d \mid y^T x_e = 0\}$ is a hyperplane separating $x_C$ and $x_D$, then there exists a point $x_Z$ on $H_e$ with the property that $x_Z$ is on the same side as $x_C$ and $x_D$ for all hyperplanes not separating $x_C$ and $x_D$. We can find such a point by intersecting $H_e$ with the line segment that connects $x_C$ and $x_D$.

As we will see later the partially ordered set $(\mathcal{L}, \leq)$ describes the face lattice of a cell decomposition of the sphere $S^{d-1}$ by pseudohyperspheres. Each sign vector corresponds to a face of the cell decomposition. We define the ***rank*** $d$ of $\mathcal{M} = (E,\mathcal{L})$ to be the (unique) length of the maximal chains in $(\mathcal{L}, \leq)$ minus one. In the case of realizable arrangements $\mathcal{S}_X$ of hyperspheres, the lattice $(\mathcal{L}_X, \leq)$ equals the face lattice of the cell complex of the arrangement (see Section 6.2.4).

## 6.2.2 COCIRCUITS

The covectors of (inclusion-)minimal support in $\mathcal{L} \backslash \{\mathbf{0}\}$ correspond to the 0-faces (= vertices) of the cell decomposition that we have just described. We call the set $\mathcal{C}^*(\mathcal{M})$ of all such minimal covectors the ***cocircuits*** of $\mathcal{M}$. An oriented matroid can be described by its set of cocircuits, as shown by the following theorem.

**THEOREM 6.2.1** *Cocircuit Characterization*

*A collection $\mathcal{C}^* \subset \{-,0,+\}^E$ is the set of cocircuits of an oriented matroid $\mathcal{M}$ if and only if it satisfies*

(CC0) $\mathbf{0} \notin \mathcal{C}^*$

(CC1) $C \in \mathcal{C}^* \implies -C \in \mathcal{C}^*$

(CC2) *for all $C, D \in \mathcal{C}^*$ we have: $\underline{C} \subset \underline{D} \Longrightarrow C = D$ or $C = -D$*

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(CC3) $C, D \in \mathcal{C}^*$, $C \neq -D$, *and* $e \in S(C,D) \implies$
*there is a* $Z \in \mathcal{C}^*$ *with* $Z^+ \subset (C^+ \cup D^+)\backslash\{e\}$ *and* $Z^- \subset (C^- \cup D^-)\backslash\{e\}$.

**THEOREM 6.2.2** *Covector/Cocircuit Translation*

*For every oriented matroid $\mathcal{M}$, one can uniquely determine the set $\mathcal{C}^*$ of cocircuits from the set $\mathcal{L}$ of covectors of $\mathcal{M}$, and conversely, as follows:*

(i) *$\mathcal{C}^*$ is the set of sign vectors with minimal support in $\mathcal{L}\backslash\{\mathbf{0}\}$:*
$\mathcal{C}^* = \{C \in \mathcal{L}\backslash\{\{\mathbf{0}\}\} \mid C' \leq C \Longrightarrow C' \in \{\mathbf{0}, C\}\}$

(ii) *$\mathcal{L}$ is the set of all sign vectors obtained by successive composition of a finite number of cocircuits from $\mathcal{C}^*$:*
$\mathcal{L} = \{C_1 \circ \ldots \circ C_k \mid k \geq 0,\ C_1, \ldots, C_k \in \mathcal{C}^*\}$.
*(The zero vector is obtained as the composition of an empty set of covectors.)*

In part (ii) of this result one may assume additionally that the sign vectors $C_i$ are ***compatible***, that is, they do not have opposite nonzero signs in any component, so that the composition is commutative.

## 6.2.3 CHIROTOPES

## GLOSSARY

***Alternating sign map:*** A map $\chi\colon E^d \longrightarrow \{-,0,+\}$ such that any transposition of two components changes the sign: $\chi(\tau_{ij}(\lambda)) = -\chi(\lambda)$.

***Chirotope:*** An alternating sign map $\chi$ that encodes the basis orientations of an oriented matroid $\mathcal{M}$ of rank $d$.

We now present an axiom system for *chirotopes*, which characterizes oriented matroids in terms of basis orientations. Here an algebraic connection to determinant identities becomes obvious. Chirotopes are the main tool for translating problems in oriented matroid theory to an algebraic setting [BS89a]. They also form a description of oriented matroids that is very practical for many algorithmic purposes (for instance in computational geometry; see Knuth [Knu92]).

**Definition:** Let $E := \{1, \ldots, n\}$ and $0 \leq d \leq n$. A ***chirotope of rank d*** is an alternating sign map $\chi\colon E^d \to \{-,0,+\}$ that satisfies

(CHI1) the map $|\chi|\colon E^d \to \{0,1\}$, $\lambda \mapsto |\chi(\lambda)|$ is a matroid, and

(CHI2) for every $\lambda \in E^{d-2}$ and $a,b,c,d \in E\backslash\lambda$ the set

$$\{\ \chi(\lambda,a,b)\cdot\chi(\lambda,c,d),\ -\chi(\lambda,a,c)\cdot\chi(\lambda,b,d),\ \chi(\lambda,a,d)\cdot\chi(\lambda,b,c)\ \}$$

either contains $\{-1,+1\}$ or equals $\{0\}$.

Where does the motivation of this axiomatization come from? If we again consider a configuration $X := (x_1, \ldots, x_n)$ of vectors in $\mathbb{R}^d$, we can observe the following identity among the $d \times d$ submatrices of $X$:

$$\begin{aligned} &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_b)\cdot\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_c, x_d) \\ -\ &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_c)\cdot\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_b, x_d) \\ +\ &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_d)\cdot\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_b, x_c) \quad = \quad 0 \end{aligned}$$

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for all $\lambda \in E^{d-2}$ and $a, b, c, d \in E\backslash\lambda$. Such a relation is called a ***three-term Grassmann–Plücker identity***. If we compare this identity to our axiomatization, we see that (CHI2) implies that

$$\begin{array}{rrcl} \chi_X : & E^d & \to & \{-, 0, +\} \\ & (\lambda_1, \ldots, \lambda_d) & \mapsto & \mathrm{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})) \end{array}$$

is consistent with these identities. More precisely, if we consider $\chi_X$ as defined above for a vector configuration $X$, the above Grassmann–Plücker identities imply that (CHI2) is satisfied. (CHI1) is also satisfied since for the vectors of $X$ the Steinitz exchange axiom holds. (In fact the exchange axiom is a consequence of higher order Grassmann–Plücker identities.)

Consequently, $\chi_X$ is a chirotope for every $X \in (\mathbb{R}^d)^n$. Thus chirotopes can be considered as a combinatorial model of the determinant values on vector configurations. The following is not easy to prove, but essential.

**THEOREM 6.2.3** *Chirotope/Cocircuit Translation*

*For each chirotope $\chi$ of rank $d$ on $E := \{1, \ldots, n\}$ the set*

$$\mathcal{C}^*(\chi) \ = \ \big\{\big(\chi(\lambda, 1), \chi(\lambda, 2), \ldots, \chi(\lambda, n)\big) \ \big| \ \lambda \in E^{d-1}\big\}$$

*forms the set of cocircuits of an oriented matroid. Conversely, for every oriented matroid $\mathcal{M}$ with cocircuits $\mathcal{C}^*$ there exists a unique pair of chirotopes $\{\chi, -\chi\}$ such that $\mathcal{C}^*(\chi) = \mathcal{C}^*(-\chi) = \mathcal{C}^*$.*

The retranslation of cocircuits into signs of bases is straightforward but needs extra notation. It is omitted here.

## 6.2.4 ARRANGEMENTS OF PSEUDOSPHERES

### GLOSSARY

***A (d-1)-sphere:*** The standard unit sphere $S^{d-1} := \{x \in \mathbb{R}^d \mid ||x|| = 1\}$, or any homeomorphic image of it.

***Pseudosphere:*** The image $s \subset S^{d-1}$ of the equator $\{x \in S^{d-1} \mid x_d = 0\}$ in the unit sphere under a self-homeomorphism $\phi\colon S^{d-1} \to S^{d-1}$. (This definition describes topologically *tame* embeddings of a $(d-2)$-sphere in $S^{d-1}$. Pseudospheres behave "nicely" in the sense that they divide $S^{d-1}$ into two open sets, its ***sides***, that are homeomorphic to open $(d-1)$-balls.)

***Oriented pseudosphere:*** A pseudosphere together with a choice of a positive side $s^+$ and a negative side $s^-$.

***Arrangement of pseudospheres:*** A set of $n$ pseudospheres in $S^{d-1}$ with the extra condition that any subset of $d+2$ or fewer pseudospheres is ***realizable***: it defines a cell decomposition of $S^{d-1}$ that is isomorphic to a decomposition by an arrangement of $d+2$ linear hyperplanes.

***Essential arrangement:*** An arrangement such that the intersection of all the pseudospheres is empty.

***Rank:*** The codimension in $S^{d-1}$ of the intersection of all the pseudospheres. For an essential arrangement in $S^{d-1}$, the rank is $d$.

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***Topological representation of $\mathcal{M} = (E, \mathcal{L})$:*** An essential arrangement of oriented pseudospheres such that $\mathcal{L}$ is the collection of sign vectors associated with the cells of the arrangement.

One of the most important interpretations of oriented matroids is given by the topological representation theorem of Folkman and Lawrence [FL78]; see also [BLS+93, Chapters 4 and 5] and [BKMS05]. It states that oriented matroids are in bijection to (combinatorial equivalence classes of) *arrangements of oriented pseudospheres*. Arrangements of pseudospheres are a topological generalization of hyperplane arrangements, in the same way in which arrangements of pseudolines generalize line arrangements. Thus every rank $d$ oriented matroid describes a certain cell decomposition of the $(d-1)$-sphere. Arrangements of pseudospheres are collections of pseudospheres that have intersection properties just like those satisfied by arrangements of proper subspheres.

**Definition:** A finite collection $\mathcal{P} = (s_1, s_2, \ldots, s_n)$ of pseudospheres in $S^{d-1}$ is an ***arrangement of pseudospheres*** if the following conditions hold (we set $E := \{1, \ldots, n\}$):

(PS1) For all $A \subset E$ the set $S_A = \bigcap_{e \in A} s_e$ is a topological sphere.

(PS2) If $S_A \not\subset s_e$, for $A \subset E, e \in E$, then $S_A \cap s_e$ is a pseudosphere in $S_A$ with sides $S_A \cap s_e^+$ and $S_A \cap s_e^-$.

Notice that this definition permits two pseudospheres of the arrangement to be identical. An entirely different, but equivalent, definition is given in the glossary.

We see that every essential arrangement of pseudospheres $\mathcal{P}$ partitions the $(d-1)$-sphere into a regular cell complex $\Gamma(\mathcal{P})$. Each cell of $\Gamma(\mathcal{P})$ is uniquely determined by a sign vector in $\{-, 0, +\}^E$ encoding the relative position with respect to each pseudosphere $s_i$. Conversely, $\Gamma(\mathcal{P})$ characterizes $\mathcal{P}$ up to homeomorphism. An arrangement of pseudospheres $\mathcal{P}$ is ***realizable*** if there exists an arrangement of proper spheres $\mathcal{S}_X$ with $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{S}_X)$.

The translation of arrangements of pseudospheres to oriented matroids is given by the topological representation theorem of Folkman and Lawrence [FL78], as follows. (For the definition of "loop," see Section 6.3.1.)

**THEOREM 6.2.4** *The Topological Representation Theorem (Pseudosphere/ Covector Translation)*

*If $\mathcal{P}$ is an essential arrangement of pseudospheres on $S^{d-1}$ then $\Gamma(\mathcal{P}) \cup \{\mathbf{0}\}$ forms the set of covectors of an oriented matroid of rank d. Conversely, for every oriented matroid $(E, \mathcal{L})$ of rank $d$ (without loops) there exists an essential arrangement of pseudospheres $\mathcal{P}$ on $S^{d-1}$ with $\Gamma(\mathcal{P}) = \mathcal{L} \backslash \{\mathbf{0}\}$.*

## 6.2.5 REALIZABILITY AND DUALITY

### GLOSSARY

***Realizable oriented matroid:*** Oriented matroid $\mathcal{M}$ such that there is a vector configuration $X$ with $\mathcal{M}_X = \mathcal{M}$.

***Realization of $\mathcal{M}$:*** A vector configuration $X$ with $\mathcal{M}_X = \mathcal{M}$.

***Orthogonality:*** Two sign vectors $C, D \in \{-, 0, +\}^E$ are *orthogonal* if the set

$$\{C_e \cdot D_e \mid e \in E\}$$

either equals $\{0\}$ or contains $\{+, -\}$. We then write $C \perp D$.

***Vector of $\mathcal{M}$:*** A sign vector that is orthogonal to all covectors of $\mathcal{M}$; a covector of the dual oriented matroid $\mathcal{M}^*$.

***Circuit of $\mathcal{M}$:*** A vector of minimal nonempty support; a cocircuit of the dual oriented matroid $\mathcal{M}^*$.

Realizability is a crucial (and hard-to-decide) property of oriented matroids that may be discussed in any of the models/axiomatizations that we have introduced: An oriented matroid given by its covectors, cocircuits, chirotope, or by a pseudosphere arrangement is realizable if there is a vector configuration or a hypersphere arrangement which produces these combinatorial data.

There is a natural duality structure relating oriented matroids of rank $d$ on $n$ elements to oriented matroids of rank $n-d$ on $n$ elements. It is an amazing fact that the existence of such a duality relation can be used to give another axiomatization of oriented matroids (see [BLS$^+$93, Section 3.4]). Here we restrict ourselves to the definition of the dual of an oriented matroid $\mathcal{M}$.

**THEOREM 6.2.5** *Duality*

*For every oriented matroid $\mathcal{M} = (E, \mathcal{L})$ of rank $d$ there is a unique oriented matroid $\mathcal{M}^* = (E, \mathcal{L}^*)$ of rank $|E| - d$ given by*

$$\mathcal{L}^* \quad = \quad \Big\{D \in \{-, 0, +\}^E \;\Big|\; C \perp D \text{ for every } C \in \mathcal{L}\Big\}.$$

*$\mathcal{M}^*$ is called the **dual** of $\mathcal{M}$. In particular, $(\mathcal{M}^*)^* = \mathcal{M}$.*

In particular, the cocircuits of the dual oriented matroid $\mathcal{M}^*$, which we call the *circuits* of $\mathcal{M}$, also determine $\mathcal{M}$. Hence the collection $\mathcal{C}(\mathcal{M})$ of all circuits of an oriented matroid $\mathcal{M}$, given by

$$\mathcal{C}(\mathcal{M}) \quad := \quad \mathcal{C}^*(\mathcal{M}^*),$$

is characterized by *the same* cocircuit axioms. Analogously, the *vectors* of $\mathcal{M}$ are obtained as the covectors of $\mathcal{M}^*$; they are characterized by the covector axioms.

An oriented matroid $\mathcal{M}$ is realizable if and only if its dual $\mathcal{M}^*$ is realizable. The reason for this is that a matrix $(I_d|A)$ represents $\mathcal{M}$ if and only if $(-A^T|I_{n-d})$ represents $\mathcal{M}^*$. (Here $I_d$ denotes a $d \times d$ identity matrix, $A \in \mathbb{R}^{d\times(n-d)}$, and $A^T \in \mathbb{R}^{(n-d)\times d}$ denotes the transpose of $A$.)

Thus for a realizable oriented matroid $\mathcal{M}_X$ the vectors represent the linear dependencies among the columns of $X$, while the circuits represent minimal linear dependencies. Similarly, in the pseudoarrangements picture, circuits correspond to minimal systems of closed hemispheres that cover the whole sphere, while vectors correspond to consistent unions of such covers that never require the use of both hemispheres determined by a pseudosphere. This provides a direct geometric interpretation of circuits and vectors.

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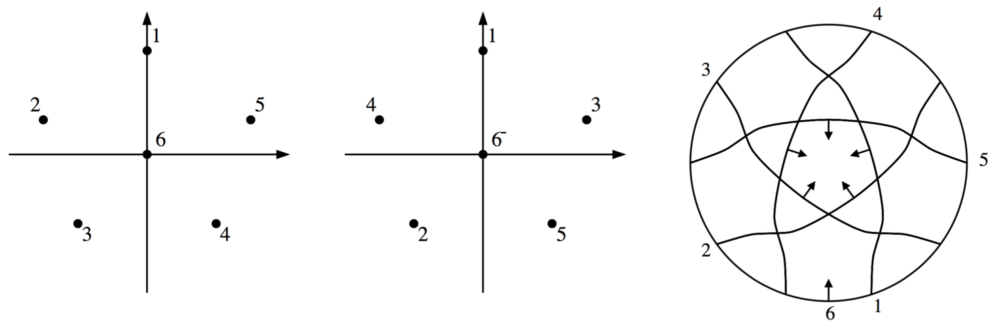
### 6.2.6 AN EXAMPLE

We close this section with an example that demonstrates the different representations of an oriented matroid. Consider the planar point configuration $X$ given in Figure 6.2.1(left).

FIGURE 6.2.1
*An example of an oriented matroid on 6 elements.*




Homogeneous coordinates for $X$ are given by

$$X := \begin{pmatrix} 0 & 3 & 1 \\ -3 & 1 & 1 \\ -2 & -2 & 1 \\ 2 & -2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The chirotope $\chi_X$ of $\mathcal{M}$ is given by the orientations:

$$\begin{array}{lllll} \chi(1,2,3)=+ & \chi(1,2,4)=+ & \chi(1,2,5)=+ & \chi(1,2,6)=+ & \chi(1,3,4)=+ \\ \chi(1,3,5)=+ & \chi(1,3,6)=+ & \chi(1,4,5)=+ & \chi(1,4,6)=- & \chi(1,5,6)=- \\ \chi(2,3,4)=+ & \chi(2,3,5)=+ & \chi(2,3,6)=+ & \chi(2,4,5)=+ & \chi(2,4,6)=+ \\ \chi(2,5,6)=- & \chi(3,4,5)=+ & \chi(3,4,6)=+ & \chi(3,5,6)=+ & \chi(4,5,6)=+ \end{array}$$

Half of the cocircuits of $\mathcal{M}$ are given in the table below (the other half is obtained by negating the data):

$$\begin{array}{lll} (0,0,+,+,+,+) & (0,-,0,+,+,+) & (0,-,-,0,+,-) \\ (0,-,-,-,0,-) & (0,-,-,+,+,0) & (+,0,0,+,+,+) \\ (+,0,-,0,+,+) & (+,0,-,-,0,-) & (+,0,-,-,+,0) \\ (+,+,0,0,+,+) & (+,+,0,-,0,+) & (+,+,0,-,-,0) \\ (+,+,+,0,0,+) & (-,+,+,0,-,0) & (-,-,+,+,0,0) \end{array}$$

Observe that the cocircuits correspond to the point partitions produced by hyperplanes spanned by points. Half of the circuits of $\mathcal{M}$ are given in the next table.

The circuits correspond to sign patterns induced by minimal linear dependencies on the rows of the matrix $X$. It is easy to check that every pair consisting of a circuit and a cocircuit fulfills the orthogonality condition.

$$\begin{array}{lll}(+,-,+,-,0,0) & (+,-,+,0,-,0) & (+,-,+,0,0,-)\\ (+,-,0,+,-,0) & (+,+,0,+,0,-) & (+,-,0,0,-,+)\\ (+,0,-,+,-,0) & (+,0,+,+,0,-) & (+,0,+,0,+,-)\\ (+,0,0,+,-,-) & (0,+,-,+,-,0) & (0,+,-,+,0,-)\\ (0,+,+,0,+,-) & (0,+,0,+,+,-) & (0,0,+,-,+,-)\end{array}$$

Figure 6.2.1(right) shows the corresponding arrangement of pseudolines. The circle bounding the configuration represents the projective line at infinity representing line 6.

An affine picture of a realization of the dual oriented matroid is given in Figure 6.2.1(middle). The minus-sign at point 6 indicates that a reorientation at point 6 has taken place. It is easy to check that the circuits and the cocircuits interchange their roles when dualizing the oriented matroid.

## 6.3 IMPORTANT CONCEPTS

In this section we briefly introduce some very basic concepts in the theory of oriented matroids. The list of topics treated here is tailored toward some areas of oriented matroid theory that are particularly relevant for applications. Thus many other topics of great importance are left out. In particular, see [BLS$^+$93, Section 3.3] for minors of oriented matroids, and [BLS$^+$93, Chapter 7] for basic constructions.

### 6.3.1 SOME BASIC CONCEPTS

In the following glossary, we list some fundamental concepts of oriented matroid theory. Each of them can be expressed in terms of any one of the representations of oriented matroids that we have introduced (covectors, cocircuits, chirotopes, pseudoarrangements), but for each of these concepts some representations are much more convenient than others. Also, each of these concepts has some interesting properties with respect to the duality operator—which may be more or less obvious, depending on the representation that one uses.

### GLOSSARY

***Direct sum:*** An oriented matroid $\mathcal{M} = (E, \mathcal{L})$ has a ***direct sum decomposition***, denoted by $\mathcal{M} = \mathcal{M}(E_1) \oplus \mathcal{M}(E_2)$, if $E$ has a partition into nonempty subsets $E_1$ and $E_2$ such that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ for two oriented matroids $\mathcal{M}_1 = (E_1, \mathcal{L}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{L}_2)$. If $\mathcal{M}$ has no direct sum decomposition, then it is ***irreducible***.

***Loops and coloops:*** A loop of $\mathcal{M} = (E, \mathcal{L})$ is an element $e \in E$ that satisfies $C_e = 0$ for all $C \in \mathcal{L}$. A coloop satisfies $\mathcal{L} \cong \mathcal{L}' \times \{-, 0, +\}$, where $\mathcal{L}'$ is obtained by deleting the $e$-components from the vectors in $\mathcal{L}$. If $\mathcal{M}$ has a direct sum decomposition with $E_2 = \{e\}$, then $e$ is either a loop or a coloop.

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***Acyclic oriented matroid:*** Oriented matroid $\mathcal{M} = (E, \mathcal{L})$ for which $(+, \ldots, +)$ is a covector in $\mathcal{L}$; equivalently, the union of the supports of all nonnegative cocircuits is $E$.

***Totally cyclic oriented matroid:*** An oriented matroid without nonnegative cocircuits; equivalently, $\mathcal{L} \cap \{0, +\}^E = \{\mathbf{0}\}$.

***Uniform:*** An oriented matroid $\mathcal{M}$ of rank $d$ on $E$ is ***uniform*** if all of its cocircuits have support of size $|E| - d + 1$, that is, they have exactly $d - 1$ zero entries. Equivalently, $\mathcal{M}$ is uniform if its chirotope has all values in $\{+, -\}$.

**THEOREM 6.3.1** *Duality II*

*Let $\mathcal{M}$ be an oriented matroid on the ground set $E$, and $\mathcal{M}^*$ its dual.*

- *$\mathcal{M}$ is acyclic if and only if $\mathcal{M}^*$ is totally cyclic. (However, "most" oriented matroids are neither acyclic nor totally cyclic!)*
- *$e \in E$ is a loop of $\mathcal{M}$ if and only if it is a coloop of $\mathcal{M}^*$.*
- *$\mathcal{M}$ is uniform if and only if $\mathcal{M}^*$ is uniform.*
- *$\mathcal{M}$ is a direct sum $\mathcal{M}(E) = \mathcal{M}(E_1) \oplus \mathcal{M}(E_2)$ if and only if $\mathcal{M}^*$ is a direct sum $\mathcal{M}^*(E) = \mathcal{M}^*(E_1) \oplus \mathcal{M}^*(E_2)$.*

Duality of oriented matroids captures, among other things, the concepts of linear programming duality [BK92] [BLS$^+$93, Chapter 10] and the concept of Gale diagrams for polytopes [Grü67, Section 5.4] [Zie95, Lecture 6]. For the latter, we note here that the vertex set of a $d$-dimensional convex polytope $P$ with $d+k$ vertices yields a configuration of $d + k$ vectors in $\mathbb{R}^{d+1}$, and thus an oriented matroid of rank $d + 1$ on $d + k$ points. Its dual is a realizable oriented matroid of rank $k - 1$, the ***Gale diagram*** of $P$. It can be modeled by a signed affine point configuration of dimension $k - 2$, called an ***affine Gale diagram*** of $P$. Hence, for "small" $k$, we can represent a (possibly high-dimensional) polytope with "few vertices" by a low-dimensional point configuration. In particular, this is beneficial in the case $k = 4$, where polytopes with "universal" behavior can be analyzed in terms of their 2-dimensional affine Gale diagrams. For further details, see Chapter 15 of this Handbook.

## 6.3.2 REALIZABILITY AND REALIZATION SPACES

### GLOSSARY

***Realization space:*** Let $\chi \colon E^d \to \{-, 0, +\}$ be a chirotope with $\chi(1, \ldots, d) = +$. The *realization space* $\mathcal{R}(\chi)$ is the set of all matrices $X \in \mathbb{R}^{d \cdot n}$ with $\chi_X = \chi$ and $x_i = e_i$ for $i = 1, \ldots, d$, where $e_i$ is the $i$th unit vector. If $\mathcal{M}$ is the corresponding oriented matroid, we write $\mathcal{R}(\mathcal{M}) = \mathcal{R}(\chi)$.

***Rational realization:*** A realization $X \in \mathbb{Q}^{d \cdot n}$; that is, a point in $\mathcal{R}(\chi) \cap \mathbb{Q}^{d \cdot n}$.

***Basic primary semialgebraic set:*** The (real) solution set of an arbitrary finite system of polynomial equations and strict inequalities with integer coefficients.

***Existential Theory of the Reals:*** The problem of solving arbitrary systems of polynomial equations and inequalities with integer coefficients.

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***Stable equivalence:*** A strong type of arithmetic and homotopy equivalence. Two semialgebraic sets are stably equivalent if they can be connected by a sequence of rational coordinate changes, together with certain projections with contractible fibers. (See [RZ95] and [Ric96] for details.) In particular, two stably equivalent semialgebraic sets have the same number of components, they are homotopy equivalent, and either both or neither of them have rational points.

One of the main problems in oriented matroid theory is to design algorithms that find a realization of a given oriented matroid if it exists. However, for oriented matroids with large numbers of points, one cannot be too optimistic, since the realizability problem for oriented matroids is NP-hard. This is one of the consequences of Mnëv's universality theorem below. An upper bound for the worst-case complexity of the realizability problem is given by the following theorem. It follows from general complexity bounds for algorithmic problems about semialgebraic sets by Basu, Pollack, and Roy [BPR96] [BPR03] (see also Chapter 37 of this Handbook).

**THEOREM 6.3.2** *Complexity of the Best General Algorithm Known*

*The realizability of a rank $d$ oriented matroid on $n$ points can be decided by solving a system of $S = \binom{n}{d}$ real polynomial equations and strict inequalities of degree at most $D = d-1$ in $K = (n-d-1)(d-1)$ variables. Thus, with the algorithms of* [BPR96]*, the number of bit operations needed to decide realizability is (in the Turing machine model of complexity) bounded by $(S/K)^K \cdot S \cdot D^{O(K)}$ in a situation where $d$ is fixed and $n$ is large.*

## THE UNIVERSALITY THEOREM

A basic observation is that all oriented matroids of rank 2 are realizable. In particular, up to change of orientations and permuting the elements in $E$ there is only one uniform oriented matroid of rank 2. The realization space of an oriented matroid of rank 2 is always stably equivalent to $\{0\}$; in particular, if $\mathcal{M}$ is uniform of rank 2 on $n$ elements, then $\mathcal{R}(\mathcal{M})$ is isomorphic to an open subset of $\mathbb{R}^{2n-4}$.

In contrast to the rank 2 case, Mnëv's universality theorem states that for oriented matroids of rank 3, the realization space can be "arbitrarily complicated." Here is what one can observe for oriented matroids on few elements:

- The realization spaces of all realizable uniform oriented matroids of rank 3 and at most 9 elements are contractible (Richter [Ric89]).
- There is a realizable rank 3 oriented matroid on 9 elements that has no realization with rational coordinates (Perles [Grü67, p. 93]).
- There is a realizable rank 3 oriented matroid on 13 elements with disconnected realization space (Tsukamoto [Tsu13]).

The universality theorem is a fundamental statement with various implications for the configuration spaces of various types of combinatorial objects.

**THEOREM 6.3.3** *Mnëv's Universality Theorem* [Mnë88]

*For every basic primary semialgebraic set $V$ defined over $\mathbb{Z}$ there is a chirotope $\chi$ of rank 3 such that $V$ and $\mathcal{R}(\chi)$ are stably equivalent.*

Although some of the facts in the following list were proved earlier than Mnëv's universality theorem, they all can be considered as consequences of the construction techniques used by Mnëv.

## CONSEQUENCES OF THE UNIVERSALITY THEOREM

1. The full field of algebraic numbers is needed to realize all oriented matroids of rank 3.
2. The realizability problem for oriented matroids is NP-hard (Mnëv [Mnë88], Shor [Sho91]).
3. The realizability problem for oriented matroids is (polynomial-time-)equivalent to the Existential Theory of the Reals (Mnëv [Mnë88]).
4. For every finite simplicial complex $\Delta$, there is an oriented matroid whose realization space is homotopy equivalent to $\Delta$.
5. Realizability of rank 3 oriented matroids cannot be characterized by excluding a finite set of "forbidden minors" (Bokowski and Sturmfels [BS89b]).
6. In order to realize all combinatorial types of integral rank 3 oriented matroids on $n$ elements, even uniform ones, in the integer grid $\{1, 2, \ldots, f(n)\}^3$, the "coordinate size" function $f(n)$ has to grow doubly exponentially in $n$ (Goodman, Pollack, and Sturmfels [GPS90]).
7. The ***isotopy problem*** for oriented matroids (Can one given realization of $\mathcal{M}$ be continuously deformed, through realizations, to another given one?) has a negative solution in general, even for uniform oriented matroids of rank 3 [JMSW89].

## 6.3.3 TRIANGLES AND SIMPLICIAL CELLS

There is a long tradition of studying *triangles* in arrangements of pseudolines. In his 1926 paper [Lev26], Levi already considered them to be important structures. There are good reasons for this. On the one hand, they form the simplest possible cells of full dimension, and are therefore of basic interest. On the other hand, if the arrangement is simple, triangles locate the regions where a "smallest" local change of the combinatorial type of the arrangement is possible. Such a change can be performed by taking one side of the triangle and "pushing" it over the vertex formed by the other two sides. It was observed by Ringel [Rin56] that any two simple arrangements of pseudolines can be deformed into one another by performing a sequence of such "triangle flips."

Moreover, the realizability of a pseudoline arrangement may depend on the situation at the triangles. For instance, if any one of the triangles in the nonrealizable example of Figure 6.1.2 other than the central one is flipped, we obtain a realizable pseudoline arrangement.

## TRIANGLES IN ARRANGEMENTS OF PSEUDOLINES

Let $\mathcal{P}$ be any arrangement of $n$ pseudolines.

1. For any pseudoline $\ell$ in $\mathcal{P}$ there are at least 3 triangles adjacent to $\ell$.
   Either the $n-1$ pseudolines different from $\ell$ intersect in one point (i.e., $\mathcal{P}$ is a ***near-pencil***), or there are at least $n-3$ triangles that are not adjacent to $\ell$. Thus $\mathcal{P}$ contains at least $n$ triangles (Levi [Lev26]).
2. $\mathcal{P}$ is ***simplicial*** if all its regions are bounded by exactly 3 (pseudo)lines.
   Except for the near-pencils, there are two infinite classes of simplicial line arrangements and 94 additional "sporadic" simplicial line arrangements (and many more simplicial pseudoarrangements) known: See Grünbaum [Grü71] with the recent updates by Grünbaum [Grü09], and Cuntz [Cun12], who also classified the simplicial pseudoarrangements up to $n = 27$ pseudolines; the first nonstretchable example occurs for $n = 15$.
3. If $\mathcal{P}$ is simple, then it contains at most $\frac{n(n-1)}{3}$ triangles.
   For infinitely many values of $n$, there exists a simple arrangement with $\frac{n(n-1)}{3}$ triangles (Harborth and Roudneff, see [Rou96]).
4. Any two simple arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ can be deformed into one another by a sequence of simplicial flips (Ringel [Rin56]).

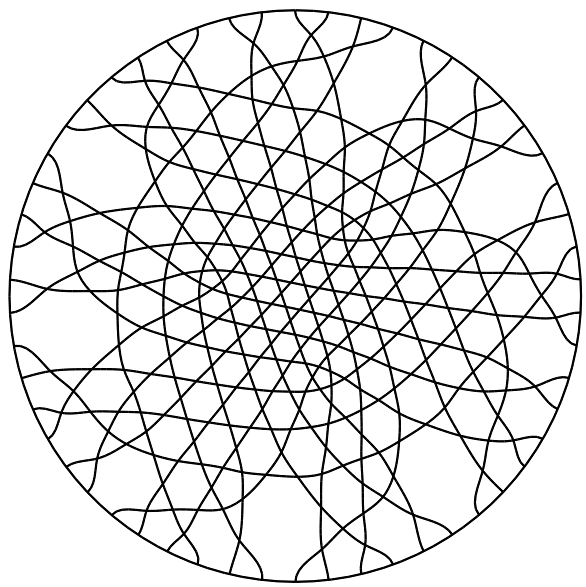

FIGURE 6.3.1
*A simple arrangement of* 28 *pseudolines with a maximal number of* 252 *triangles.*

Every arrangement of pseudospheres in $S^{d-1}$ has a centrally symmetric representation. Thus we can always derive an arrangement of projective pseudohyperplanes (pseudo $(d-2)$-planes in $\mathbb{RP}^{d-1}$) by identifying antipodal points. The proper analogue for the triangles in rank 3 are the $(d-1)$-simplices in projective arrangements of pseudohyperplanes in rank $d$, i.e., the regions bounded by the minimal number, $d$, of pseudohyperplanes. We call an arrangement ***simple*** if no more than $d-1$ planes meet in a point.

It was conjectured by Las Vergnas in 1980 [Las80] that (as in the rank 3 case) any two simple arrangements can be transformed into each other by a sequence of flips of simplicial regions. In particular this requires that every simple arrangement contain *at least one* simplicial region (which was also conjectured by Las Vergnas). If we consider the case of realizable arrangements only, it is not difficult to prove that any two members in this subclass can be connected by a sequence of flips of simplicial regions and that each realizable arrangement contains at least one simplicial cell. In fact, Shannon [Sha79] proved that every arrangement (even the nonsimple ones) of $n$ projective hyperplanes in rank $d$ contains at least $n$ simplicial regions. More precisely, for every hyperplane $h$ there are at least $d$ simplices adjacent to $h$ and at least $n-d$ simplices not adjacent to $h$. The contrast between the Las Vergnas conjecture and the results known for the nonrealizable case is dramatic.

### SIMPLICIAL CELLS IN PSEUDOARRANGEMENTS

1. There is an arrangement of 8 pseudoplanes in rank 4 having only 7 simplicial regions (Altshuler and Bokowski [ABS80], Roudneff and Sturmfels [RS88]).
2. Every rank 4 arrangement with $n < 13$ pseudoplanes has at least one simplicial region (Bokowski and Rohlfs [BR01]).
3. For every $k > 2$ there is a rank 4 arrangement of $4k$ pseudoplanes having only $3k+1$ simplicial regions. (This result of Richter-Gebert [Ric93] was improved by Bokowski and Rohlfs [BR01] to arrangements of $7k+c$ pseudoplanes with $5k+c$ simplicial regions, for $k \geq 0$ and $c \geq 4$.)
4. There is a rank 4 arrangement consisting of 20 pseudoplanes for which one plane is not adjacent to any simplicial region (Richter-Gebert [Ric93]; improved to 17 pseudoplanes by Bokowski and Rohlfs [BR01]).

### OPEN PROBLEMS

The topic of simplicial cells is interesting and rich in structure even in rank 3. The case of higher dimensions is full of unsolved problems and challenging conjectures. These problems are relevant for various problems of great geometric and topological interest, such as the structure of spaces of triangulations. Three key problems are:

1. Classify simplical arrangements. Is it true, at least, that there are only finitely many types of simplicial arrangements of straight lines outside the three known infinite families?
2. Does every arrangement of pseudohyperplanes contain at least one simplicial region?
3. Is it true that any two simple arrangements (of the same rank and the same number of pseudohyperplanes) can be transformed into one another by a sequence of flips at simplex regions?

## 6.4 APPLICATIONS IN POLYTOPE THEORY

Oriented matroid theory has become a very important structural tool for the theory of polytopes, but also for the algorithmic treatment and classification of polytopes. In this section we give a brief overview of fundamental concepts and of results.

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## MATROID POLYTOPES

The convexity properties of a point configuration $X$ are modeled superbly by the oriented matroid $\mathcal{M}_X$. The combinatorial versions of many theorems concerning convexity also hold on the level of general (including nonrealizable) oriented matroids. For instance, there are purely combinatorial versions of Carathéodory's, Radon's, and Helly's theorems [BLS$^+$93, Section 9.2].

In particular, oriented matroid theory provides an entirely combinatorial model of convex polytopes, commonly known as "matroid polytopes," although "oriented matroid polytope" might be more appropriate. (Warning: In Combinatorial Optimization, an entirely different object, the 0/1-polytope spanned by the incidence vectors of the bases of an (unoriented) matroid is also called a "matroid polytope.")

**Definition:** The ***face lattice*** (sometimes called the ***Las Vergnas face lattice***) of an acyclic oriented matroid $\mathcal{M} = (E, \mathcal{L})$ is the set

$$\mathrm{FL}(\mathcal{M}) := \{C^0 \mid C \in \mathcal{L} \cap \{0, +\}^E\},$$

with the partial order of sign vectors induced from $(\mathcal{L}, \leq)$, which coincides with the partial ordered by inclusion of supports. The elements of $\mathrm{FL}(\mathcal{M})$ are the ***faces*** of $\mathcal{M}$. The acyclic oriented matroid $\mathcal{M} = (E, \mathcal{L})$ is a ***matroid polytope*** if $\{e\}$ is a face for every $e \in E$.

Every polytope gives rise to a matroid polytope: If $P \subset \mathbb{R}^d$ is a $d$-polytope with $n$ vertices, then the canonical embedding $x \mapsto \binom{x}{1}$ creates a vector configuration $X_P$ of rank $d+1$ from the vertex set of $P$. The oriented matroid of $X_P$ is a matroid polytope $\mathcal{M}_P$, whose face lattice $\mathrm{FL}(\mathcal{M})$ is canonically isomorphic to the face lattice of $P$.

Matroid polytopes provide a very precise model of (the combinatorial structure of) convex polytopes. In particular, the topological representation theorem implies that *every* matroid polytope of rank $d$ is the face lattice of a regular piecewise linear (PL) cell decomposition of a $(d-2)$-sphere. Thus matroid polytopes form an excellent combinatorial model for convex polytopes: In fact, much better than the model of PL spheres (which does not have an entirely combinatorial definition).

However, the construction of a polar fails in general for matroid polytopes. The cellular spheres that represent matroid polytopes have dual cell decompositions (because they are piecewise linear), but this dual cell decomposition is not in general a matroid polytope, even in rank 4 (Billera and Munson [BM84]; Bokowski and Schuchert [BS95]). In other words, the order dual of the face lattice of a matroid polytope (as an abstract lattice) is *not in general* the face lattice of a matroid polytope. (Matroid polytopes form an important tool for polytope theory, not only because of the parts of polytope theory that work for them, but also because of those that fail.)

For every matroid polytope one has the dual oriented matroid (which is totally cyclic, hence not a matroid polytope). In particular, the setup for Gale diagrams generalizes to the framework of matroid polytopes; this makes it possible to also include nonpolytopal spheres in a discussion of the realizability properties of polytopes. This amounts to perhaps the most powerful single tool ever developed for polytope theory. It leads to, among other things, the classification of $d$-dimensional polytopes with at most $d+3$ vertices, the proof that all matroid polytopes of rank $d+1$ with at most $d+3$ vertices are realizable, and the construction of nonrational polytopes as well as of nonpolytopal spheres with $d+4$ vertices.

## ALGORITHMIC APPROACHES TO POLYTOPE CLASSIFICATION

For a long time there has been substantial work in classifying the combinatorial types of $d$-dimensional polytopes with a "small" number $n$ of vertices, for $d \geq 4$.

For $d = 3$ the enumeration problem is reduced to the enumeration of 3-connected planar graphs, by Steinitz's Theorem 15.1.3. For $n \leq d + 3$, it may be solved by the "Gale diagram" technique introduced by Perles in the 1960s, which in retrospect may be seen as an incarnation of oriented matroid duality; see Section 15.1.7. For $d = 4$ one can try to do the enumeration via Schlegel diagrams; thus Brückner (1910) attempted to classify the simplicial 4-polytopes with 8 vertices. His work was corrected and completed in the 1960s, see [Grü67].

For the following discussion we restrict our attention to the simplicial case—there are additional technical problems to deal with in the nonsimplicial case, and very little work has been done there as yet (see e.g., Brinkmann & Ziegler [BZ17]).

In the simplicial case, at the core of the enumeration problem lies the following hierarchy:

$$\begin{pmatrix}\text{simplicial}\\ \text{spheres}\end{pmatrix} \supset \begin{pmatrix}\text{uniform}\\ \text{matroid polytopes}\end{pmatrix} \supset \begin{pmatrix}\text{simplicial}\\ \text{convex polytopes}\end{pmatrix}.$$

The classical plan of attack from the 1970s and 1980s, when oriented matroid technology in the current form was not yet available, was to first enumerate all isomorphism types of simplicial spheres with given parameters. Then for each sphere one would try to decide realizability. This has been successfully completed for the classification of all simplicial 3-spheres with 9 vertices (Altshuler, Bokowski, and Steinberg [ABS80]) and of all neighborly 5-spheres with 10 vertices (Bokowski and Shemer [BS87]) into polytopes and nonpolytopes.

In an alternative approach to enumerate all polytopes in a class, largely due to Bokowski and Sturmfels [BS89a], one tries to bypass the first step of enumeration of spheres and enumerates directly all possible oriented matroids/matroid polytopes, and then tries to decide realizability for each single type. Thus one has to effectively deal with three problems:

**(1)** enumeration of oriented matroids/matroid polytopes,
**(2)** proving nonrealizability, or
**(3)** proving realizability.

**(1)** For the *enumeration problem*, Finschi & Fukuda [FF02] developed an effective approach to generate oriented matroids (including the nonuniform ones!) through single element extensions: Take an oriented matroid and try to add an element. Their algorithms relied on cocircuit graphs. It was later observed that the set of possible single element extensions can be viewed as the set of solutions of a SAT problem. Moreover, SAT-solvers are readily available "off the shelf" and thus much easier to employ than the special purpose software that had previously been developed for such purposes. For example, Miyata & Padrol [MP15] use this as a key step in the enumeration of neighborly oriented matroids. (The use of SAT-solvers in oriented matroid theory had been pioneered by Schewe [Sch10], who first used them for proving that a geometric structure, such as a polytope or a sphere, does not admit a compatible oriented matroid.)

**(2)** For *proving nonrealizability* of oriented matroids, Biquadratic Final Polynomials (BFPs), introduced by Bokowski and Richter-Gebert [BR90], usually outperform all the other methods. These can be found effectively by linear programming.

(This was up to now used mostly for the uniform case, but may work even more effectively in nonuniform situations; see Brinkmann and Ziegler [BZ17].) Moreover, nonrealizable oriented matroids that do not have a BFP seem to be rare for the parameters of the enumeration problems within reach. (It had been observed by Dress and Sturmfels that every nonrealizable oriented matroid has a final polynomial proof for this fact, but the final polynomials typically are huge and cannot be found efficiently.) Moreover, recent works by Firsching and others have exploited that biquadratic final polynomials typically use only a partial oriented matroid, so BFP proofs can be based on incomplete enumeration trees.

**(3)** For *proving realizability* of oriented matroids, exact methods—which might solve general semialgebraic problems—are not available or inefficient. Randomized methods can be employed to find realizations, but it seems hard to employ them for nonuniform oriented matroids; see Fukuda, Miyata and Moriyama [FMM13].

Firsching [Fir17] demonstrated recently that current nonlinear optimization software can be used very efficiently to find realizations. Those then (as one is using *numerical* software with rounding errors) have to be checked in exact arithmetic. This turned out to work very well for simplicial polytopes, but also in nonuniform situations, say for inscribed realizations with all vertices on a sphere. If the program does not find a solution or does not terminate (which happens more often), this does not yield a proof, but may be interpreted as suggesting that the oriented matroid might be not realizable.

Another important tool in classification efforts is the use of constructions that preserve realizability, such as stackings, or lexicographic extensions. The latter is in particular useful in the context of neighborly polytopes, see Padrol [Pad13].

Recently there has been a number of complete classification results based on the methods we have mentioned. This includes the classifications of

- simplicial 4-polytopes with 10 vertices [Fir17],
- simplicial 5-polytopes with 9 vertices [FMM13], and
- neighborly 8-polytopes with 12 vertices [MP15].

A comprehensive overview table can be found in [Fir15, p. 17].

Finally, let us note that in the passages from spheres to oriented matroids and to polytopes there are considerable subtleties involved that lead to important structural insights. For a given simplicial sphere, the following may apply:

- There may be *no* matroid polytope that supports it. In this case the sphere is called ***nonmatroidal***. The Barnette sphere [BLS$^+$93, Proposition 9.5.3] is an example.
- There may be *exactly one* matroid polytope. In this (important) case the sphere is called ***rigid***. That is, a matroid polytope $\mathcal{M}$ is rigid if $\mathrm{FL}(\mathcal{M}') = \mathrm{FL}(\mathcal{M})$ already implies $\mathcal{M}' = \mathcal{M}$. For rigid matroid polytopes the face lattice uniquely defines the oriented matroid, and thus every statement about the matroid polytope yields a statement about the sphere. In particular, the matroid polytope and the sphere have the same realization space.

  Rigid matroid polytopes are a priori rare; however, the *Lawrence construction* [BLS$^+$93, Section 9.3] [Zie95, Section 6.6] associates with every oriented matroid $\mathcal{M}$ on $n$ elements in rank $d$ a rigid matroid polytope $\Lambda(\mathcal{M})$ with $2n$ vertices of rank $n + d$. The realizations of $\Lambda(\mathcal{M})$ can be retranslated into realizations of $\mathcal{M}$.

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Furthermore, even-dimensional neighborly polytopes are rigid (Shemer [She82], Sturmfels [Stu88]), which is a key property for all approaches to the classification of neighborly polytopes with few vertices, but also for proving the universality for simplicial polytopes (see Theorem 6.3.4 below).

- There may be *many* matroid polytopes.

The situation is similarly complex for the second step, from matroid polytopes to convex polytopes. In fact, for each matroid polytope the following may apply:

- There may be *no* convex polytope—this is the case for a nonrealizable matroid polytope. These exist already with relatively few vertices; namely in rank 5 with 9 vertices [BS95], and in rank 4 with 10 vertices [BLS⁺93, Proposition 9.4.5].
- There may be essentially *only one*—this is the rare case where the matroid polytope is "projectively unique" (cf. Adiprasito and Ziegler [AZ15]).
- There may be *many* convex polytopes—the space of all polytopes for a given matroid polytope is the realization space of the oriented matroid, and this may be "arbitrarily complicated." This is made precise by Mnëv's universality theorem [Mnë88]. Note that for simplicial polytopes, this has been claimed since the 1980s, but has been proven only recently by Adiprasito and Padrol [AP17]. (One can prove universality for the realization spaces of uniform oriented matroids of rank 3 from the nonuniform case by a scattering technique [BS89a, Thm. 6.2]; for polytopes, this technique does not suffice, as Lawrence extensions (see Chapter 15) destroy simpliciality.)

**THEOREM 6.4.1** *The Universality Theorem for Polytopes* [Mnë88] [AP17]

*For every [open] basic primary semialgebraic set $V$ defined over $\mathbb{Z}$ there is an integer $d$ and a [simplicial] $d$-dimensional polytope $P$ on $d+4$ vertices such that $V$ and the realization space of $P$ are stably equivalent.*

## 6.5 SOURCES AND RELATED MATERIAL

### FURTHER READING

The basic theory of oriented matroids was introduced in two fundamental papers, Bland and Las Vergnas [BL78] and Folkman and Lawrence [FL78]. We refer to the monograph by Björner, Las Vergnas, Sturmfels, White, and Ziegler [BLS⁺93] for a broad introduction, and for an extensive development of the theory of oriented matroids. An extensive bibliography is given in Ziegler [Zie96+]. Other introductions and basic sources of information include Bachem and Kern [BK92], Bokowski [Bok93] and [Bok06], Bokowski and Sturmfels [BS89a], and Ziegler [Zie95, Lectures 6 and 7].

### RELATED CHAPTERS

Chapter 5: Pseudoline arrangements
Chapter 15: Basic properties of convex polytopes

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Chapter 16: Subdivisions and triangulations of polytopes
Chapter 28: Arrangements
Chapter 37: Computational and quantitative real algebraic geometry
Chapter 49: Linear programming
Chapter 60: Geometric applications of the Grassmann–Cayley algebra

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