55 GRAPH DRAWING

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INTRODUCTION

Graph drawing addresses the problem of constructing geometric representations of graphs, and has important applications to key computer technologies such as software engineering, database systems, visual interfaces, and computer-aided design. Research on graph drawing has been conducted within several diverse areas, including discrete mathematics (topological graph theory, geometric graph theory, order theory), algorithmics (graph algorithms, data structures, computational geometry, VLSI), and human-computer interaction (visual languages, graphical user interfaces, information visualization). This chapter overviews aspects of graph drawing that are especially relevant to computational geometry. Basic definitions on drawings and their properties are given in Section 55.1. Bounds on geometric and topological properties of drawings (e.g., area and crossings) are presented in Section 55.2. Section 55.3 deals with the time complexity of fundamental graph drawing problems. An example of a drawing algorithm is given in Section 55.4. Techniques for drawing general graphs are surveyed in Section 55.5.

55.1 DRAWINGS AND THEIR PROPERTIES

TYPES OF GRAPHS

First, we define some terminology on graphs pertinent to graph drawing. Throughout this chapter let n and m be the number of graph vertices and edges respectively, and d the maximum vertex degree (i.e., number of edges incident to a vertex).

GLOSSARY

Degree-k graph: Graph with maximum degree $d \leq k$.

Digraph: Directed graph, i.e., graph with directed edges.

Acyclic digraph: Digraph without directed cycles.

Transitive edge: Edge (u, v) of a digraph is transitive if there is a directed path from u to v not containing edge (u, v).

Reduced digraph: Digraphs without transitive edges.

Source: Vertex of a digraph without incoming edges.

Sink: Vertex of a digraph without outgoing edges.

st-digraph: Acyclic digraph with exactly one source and one sink, which are joined by an edge (also called ***bipolar digraph***).

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Connected graph: Graph in which any two vertices are joined by a path.

Biconnected graph: Graph in which any two vertices are joined by two vertex-disjoint paths.

Triconnected graph: Graph in which any two vertices are joined by three (pairwise) vertex-disjoint paths.

Layered (di)graph: (Di)graph whose vertices are partitioned into sets, called layers, such that no two vertices in the same layer are adjacent.

k-layered (di)graph: Layered (di)graph with k layers.

Tree: Connected graph without cycles.

Directed Tree: Digraph whose underlying undirected graph is a tree.

Rooted tree: Directed tree with a distinguished vertex, the ***root***, such that each vertex lies on a directed path to the root. A rooted tree is also viewed as a layered digraph where the layers are sets of vertices at the same distance from the root.

Binary tree: Rooted tree where each vertex has at most two incoming edges.

Ternary tree: Rooted tree where each vertex has at most three incoming edges.

Series-parallel digraph (SP digraph): A digraph with a single source s and a single sink t recursively defined as follows: (i) a single edge (s, t) is a series-parallel digraph. Given two series-parallel digraphs G' and G'' with sources s' and s'', respectively and sinks t' and t'', respectively, (ii) the digraph obtained by identifying t' with s'' is a series-parallel digraph; (iii) the digraph obtained by identifying s' with s'' and t' with t'' is a series-parallel digraph. The series-parallel digraphs defined above are often called ***two-terminal series parallel digraphs***. Throughout this section series-parallel digraphs have no multiple edges.

Series-parallel graph (SP graph): The underlying undirected graph of a series-parallel digraph.

Bipartite (di)graph: (Di)graph whose vertices are partitioned into two sets and each edge connects vertices in different sets. A bipartite (di)graph is also viewed as a 2-layered (di)graph.

TYPES OF DRAWINGS

In a drawing of a graph one has to geometrically represent the vertices and their adjacencies (edges). This can be done in several different ways. In the most common types of drawing, vertices are represented by points (or by geometric figures such as circles or rectangles) and edges are represented by curves such that any two edges intersect at most in a finite number of points. In other types of drawings vertices can be represented by various geometric objects (segments, curves, polygons) while adjacencies can be represented by intersections, contacts, or visibility of the objects representing the vertices.

GLOSSARY

Polyline drawing: Each edge is a polygonal chain (Figure 55.1.1(a)).

Straight-line drawing: Each edge is a straight-line segment (Figure 55.1.1(b)).

Orthogonal drawing: Each edge is a chain of horizontal and vertical segments (Figure 55.1.1(c)).

Bend: In a polyline drawing, point where two segments belonging to the same edge meet (Figure 55.1.1(a)).

Orthogonal Representation: Description of an orthogonal drawing in terms of bends along each edge and angles around each vertex with no information about the length of the segments that connect vertices and bends.

Crossing: Intersection point of two edges that is not a common vertex nor a touching (tangential) point (Figure 55.1.1(b)).

Grid drawing: Polyline drawing such that vertices and bends have integer coordinates.

Planar drawing: Drawing where no two edges cross (see Figure 55.1.1(d)).

Planar (di)graph: (Di)graph that admits a planar drawing.

Face: A connected region of the plane defined by a planar drawing, where the unbounded region is called the ***external face***.

Embedded (di)graph: Planar (di)graph with a prespecified topological embedding (i.e., set of faces), which must be preserved in the drawing.

Outerplanar (di)graph: A planar (di)graph that admits a planar drawing with all vertices on the boundary of the external face.

Convex drawing: Planar straight-line drawing such that the boundary of each face is a convex polygon.

Upward drawing: Drawing of a digraph where each edge is monotonically non-decreasing in the vertical direction (see Figure 55.1.1(d)).

Upward planar digraph: Digraph that admits an upward planar drawing.

Layered drawing: Drawing of a layered graph such that vertices in the same layer lie on the same horizontal line (also called ***hierarchical drawing***).

Dominance drawing: Upward drawing of an acyclic digraph such that there exists a directed path from vertex u to vertex v if and only if $x(u) \leq x(v)$ and $y(u) \leq y(v)$, where $x(\cdot)$ and $y(\cdot)$ denote the coordinates of a vertex.

hv-drawing: Upward orthogonal straight-line drawing of a binary tree such that the drawings of the subtrees of each node are separated by a horizontal or vertical line.

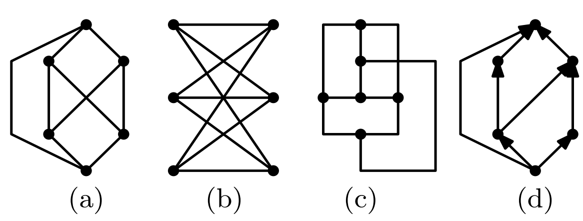

FIGURE 55.1.1
Types of drawings: (a) *polyline drawing of* $K_{3,3}$*;* (b) *straight-line drawing of* $K_{3,3}$*;* (c) *orthogonal drawing of* $K_{3,3}$*;* (d) *planar upward drawing of an acyclic digraph.*

Straight-line and orthogonal drawings are special cases of polyline drawings. Polyline drawings provide great flexibility since they can approximate drawings with curved edges. However, edges with more than two or three bends may be difficult to "follow" for the eye. Also, a system that supports editing of polyline

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drawings is more complicated than one limited to straight-line drawings. Hence, depending on the application, polyline or straight-line drawings may be preferred. If vertices are represented by points, orthogonal drawings are possible only for graphs of maximum vertex degree 4.

PROPERTIES OF DRAWINGS

GLOSSARY

Area: Area of the smallest axis-aligned rectangle (***bounding box***) containing the drawing. This definition assumes that the drawing is constrained by some resolution rule that prevents it from being reduced by an arbitrary scaling (e.g., requiring a grid drawing, or stipulating a minimum unit distance between any two vertices).

Total edge length: Sum of the lengths of the edges.

Maximum edge length: Maximum length of an edge.

Curve complexity: Maximum number of bends along an edge of a polyline drawing.

Angular resolution: Smallest angle formed by two edges, or segments of edges, incident on the same vertex or bend, in a polyline drawing.

Perfect angular resolution: A drawing has perfect angular resolution if for every vertex v the angle formed by any two consecutive edges around v is $\frac{2\pi}{d(v)}$, where $d(v)$ is the degree of v.

Aspect ratio: Ratio of the longest to the shortest side of the smallest rectangle with horizontal and vertical sides covering the drawing.

There are infinitely many drawings for a graph. In drawing a graph, we would like to take into account a variety of properties. For example, planarity and the display of symmetries are highly desirable in visualization applications. Also, it is customary to display trees and acyclic digraphs with upward drawings. In general, to avoid wasting valuable space on a page or a computer screen, it is important to keep the area of the drawing small. In this scenario, many graph drawing problems can be formalized as multi-objective optimization problems (e.g., construct a drawing with minimum area and minimum number of crossings), so that tradeoffs are inherent in solving them. Typically, it is desirable to maximize the angular resolution and to minimize the other measures.

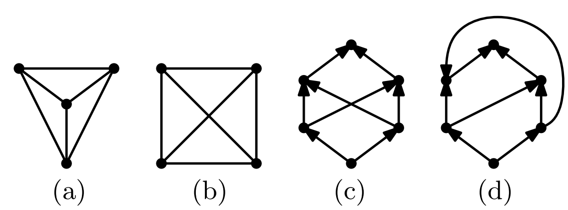

FIGURE 55.1.2
(a–b) *Tradeoff between planarity and symmetry in drawing K_4.* (c–d) *Tradeoff between planarity and upwardness in drawing an acyclic digraph G.*

The following examples illustrate two typical tradeoffs in graph drawing problems. Figure 55.1.2(a–b) shows two drawings of K_4, the complete graph on four vertices. The drawing of part (a) is planar, while the drawing of part (b) "maximizes symmetries." It can be shown that no drawing of K_4 is optimal with respect to both criteria, i.e., the maximum number of symmetries cannot be achieved by a planar drawing. Figure 55.1.2(c–d), shows two drawing of the same acyclic digraph G. The drawing of part (c) is upward, while the drawing of part (d) is planar. It can be shown that there is no drawing of G which is both planar and upward.

55.2 BOUNDS ON DRAWING PROPERTIES

For various classes of graphs and drawing types, many universal/existential upper and lower bounds for specific drawing properties have been discovered. Such bounds typically exhibit tradeoffs between drawing properties. A universal bound applies to all the graphs of a given class. An existential bound applies to infinitely many graphs of the class. In the following tables, the abbreviations PSL, PSLO, PO, and PPL are used for planar straight-line, planar straight-line orthogonal, planar orthogonal, and planar polyline, respectively.

BOUNDS ON THE AREA

Tables 55.2.1 and 55.2.2 summarize selected universal upper bounds and existential lower bounds on the area of drawings of trees and graphs, respectively. The following comments apply to Tables 55.2.1 and 55.2.2, where specific rows of the table are indicated within parentheses. Linear or almost-linear bounds on the area can be achieved for several families of trees (1–7, 10–14, and 17–27); typically, superlinear lower bounds are associated with order preserving drawings (6, 9, 12, 14, 16, 19, 26). For directed trees, if a given embedding must be preserved, exponential area is required (28). See Table 55.2.5 for tradeoffs between area and aspect ratio in drawings of trees. Planar graphs admit planar drawings with quadratic area both in the straight-line, polyline and orthogonal model (10–12). For planar straight-line drawings, outerplanar graphs are the only class of graphs for which a sublinear upper bound is known (4). For polyline drawings this is true also for series-parallel graphs (5 and 7). Series-parallel graphs are the only subclass of planar graphs for which a superlinear lower bound is known both for straight-line and polyline drawings (6–7). If drawings are not required to be planar, linear area can be achieved for planar graphs (13). If, however, the drawing is required to be orthogonal, then superlinear lower bounds exists both for planar and non-planar graphs (2 and 3). In this case, almost linear area can be achieved for planar graphs (2), while linear area is possible for outerplanar graphs (1). Studies about the nature of the crossings in compact straight-line drawings of planar graphs are presented in [DDLM12, DDLM13]. Upward planar drawings provide an interesting tradeoff between area and having straight-line edges or not (15–24). Indeed, if a straight-line drawing is required, the area can become exponential even for subclasses of upward planar digraphs like outerplanar or bipartite DAGs (15, 18, 20, 22). In these cases a quadratic area bound is achieved only at the expense of a linear number of bends (16, 19, 21, 24). Other cases for which a polynomial bound is known are series-parallel graphs, when one is allowed to change the embedding (17), and upward dominance drawings of reduced planar st-graphs (23).

TABLE 55.2.1 Universal upper and existential lower bounds on the area of drawings of trees.

	CLASS OF GRAPHS	DRAWING TYPE	AREA	
1	Fibonacci tree	strict upw PSL	$\Omega(n)$ trivial	$O(n)$ [Tre96]
2	AVL tree	strict upw PSL	$\Omega(n)$ trivial	$O(n)$ [CPP98]
3	Balanced binary tree	strict upw PSL	$\Omega(n)$ trivial	$O(n)$ [CP98]
4	Binary tree	PSL	$\Omega(n)$ trivial	$O(n)$ [GR04]
5	Binary tree	ord pres PSL	$\Omega(n)$ trivial	$O(n \log \log n)$ [GR03a]
6	Binary tree	strict upw ord pres PSL	$\Omega(n \log n)$ [CDP92]	$O(n \log n)$ [GR03a]
7	Binary tree	PSLO	$\Omega(n)$ trivial	$O(n \log \log n)$ [SKC00]
8	Binary tree	ord pres PSLO	$\Omega(n)$ trivial	$O(n^{1.5})$ [Fra08b]
9	Binary tree	upw ord pres PSLO	$\Omega(n^2)$ [Fra08b]	$O(n^2)$ [Fra08b]
10	Binary tree	ord pres PO	$\Omega(n)$ trivial	$O(n)$ [DT81]
11	Binary tree	upw PO	$\Omega(n \log \log n)$ [GGT96]	$O(n \log \log n)$ [GGT96, Kim95, SKC00]
12	Binary tree	upw ord pres PO	$\Omega(n \log n)$ [GGT96]	$O(n \log n)$ [Kim95]
13	Binary tree	upw PPL	$\Omega(n)$ trivial	$O(n)$ [GGT96]
14	Binary tree	upw ord pres PPL	$\Omega(n \log n)$ [GGT96]	$O(n \log n)$ [GGT96]
15	Ternary tree	PSLO	$\Omega(n)$ trivial	$O(n^{1.631})$ [Fra08b]
16	Ternary tree	ord pres PSLO	$\Omega(n^2)$ [Fra08b]	$O(n^2)$ [Fra08b]
17	Ternary tree	PO	$\Omega(n)$ trivial	$O(n)$ [Val81]
18	Ternary tree	ord pres PO	$\Omega(n)$ trivial	$O(n)$ [DT81]
19	Ternary tree	upw ord pres PO	$\Omega(n \log n)$ [GGT96]	$O(n \log n)$ [Kim95]
20	deg-$O(1)$ rooted tree	upw PSL	$\Omega(n)$ trivial	$O(n \log \log n)$ [SKC00]
21	deg-$O(n^{\frac{a}{2}})$ rooted tree	PSL	$\Omega(n)$ trivial	$O(n)$ [GR03b]
22	deg-$O(n^a)$ rooted tree	upw PPL	$\Omega(n)$ trivial	$O(n)$ [GGT96]
23	Rooted tree	ord pres PSL	$\Omega(n)$ trivial	$O(n \log n)$ [GR03a]
24	Rooted tree	upw PSL	$\Omega(n)$ trivial	$O(n \log n)$ [CDP92]
25	Rooted tree	strict upw PSL	$\Omega(n \log n)$ [CDP92]	$O(n \log n)$ [CDP92]
26	Rooted tree	strict upw ord pres PSL	$\Omega(n \log n)$ [CDP92]	$O(n 4^{\sqrt{2 \log_2 n}})$ [Cha02]
27	Directed trees	upw PSL	$\Omega(n \log n)$ [CDP92]	$O(n \log n)$ [Fra08a]
28	Directed trees	upw ord pres PSL	$\Omega(b^n)$ [Fra08a]	$O(c^n)$ [GT93]

Note: n is the number of vertices, a, b, and c are constants such that $0 \leq a < 1$, $1 < b < c$.
All bounds assume grid drawings.

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TABLE 55.2.2 Universal upper and existential lower bounds on the area of drawings of graphs.

	CLASS OF GRAPHS	DRAWING TYPE	AREA	
1	Outerpl deg-4 graph	orthogonal	$\Omega(n)$ trivial	$O(n)$ [Lei80]
2	Planar deg-4 graph	orthogonal	$\Omega(n \log n)$ [Lei84]	$O(n \log^2 n)$ [Lei80, Val81]
3	Deg-4 graph	orthogonal	$\Omega(n^2)$ [Val81]	$O(n^2)$ [BK98a, BS15, PT98, Sch95, Val81]
4	Outerplanar graph	PSL	$\Omega(n)$ trivial	$O(n^{1.48})$ [DF09]
5	Outerplanar graph	PPL	$\Omega(n)$ trivial	$O(n \log n)$ [Bie11]
6	SP graph	PSL	$\Omega(n2^{\sqrt{\log_2 n}})$ [Fra10]	$O(n^2)$ [FPP90, Sch90]
7	SP graph	PPL	$\Omega(n2^{\sqrt{\log_2 n}})$ [Fra10]	$O(n^{1.5})$ [Bie11]
8	Triconn pl graph	convex PSL	$\Omega(n^2)$ [FPP90, FP08, MNRA11, Val81]	$O(n^2)$ [BFM07, CK97, DTV99, ST92]
9	Triconn pl graph	strict convex PSL	$\Omega(n^3)$ [And63, BP92, BT04, Rab93]	$O(n^4)$ [BR06]
10	Planar graph	PSL	$\Omega(n^2)$ [FPP90]	$O(n^2)$ [FPP90, Sch90]
11	Planar graph	PPL	$\Omega(n^2)$ [FPP90]	$O(n^2)$ [DT88, DTT92, Kan96]
12	Planar graph	PO	$\Omega(n^2)$ [FPP90]	$O(n^2)$ [BK98a, Kan96, Tam87, TT89]
13	Planar graph	straight-line	$\Omega(n)$ trivial	$O(n)$ [Woo05]
14	General graph	polyline	$\Omega(n)$ trivial	$O((n+\chi)^2)$ [BK98a, Kan96, Tam87, TT89]
15	Outerplanar DAG	upw PSL	$\Omega(b^n)$ [Fra08a]	$O(c^n)$ [GT93]
16	Outerplanar DAG	upw PPL	$\Omega(n^2)$ [Fra08a]	$O(n^2)$ [DT88, DTT92]
17	SP digraph	upw PSL	$\Omega(n^2)$ trivial	$O(n^2)$ [BCD+94]
18	Embed SP digraph	upw PSL	$\Omega(b^n)$ [BCD+94]	$O(c^n)$ [GT93]
19	Embed SP digraph	upw PPL	$\Omega(n^2)$ trivial	$O(n^2)$ [DT88, DTT92]
20	Bipartite DAG	upw PSL	$\Omega(b^n)$ [Fra08a]	$O(c^n)$ [GT93]
21	Bipartite DAG	upw PPL	$\Omega(n^2)$ [Fra08a]	$O(n^2)$ [DT88, DTT92]
22	Upward pl digraph	upw PSL	$\Omega(b^n)$ [DTT92]	$O(c^n)$ [GT93]
23	Reduced pl *st*-digraph	upw PSL dominance	$\Omega(n^2)$ [FPP90]	$O(n^2)$ [DTT92]
24	Upward pl digraph	upw PPL	$\Omega(n^2)$ [FPP90]	$O(n^2)$ [DT88, DTT92]

Note: n is the number of vertices, χ is the number of crossings in the drawing, a, b, and c are constants such that $0 \le a < 1$, $1 < b < c$. All bounds assume grid drawings.

BOUNDS ON THE ANGULAR RESOLUTION

Table 55.2.3 summarizes selected universal lower bounds and existential upper bounds on the angular resolution of drawings of graphs. The bounds of the first row are stated for $n \geq 5$ because any planar straight-line drawing of K_4 has angular resolution lower than $\frac{\pi}{4}$.

TABLE 55.2.3 Universal lower and existential upper bounds on angular resolution.

CLASS OF GRAPHS	DRAWING TYPE	ANGULAR RESOLUTION	
deg-3 plan graph †	PSL	$\geq \frac{\pi}{4}$ [DLM14]	$\leq \frac{\pi}{4}$ [DLM14]
SP graph	PSL	$\Omega\left(\frac{1}{d}\right)$ [LLMN13]	$O\left(\frac{1}{d}\right)$ trivial
General graph	straight-line	$\Omega\left(\frac{1}{d^2}\right)$ [FHH$^+$93]	$O\left(\frac{\log d}{d^2}\right)$ [FHH$^+$93]
Planar graph	straight-line	$\Omega\left(\frac{1}{d}\right)$ [FHH$^+$93]	$O\left(\frac{1}{d}\right)$ trivial
Planar graph	PSL	$\Omega\left(\frac{1}{c^d}\right)$ [MP94]	$O\left(\sqrt{\frac{\log d}{d^3}}\right)$ [GT94]
Planar graph	PSL	$\Omega\left(\frac{1}{n^2}\right)$ [FPP90, Sch90]	$O\left(\frac{1}{n}\right)$ trivial
Planar graph	PPL	$\Omega\left(\frac{1}{d}\right)$ [Kan96]	$O\left(\frac{1}{d}\right)$ trivial

Note: n is the number of vertices, d is the maximum vertex degree c is a constant such that $c > 1$.
† $n \geq 5$;

BOUNDS ON THE NUMBER OF BENDS

Table 55.2.4 summarizes selected universal upper bounds and existential lower bounds on the total number of bends and on the curve complexity of orthogonal drawings. Some bounds are stated for $n \geq 5$ or $n \geq 7$ because the maximum number of bends is at least two for K_4 and at least three for the skeleton graph of an octahedron, in any planar orthogonal drawing.

TABLE 55.2.4 Orthogonal drawings: universal upper and existential lower bounds on the number of bends.

CLASS OF GRAPHS	DRAWING TYPE	TOTAL NUM. OF BENDS		CC	REF
deg-4 †	orthog	$\geq n$	$\leq 2n+2$	2	[BK98a]
Planar deg-4 †	orthog planar	$\geq 2n-2$	$\leq 2n+2$	2	[BK98a, TTV91]
Embed deg-4	orthog planar	$\geq 2n-2$	$\leq \frac{12}{5}n+2$	3	[EG95, LMS91, TT89, TTV91]
Biconn embed deg-4	orthog planar	$\geq 2n-2$	$\leq 2n+2$	3	[EG95, LMS91, TT89, TTV91]
Triconn embed deg-4	orthog planar	$\geq \frac{4}{3}(n-1)+2$	$\leq \frac{3}{2}n+4$	2	[Kan96]
Embed deg-3 ‡	orthog planar	$\geq \frac{1}{2}n+1$	$\leq \frac{1}{2}n+1$	1	[Kan96, LMPS92]

Note: CC stands for curve complexity, while n is the number of vertices. † $n \geq 7$; ‡ $n \geq 5$.

TRADEOFF BETWEEN AREA AND ASPECT RATIO

The ability to construct area-efficient drawings is essential in practical visualization applications, where screen space is at a premium. However, achieving small area is not enough, e.g., a drawing with high aspect ratio may not be conveniently placed on a workstation screen, even if it has modest area. Hence, it is important to keep the aspect ratio small. Ideally, one would like to obtain small area for any given aspect ratio in a wide range. This would provide graphical user interfaces with the flexibility of fitting drawings into arbitrarily shaped windows. A variety of tradeoffs for the area and aspect ratio arise even when drawing graphs with a simple structure, such as trees. Table 55.2.5 summarizes selected universal bounds that can be simultaneously achieved on the area and the aspect ratio of various types of drawings of trees. Only for a few cases there exist algorithms that guarantee efficient area performance and that can accept any user-specified aspect ratio in a given range. For such cases, the aspect ratio in Table 55.2.5 is given as an interval.

TABLE 55.2.5 Trees: Universal upper bounds simultaneously achievable for area and aspect ratio.

CLASS OF GRAPHS	DRAWING TYPE	AREA	ASPECT RATIO	REF
Binary tree	PSL	$O(n)$	$[O(1), O(n^b)]$	[GR04]
Binary tree	ord pres PSL	$O(n \log n)$	$\left[O(1), O\left(\frac{n}{\log n}\right)\right]$	[GR04]
Binary tree	ord pres PSL	$O(n \log\log n)$	$O\left(\frac{n \log\log n}{\log^2 n}\right)$	[GR04]
Binary tree	upw ord pres PSL	$O(n \log n)$	$O\left(\frac{n}{\log n}\right)$	[GR04]
Binary tree	PSLO	$O(n \log\log n)$	$\left[O(1), O\left(\frac{n \log\log n}{\log^2 n}\right)\right]$	[SKC00]
Binary tree	upward PO	$O(n \log\log n)$	$O\left(\frac{n \log\log n}{\log^2 n}\right)$	[GGT96]
Binary tree	upward PSLO	$O(n \log n)$	$\left[O(1), O\left(\frac{n}{\log n}\right)\right]$	[CGKT02]
deg-4 tree	orthogonal	$O(n)$	$O(1)$	[Lei80, Val81]
deg-4 tree	orthogonal, leaves on hull	$O(n \log n)$	$O(1)$	[BK80]
Rooted deg-$O(n^a)$ tree	upward PPL	$O(n)$	$[O(1), O(n^b)]$	[GGT96]
Rooted tree	upward PSL layered	$O(n^2)$	$O(1)$	[RT81]
Rooted tree	upward PSL	$O(n \log n)$	$O\left(\frac{n}{\log n}\right)$	[CDP92]

Note: n is the number of vertices, a and b are constants such that $0 \le a, b < 1$.
All bounds assume grid drawings.

While upward planar straight-line drawings are the most natural way of visualizing rooted trees, the existing drawing techniques are unsatisfactory with respect to either the area requirement or the aspect ratio. Regarding polyline drawings, linear area can be achieved with a prescribed aspect ratio. However, for rooted trees, straight-line drawing remains by far the most used convention. For non-upward drawings of trees, linear area and optimal aspect ratio are possible for planar orthogonal drawings, and a small (logarithmic) amount of extra area is needed if the leaves are constrained to be on the convex hull of the drawing (e.g., pins on the boundary of a VLSI circuit). However, the non-upward drawing methods for rooted trees are better suited for VLSI layout than for visualization applications.

TRADEOFF BETWEEN AREA AND ANGULAR RESOLUTION

Table 55.2.6 summarizes selected universal bounds that can be simultaneously achieved on the area and the angular resolution of drawings of graphs. Universal lower bounds on the angular resolution exist that depend only on the degree of the graph. Also, substantially better bounds can be achieved by drawing a planar graph with bends or in a nonplanar way. Concerning trade-offs between area and angular resolution, Garg and Tamassia [GT94] proved that for any chosen angular resolution ρ, there exists a planar graph such that any planar straight-line drawing with angular resolution ρ has area $\Omega(a^{\rho n})$, for a constant $a > 1$. Duncan et al. [DEG+13] proved that there are trees that require exponential area for any order preserving planar straight-line drawing having perfect angular resolution. Duncan et al. also proved that perfect angular resolution and polynomial area can be simultaneously achieved for trees if order is not preserved or if the edges are drawn as circular arcs.

TABLE 55.2.6 Universal upper bounds for area and lower bounds for angular resolution, simultaneously achievable.

CLASS OF GRAPHS	DRAWING TYPE	AREA	ANGULAR RESOLUTION	REF
Tree	PSL	$O(n^8)$	$\Omega\left(\frac{1}{d}\right)$	[DEG+13]
Planar graph	SL grid	$O(d^6 n)$	$\Omega\left(\frac{1}{d^2}\right)$	[FHH+93]
Planar graph	SL grid	$O(d^3 n)$	$\Omega\left(\frac{1}{d}\right)$	[FHH+93]
Planar graph	PSL grid	$O(n^2)$	$\Omega\left(\frac{1}{n^2}\right)$	[FPP90, Sch90]
Planar graph	PSL grid	$O(b^n)$	$\Omega\left(\frac{1}{c^d}\right)$	[MP94]
Planar graph	PPL grid	$O(n^2)$	$\Omega\left(\frac{1}{d}\right)$	[Kan96]

Note: n is the number of vertices, d is the maximum vertex degree, b and c are constants such that $b > 1$ and $c > 1$.

OPEN PROBLEMS

1. Determine the area requirement of planar straight-line orthogonal drawings of binary and ternary trees. There are currently wide gaps between the known upper and lower bounds (Table 55.2.1, rows 8 and 15).

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2. Determine the area requirement of (upward) planar straight-line drawings of trees. There is currently an $O(\log n)$ gap between the known upper and lower bounds (Table 55.2.1, row 24).

3. Determine the area requirement of (outer)planar straight-line grid drawings of outerplanar graphs. There is currently an $O(n^{0.48})$ gap between the known upper and lower bounds (Table 55.2.2, row 4).

4. Determine the area requirement of planar straight-line grid drawing of series-parallel graphs. In particular it would be interesting to prove a subquadratic upper bound (Table 55.2.2, row 6).

5. Determine the area requirement of orthogonal (or, more generally, polyline) nonplanar drawings of planar graphs. There is currently an $O(\log n)$ gap between the known upper and lower bounds (Table 55.2.2, row 2).

6. Close the gap between the $\Omega(\frac{1}{d^2})$ universal lower bound and the $O(\frac{\log d}{d^2})$ existential upper bound on the angular resolution of straight-line drawings of general graphs (Table 55.2.3).

7. Close the gap between the $\Omega(\frac{1}{c^d})$ universal lower bound and the $O(\sqrt{\frac{\log d}{d^3}})$ existential upper bound on the angular resolution of planar straight-line drawings of planar graphs (Table 55.2.3).

8. Determine the best possible aspect ratio and area that can be simultaneously achieved for (upward) planar straight-line drawings of trees (Table 55.2.5).

55.3 COMPLEXITY OF GRAPH DRAWING PROBLEMS

Tables 55.3.1–55.3.4 summarize selected results on the time complexity of some fundamental graph drawing problems.

It is interesting that apparently similar problems exhibit very different time complexities. For example, while planarity testing can be done in linear time, upward planarity testing is NP-hard. Note that, as illustrated in Figure 55.1.2 (c–d), planarity and acyclicity are necessary but not sufficient conditions for upward planarity. While many efficient algorithms exist for constructing drawings of trees and planar graphs with good universal area bounds, exact area minimization for most types of drawings is NP-hard, even for trees.

OPEN PROBLEMS

1. Reduce the time complexity of upward planarity testing for embedded digraphs (which is currently $O(n^2)$), biconnected series-parallel digraphs (currently $O(n^4)$), and biconnected outerplanar digraphs (currently $O(n^2)$), or prove a superlinear lower bound (Table 55.3.1).

2. Reduce the time complexity of computing a planar straight-line drawing of an outerplanar graph such that the vertices are represented by a set of given points in general position (currently $O(n \log^3 n)$) or prove an $\omega(n \log n)$ lower bound (Table 55.3.2).

TABLE 55.3.1 Time complexity of some fundamental graph drawing problems: general graphs and digraphs.

CLASS OF GRAPHS	PROBLEM	TIME COMPLEXITY	
General graph	minimize crossings	NP-hard [GJ83]	
2-layered graph	minimize crossings in layered drawing with preassigned order on one layer	NP-hard [EW94]	
General graph	planarity testing and computing a planar embedding	$\Omega(n)$ trivial	$O(n)$ [BL76, CNAO85, FR82, ET76, HT74, LEC67]
General graph	maximum planar subgraph	NP-hard [GJ79]	
General graph	maximal planar subgraph	$\Omega(n+m)$ trivial	$O(n+m)$ [CHT93, Dji95, DT89, La94]
General graph	test the existence of a drawing where each edge is crossed at most once	NP-hard [GB07, KM09]	
General graph with $m = 4n - 8$	test the existence of a drawing where each edge is crossed at most once	$\Omega(n)$ trivial	$O(n^3)$ † [CGP06]
General graph	test the existence of a straight-line drawing where edges cross forming right angles	NP-hard [ABS12]	
2-layered graph	test the existence of a straight-line layered drawing where edges cross forming right angles	$\Omega(n)$ trivial	$O(n)$ [DDEL14]
General digraph	upward planarity testing	NP-hard [GT95]	
Embedded digraph	upward planarity testing	$\Omega(n)$ trivial	$O(n^2)$ [BDLM94]
Biconnected series-parallel digraphs	upward planarity testing	$\Omega(n)$ trivial	$O(n^4)$ [DGL10]
Biconnected outerplanar digraphs	upward planarity testing	$\Omega(n)$ trivial	$O(n^2)$ [Pap95]
Biconnected bipartite digraphs	upward planarity testing	$\Omega(n)$ trivial	$O(n)$ [DLR90]
Single-source digraph	upward planarity testing	$\Omega(n)$ trivial	$O(n)$ [BDMT98, HL96]
General graph	draw as the intersection graph of a set of unit diameter disks in the plane	NP-hard [BK98b]	

Note: n is the number of vertices, m is the number of edges.
†Brandenburg [Bra15] recently announced an $O(n)$ time algorithm for this problem.

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TABLE 55.3.2 Time complexity of some fundamental graph drawing problems: Planar graphs and digraphs.

CLASS OF GRAPHS	PROBLEM	TIME COMPLEXITY	
Planar graph	planar straight-line drawing with prescribed edge lengths	NP-hard [EW90]	
Planar graph	planar straight-line drawing with maximum angular resolution	NP-hard [Gar95]	
Embedded graph	test the existence of a planar straight-line drawing with prescribed angles between pairs of consecutive edges incident on a vertex	NP-hard [Gar95]	
Maximal planar graph	test the existence of a planar straight-line drawing with prescribed angles between pairs of consecutive edges incident on a vertex	$\Omega(n)$ trivial	$O(n)$ [DV96]
Planar graph	planar straight-line grid drawing with $O(n^2)$ area and $O(1/n^2)$ angular resolution	$\Omega(n)$ trivial	$O(n)$ [FPP90, Sch90]
Planar graph	planar polyline drawing with $O(n^2)$ area, $O(n)$ bends, and $O(1/d)$ angular resolutions	$\Omega(n)$ trivial	$O(n)$ [Kan96]
Triconn planar graph	planar straight-line convex grid drawing with $O(n^2)$ area and $O(1/n^2)$ angular resolution	$\Omega(n)$ trivial	$O(n)$ [Kan96]
Triconn planar graph	planar straight-line strictly convex drawing	$\Omega(n)$ trivial	$O(n)$ [CON85, Tut60, Tut63]
Reduced planar *st*-digraph	upward planar grid straight-line dominance drawing with minimum area	$\Omega(n)$ trivial	$O(n)$ [DTT92]
Upward planar digraph	upward planar polyline grid drawing with $O(n^2)$ area and $O(n)$ bends	$\Omega(n)$ trivial	$O(n)$ [DT88, DTT92]
Planar graph	planar straight-line drawing such that the vertices are represented by a set of given points	NP-hard [Cab06]	
Outerplanar graph	planar straight-line drawing such that the vertices are represented by a set of given points in general position	$\Omega(n \log n)$ [BMS97]	$O(n \log^3 n)$ [Bos02]
Planar graph	planar drawing such that the vertices are collinear and each edge has at most one bend	NP-hard [BK79]	
Series-parallel (di)graph	(upward) planar drawing such that the vertices are collinear and each edge has at most one bends	$\Omega(n)$ trivial	$O(n)$ [DDLW06]
Planar graph	planar drawing such that the vertices are collinear and each edge has at most two bends	$\Omega(n)$ trivial	$O(n)$ [DDLW05]

Note: n is the number of vertices.

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TABLE 55.3.3 Time complexity of some fundamental graph drawing problems: Planar graphs and digraphs.

CLASS OF GRAPHS	PROBLEM	TIME COMPLEXITY	
Planar deg-4 graph	planar orthogonal grid drawing with minimum number of bends	NP-hard [GT95]	
Planar biconnected deg-3 graph	planar orthogonal grid drawing with minimum number of bends and $O(n^2)$ area	$\Omega(n)$ trivial	$O(n^5 \log n)$ [DLV98]
Embedded deg-3 graph	planar orthogonal grid drawing with minimum number of bends (and $O(n^2)$ area)	$\Omega(n)$ trivial	$O(n)$ [RN02]
Planar biconnected deg-4 series-parallel graph	planar orthogonal grid drawing with minimum number of bends and $O(n^2)$ area	$\Omega(n)$ trivial	$O(n^4)$ [DLV98]
Planar biconnected deg-3 series-parallel graph	planar orthogonal grid drawing with minimum number of bends and $O(n^2)$ area	$\Omega(n)$ trivial	$O(n^3)$ [DLV98]
Embedded deg-4 graph	planar orthogonal grid drawing with minimum number of bends and $O(n^2)$ area	$\Omega(n)$ trivial	$O(n^{3/2})$ [CK12]
Planar deg-4 graph	planar orthogonal grid drawing with $O(n^2)$ area and $O(n)$ bends	$\Omega(n)$ trivial	$O(n)$ [BK98a, Kan96, TT89]
Embedded deg-4 graph	test the existence of a PSLOg drawing with rectangular faces	$\Omega(n)$ trivial	$O\left(\frac{n^{1.5}}{\log n}\right)$ [MHN06]
Planar deg-3 graph	test the existence of a PSLOg drawing with rectangular faces	$\Omega(n)$ trivial	$O(n)$ [RNG04]
Planar deg-3 graph	test the existence of a PSLOg drawing	$\Omega(n)$ trivial	$O(n)$ [RNN03]
Deg-3 series-parallel graph	test the existence of a planar orthogonal grid with no bends	$\Omega(n)$ trivial	$O(n)$ [REN06]
Planar orthog rep	planar orthogonal grid drawing with minimum area	NP-hard [Pat01]	
Note: n is the number of vertices.			

3. Reduce the time complexity of bend minimization for planar orthogonal drawings of degree-3 graphs and degree-3 and degree-4 series-parallel graphs (Table 55.3.3).
4. Reduce the time complexity of bend minimization for planar orthogonal drawings of embedded graphs (currently $O(n^{3/2})$), or prove a superlinear lower bound (Table 55.3.3).
5. Reduce the time complexity of testing the existence of a planar straight-line orthogonal drawing with rectangular faces (currently $O(n^{1.5}/\log n)$), or prove a superlinear lower bound (Table 55.3.3).
6. Reduce the time complexity of area minimization of hv-drawings of binary trees (from $O(n\sqrt{n \log n})$), or prove a superlinear lower bound (Table 55.3.4).

TABLE 55.3.4 Time complexity of some fundamental graph drawing problems: trees.

CLASS OF GRAPHS	PROBLEM	TIME COMPLEXITY	
Tree	draw as the Euclidean minimum spanning tree of a set of points in the plane	NP-hard [EW96]	
degree-4 tree	minimize area in planar orthogonal grid drawing	NP-hard [Bra90, DLT85, KL85, Sto84]	
degree-4 tree	minimize total/maximum edge length in planar orthogonal grid drawing	NP-hard [BC87, Bra90, Gre89]	
Rooted tree	minimize area in a planar straight-line upward layered grid drawing that displays symmetries and isomorphisms of subtrees	NP-hard [SR83]	
Rooted tree	minimize area in a planar straight-line upward layered drawing that displays symmetries and isomorphisms of subtrees	$\Omega(n)$ trivial	$O(n^k)$, $k \geq 1$ [SR83]
Binary tree	minimize area in hv-drawing	$\Omega(n)$ trivial	$O(n\sqrt{n \log n})$ [ELL92]
Rooted tree	planar straight-line upward layered grid drawing with $O(n^2)$ area	$\Omega(n)$ trivial	$O(n)$ [RT81]
Rooted tree	planar polyline upward grid drawing with $O(n)$ area	$\Omega(n)$ trivial	$O(n)$ [GGT93]
Tree	planar straight-line drawing such that the vertices are represented by a set of given points in general position	$\Omega(n \log n)$ [BMS97]	$O(n \log n)$ [BMS97]

Note: n is the number of vertices, m is the number of edges.

55.4 EXAMPLE OF A GRAPH DRAWING ALGORITHM

In this section we outline the algorithm in [Tam87] for computing, for an embedded degree-4 graph G, a planar orthogonal grid drawing with the minimum number of bends and using $O(n^2)$ area (see Table 55.3.2). This algorithm is the core of a practical drawing algorithm for general graphs (see Section 55.5 and Figure 55.4.1 (d)). The algorithm consists of two main phases:

1. Computation of an orthogonal representation for G, where only the bends and the angles of the orthogonal drawing are defined.

2. Assignment of integer lengths to the segments of the orthogonal representation.

Phase 1 uses a transformation into a network flow problem (Figure 55.4.1 (a–c)), where each unit of flow is associated with a right angle in the orthogonal drawing. Hence, angles are viewed as a commodity that is produced by the vertices, transported across faces by the edges through their bends, and eventually consumed by the faces. From the embedded graph G we construct a flow network N as follows. The nodes of network N are the vertices and faces of G. Let $\deg(f)$ denote the number of edges of the circuit bounding face f. Each vertex v supplies $\sigma(v) = 4$ units of flow, and each face f consumes $\tau(f)$ units of flow, where

$$\tau(f) = \begin{cases} 2\deg(f) - 4 & \text{if } f \text{ is an internal face} \\ 2\deg(f) + 4 & \text{if } f \text{ is the external face.} \end{cases}$$

By Euler's formula, $\sum_v \sigma(v) = \sum_f \tau(f)$, i.e., the total supply is equal to the total consumption.

Network N has two types of arcs:

- arcs of the type (v, f), where f is a face incident on vertex v; the flow in (v, f) represents the angle at vertex v in face f, and has lower bound 1, upper bound 4, and cost 0;
- arcs of the type (f, g), where face f shares an edge e with face g; the flow in (f, g) represents the number of bends along edge e with the right angle inside face f, and has lower bound 0, upper bound $+\infty$, and cost 1.

The conservation of flow at the vertices expresses the fact that the sum of the angles around a vertex is equal to 2π. The conservation of flow at the faces expresses the fact that the sum of the angles at the vertices and bends of an internal face is equal to $\pi(p - 2)$, where p is the number of such angles. For the external face, the above sum is equal to $\pi(p + 2)$. It can be shown that every feasible flow ϕ in network N corresponds to an admissible orthogonal representation for graph G, whose number of bends is equal to the cost of flow ϕ. Hence, an orthogonal representation for G with the minimum number of bends can be computed from a minimum-cost flow in G. This flow can be computed in $O(n^{1.5})$ time [CK12]. Phase 2 uses a simple compaction strategy derived from VLSI layout, where the lengths of the horizontal and vertical segments are computed independently after a preliminary refinement of the orthogonal representation that decomposes each face into rectangles. The resulting drawing is shown in Figure 55.4.1 (d).

55.5 TECHNIQUES FOR DRAWING GRAPHS

In this section we outline some of the most successful techniques that have been devised for drawing general graphs.

PLANARIZATION

The planarization approach is motivated by the availability of many efficient and well-analyzed drawing algorithms for planar graphs (see Table 55.3.2). If the graph is nonplanar, it is transformed into a planar graph by means of a preliminary planarization step that replaces each crossing with a fictitious vertex. The planarization approach consists of two main steps: in the first step a maximal planar subgraph G' of the input graph G is computed; in the second step, all the edges of G that are not in G' are added to G' and the crossings formed by each added edge are replaced with dummy vertices. Clearly when adding an edge one wants to produce as few crossings as possible. The two optimization problems arising in the two steps of the planarization approach, i.e., the maximum planar subgraph problem and the

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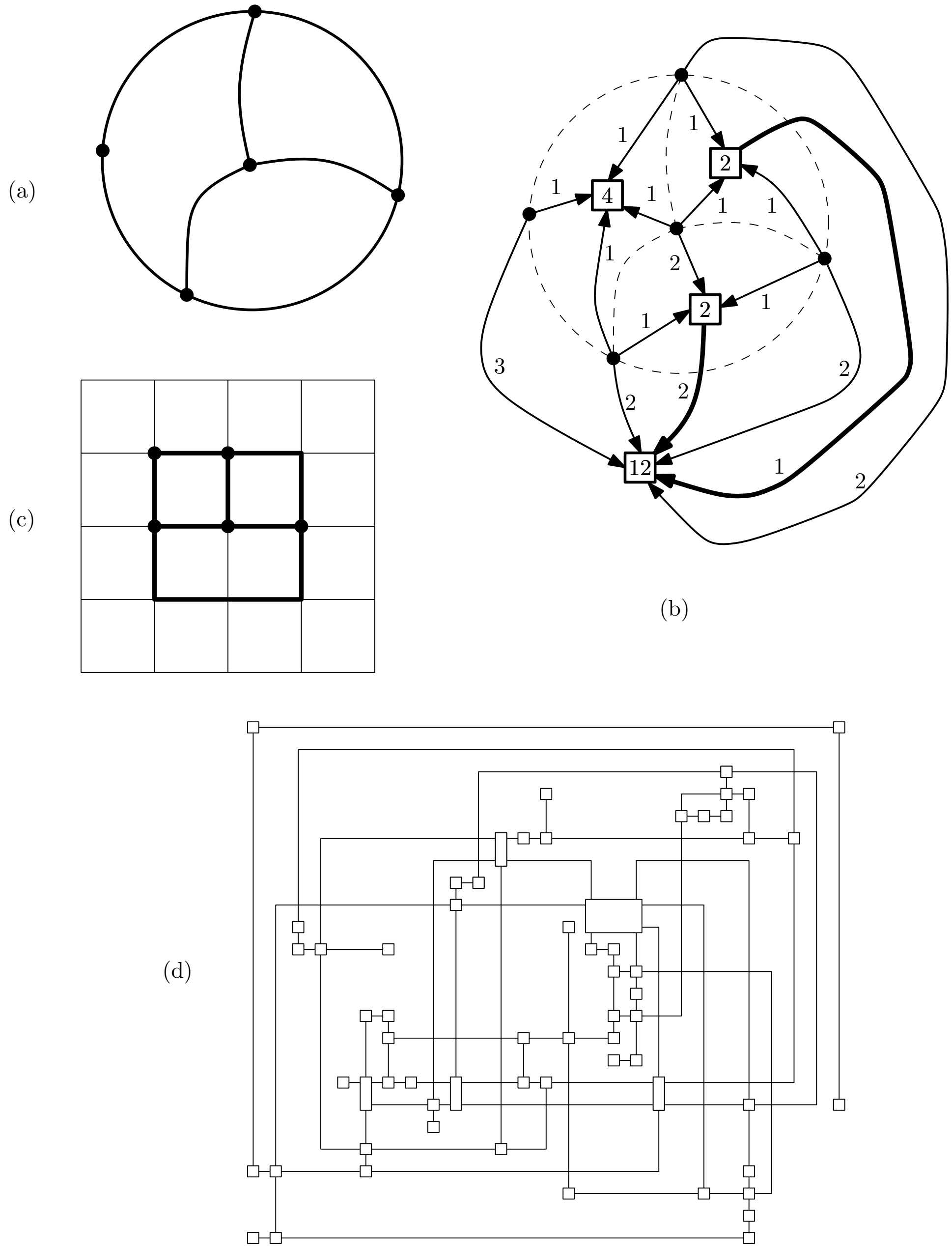
FIGURE 55.4.1

(a) *Embedded graph* G. (b) *Minimum cost flow in network* N*: the flow is shown next to each arc; arcs with zero flow are omitted; arcs with unit cost are drawn with thick lines; a face* f *is represented by a box labeled with* $\tau(f)$. (c) *Planar orthogonal grid drawing of* G *with minimum number of bends.* (d) *Orthogonal grid drawing of a nonplanar graph produced by a drawing method for general graphs based on the algorithm of Section* 55.4.



edge insertion problem, are NP-hard. Hence, existing planarization algorithms use heuristics. The best available heuristic for the maximum planar subgraph problem is described in [JM96]. This method has a solid theoretical foundation in polyhedral combinatorics, and achieves good results in practice. A sophisticated algorithm for edge insertion (that inserts each edge minimizing the number of crossings over all possible embeddings of the planar subgraph) is described in [GMW05]. See also [BCG$^+$13] for more references.

A successful drawing algorithm based on the planarization approach and a bend-minimization method [Tam87] is described in [TDB88] (Figure 55.4.1(d) was generated by this algorithm). It has been widely used in software visualization systems.

LAYERING

The layering approach for constructing polyline drawings of directed graphs transforms the digraph into a layered digraph and then constructs a layered drawing. A typical algorithm based on the layering approach consists of the following main steps:

1. Assign each vertex to a layer, with the goal of maximizing the number of edges oriented upward.
2. Insert fictitious vertices along the edges that cross layers, so that each edge in the resulting digraph connects vertices in consecutive layers. (The fictitious vertices will be displayed as bends in the final drawing.)
3. Permute the vertices on each layer with the goal of minimizing crossings.
4. Adjust the positions of the vertices in each layer with the goal of distributing the vertices uniformly and minimizing the number of bends.

Most of the subproblems involved in the various steps are NP-hard, hence heuristics must be used. The layering approach was pioneered by Sugiyama et al. [STT81] and since then a lot of research has been devoted to all optimization problems in each of the four steps above (see, e.g., [BBBH10, BK02, BWZ10, CGMW10, CGMW11, EK86, ELS93, EW94, GKNV93, HN02, JM97, MSM99, Nag05, NY04, TNB04]). See also [HN13] for more references.

FORCE DIRECTED

This approach uses a physical model where the vertices and edges of the graph are viewed as objects subject to various forces. Starting from an initial configuration (which can be randomly defined or suitably chosen), the physical system evolves into a final configuration of minimum energy, which yields the drawing. Rather than solving a system of differential equations, the evolution of the system is usually simulated using numerical methods (e.g., at each step, the forces are computed and corresponding incremental displacements of the vertices are performed).

Drawing algorithms based on the physical simulation approach are often able to detect and display symmetries in the graph. However, their running time is typically high. The physical simulation approach was pioneered in [Ead84, KS80].

Sophisticated developments and applications include [BP07, DH96, DM14, EH00, FR91, GGK04, GKN05, HJ05, HK02, KK89]. See also [Kob13] for additional references.

55.6 SOURCES AND RELATED MATERIAL

Several books devoted to graph drawing are published [DETT99, JM03, Kam89, KW01, NR04, Sug02, Tam13]. Among the early books, [Kam89] describes declarative approaches to graph drawing;[Sug02], motivated by software engineering applications, mostly focuses on layered drawings. [DETT99] is the first book that collects different techniques for graph drawing; [NR04] focuses on planar graphs. [KW01], [JM03], and [Tam13] are collections of surveys by different authors; [JM03] is devoted to graph drawing software and libraries while [Tam13] is the most recent handbook on Graph Drawing and Network Visualization. Sites with pointers to graph drawing resources and tools include the Web site `http://graphdrawing.org`, the Graph drawing e-print archive (`http://gdea.informatik.uni-koeln.de/`), and the Graph-Archive (`http://www.graph-archive.org/doku.php`).

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 10: Geometric graph theory
Chapter 23: Computational topology of graphs on surfaces
Chapter 27: Voronoi diagrams and Delaunay triangulations
Chapter 29: Triangulations and mesh generation

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