5 PSEUDOLINE ARRANGEMENTS

Stefan Felsner and Jacob E. Goodman

INTRODUCTION

Pseudoline arrangements generalize in a natural way arrangements of straight lines, discarding the straightness aspect, but preserving their basic topological and combinatorial properties. Elementary and intuitive in nature, at the same time, by the Folkman-Lawrence topological representation theorem (see Chapter 6), they provide a concrete geometric model for oriented matroids of rank 3.

After their explicit description by Levi in the 1920's, and the subsequent development of the theory by Ringel in the 1950's, the major impetus was given in the 1970's by Grünbaum's monograph *Arrangements and Spreads*, in which a number of results were collected and a great many problems and conjectures posed about arrangements of both lines and pseudolines. The connection with oriented matroids discovered several years later led to further work. The theory is by now very well developed, with many combinatorial and topological results and connections to other areas as for example algebraic combinatorics, as well as a large number of applications in computational geometry. In comparison to arrangements of lines arrangements of pseudolines have the advantage that they are more general and allow for a purely combinatorial treatment.

Section 5.1 is devoted to the basic properties of pseudoline arrangements, and Section 5.2 to related structures, such as arrangements of straight lines, configurations (and generalized configurations) of points, and allowable sequences of permutations. (We do not discuss the connection with oriented matroids, however; that is included in Chapter 6.) In Section 5.3 we discuss the stretchability problem. Section 5.4 summarizes some combinatorial results known about line and pseudoline arrangements, in particular problems related to the cell structure of arrangements. Section 5.5 deals with results of a topological nature and Section 5.6 with issues of combinatorial and computational complexity. Section 5.8 with several applications, including sweeping arrangements and pseudotriangulations.

Unless otherwise noted, we work in the real projective plane $\mathbf{P}^2$.

5.1 BASIC PROPERTIES

GLOSSARY

Arrangement of lines: A labeled set of lines not all passing through the same point (the latter is called a ***pencil***).

Pseudoline: A simple closed curve whose removal does not disconnect $\mathbf{P}^2$.

Arrangement of pseudolines: A labeled set of pseudolines not a pencil, every pair meeting no more than once (hence exactly once and crossing).

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Isomorphic arrangements: Two arrangements such that the mapping induced by their labelings is an isomorphism of the cell complexes into which they partition $\mathbf{P}^2$. (Isomorphism classes of pseudoline arrangements correspond to reorientation classes of oriented matroids of rank 3; see Chapter 6.)

Stretchable: A pseudoline arrangement isomorphic to an arrangement of straight lines. Figure 5.1.1 illustrates what was once believed to be an arrangement of straight lines, but which was later proven not to be stretchable. We will see in Section 5.6 that most pseudoline arrangements, in fact, are not stretchable.

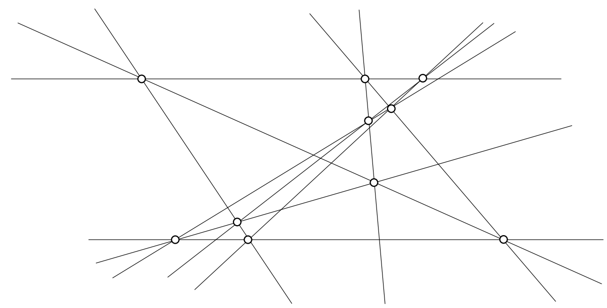

FIGURE 5.1.1
An arrangement of 10 *pseudolines, each containing* 3 *triple points; the arrangement is nonstretchable.*

Vertex: The intersection of two or more pseudolines in an arrangement.

Simple arrangement: An arrangement (of lines or pseudolines) in which there is no vertex where three or more pseudolines meet.

Euclidean arrangement of pseudolines: An arrangement of x-monotone curves in the Euclidean plane, every pair meeting exactly once and crossing there. In this case the canonical labeling of the pseudolines is using $1, \ldots, n$ in upward order on the left and in downward order on the right.

Wiring diagram: A Euclidean arrangement of pseudolines consisting of piecewise linear "wires." The wires (pseudolines) are horizontal except for small neighborhoods of their crossings with other wires; see Figure 5.1.2 for an example.

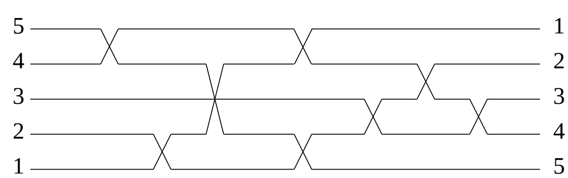

FIGURE 5.1.2
A wiring diagram.

A fundamental tool in working with arrangements of pseudolines, which takes the place of the fact that two points determine a line, is the following.

THEOREM 5.1.1 *Levi Enlargement Lemma* [Lev26]

If $\mathcal{A} = \{L_1, \ldots, L_n\}$ is an arrangement of pseudolines and $p, q \in \mathbf{P}^2$ are two distinct points not on the same member of $\mathcal{A}$, there is a pseudoline L passing through p and q such that $\mathcal{A} \cup \{L\}$ is an arrangement.

Theorem 5.1.1 has been shown by Goodman and Pollack [GP81b] not to extend

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to arrangements of pseudohyperplanes. It has, however, been extended in [SH91] to the case of "2-intersecting curves" with three given points. But it does not extend to k-intersecting curves with $k+1$ given points for $k>2$.

The Levi Enlargement Lemma is used to prove generalizations of a number of convexity results on arrangements of straight lines, duals of statements perhaps better known in the setting of configurations of points: Helly's theorem, Radon's theorem, Carathéodory's theorem, Kirchberger's theorem, the Hahn-Banach theorem, the Krein-Milman theorem, and Tverberg's generalization of Radon's theorem (cf. Chapter 4). To state two of these we need another definition. If $\mathcal{A}$ is an arrangement of pseudolines and p is a point not contained in any member of $\mathcal{A}$, $L \in \mathcal{A}$ is in the ***p*-convex hull** of $\mathcal{B} \subset \mathcal{A}$ if every path from p to a point of L meets some member of $\mathcal{B}$.

THEOREM 5.1.2 *Helly's theorem for pseudoline arrangements* [GP82a]

If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are subsets of an arrangement $\mathcal{A}$ of pseudolines, and p is a point not on any pseudoline of $\mathcal{A}$ such that, for any i, j, k, $\mathcal{A}$ contains a pseudoline in the p-convex hull of each of $\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k$, then there is an extension $\mathcal{A}'$ of $\mathcal{A}$ containing a pseudoline lying in the p-convex hull of each of $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

THEOREM 5.1.3 *Tverberg's theorem for pseudoline arrangements* [Rou88b]

If $\mathcal{A} = \{L_1, \ldots, L_n\}$ is a pseudoline arrangement with $n \geq 3m-2$, and p is a point not on any member of $\mathcal{A}$, then $\mathcal{A}$ can be partitioned into subarrangements $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and extended to an arrangement $\mathcal{A}'$ containing a pseudoline lying in the p-convex hull of $\mathcal{A}_i$ for every $i = 1, \ldots, m$.

Some of these convexity theorems, but not all, extend to higher dimensional arrangements; see [BLS$^+$99, Sections 9.2,10.4] and Section 26.3 of this Handbook.

Planar graphs admit straight line drawings, from this it follows that the pseudolines in an arrangement may be drawn as polygonal lines, with bends only at vertices. Eppstein [Epp14] investigates such drawings on small grids. Related is the following by now classical representation, which will be discussed further in Section 5.2.

THEOREM 5.1.4 [Goo80]

Every arrangement of pseudolines is isomorphic to a wiring diagram.

Theorem 5.1.4 is used in proving the following duality theorem, which extends to the setting of pseudolines the fundamental duality theorem between lines and points in the projective plane.

THEOREM 5.1.5 [Goo80]

Given an arrangement $\mathcal{A}$ of pseudolines and a set $\mathcal{S}$ of points in $\mathbf{P}^2$, there is a point set $\hat{\mathcal{A}}$ and a pseudoline arrangement $\hat{\mathcal{S}}$ so that a point $p \in \mathcal{S}$ lies on a pseudoline $L \in \mathcal{A}$ if and only if the dual point $\hat{L}$ lies on the dual pseudoline $\hat{p}$.

THEOREM 5.1.6 [AS05]

For Euclidean arrangements, the result of Theorem 5.1.5 *holds with the additional property that the duality preserves above-below relationships as well.*

5.2 RELATED STRUCTURES

GLOSSARY

Circular sequence of permutations: A doubly infinite sequence of permutations of $1, \ldots, n$ associated with an arrangement $\mathcal{A}$ of lines $L_1, \ldots, L_n$ by sweeping a directed line across $\mathcal{A}$; see Figure 5.2.3 and the corresponding sequence below.

Local equivalence: Two circular sequences of permutations are locally equivalent if, for each index i, the order in which it switches with the remaining indices is either the same or opposite.

Local sequence of unordered switches: In a Euclidean arrangement (wiring diagram), the permutation α_i given by the order in which the remaining pseudolines cross the ith pseudoline of the arrangement. In Figure 5.1.2, for example, α_1 is $(2, \{3, 5\}, 4)$.

Configuration of points: A (labeled) family $\mathcal{S} = \{p_1, \ldots, p_n\}$ of points, not all collinear, in $\mathbf{P}^2$.

Order type of a configuration $\mathcal{S}$: The mapping that assigns to each ordered triple i, j, k in $\{1, \ldots, n\}$ the orientation of the triple (p_i, p_j, p_k).

Combinatorial equivalence: Configurations $\mathcal{S}$ and $\mathcal{S}'$ are combinatorially equivalent if the set of permutations of $1, \ldots, n$ obtained by projecting $\mathcal{S}$ onto every line in general position agrees with the corresponding set for $\mathcal{S}'$.

Generalized configuration: A finite set of points in $\mathbf{P}^2$, together with an arrangement of pseudolines such that each pseudoline contains at least two of the points and for each pair of points there is a connecting pseudoline. Also called a ***pseudoconfiguration***. For an example see Figure 5.2.1.

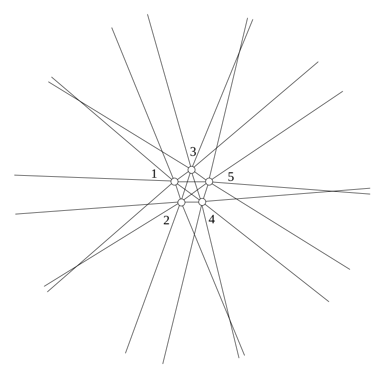

FIGURE 5.2.1
A generalized configuration of 5 points, this example is known as the ***bad pentagon****.*

Allowable sequence of permutations: A doubly infinite sequence of permutations of $1, \ldots, n$ satisfying the three conditions of Theorem 5.2.1. It follows from those conditions that the sequence is periodic of length $\leq n(n-1)$, and that its period has length $n(n-1)$ if and only if the sequence is ***simple***, i.e., each move consists of the switch of a single pair of indices.

ARRANGEMENTS OF STRAIGHT LINES

Much of the work on pseudoline arrangements has been motivated by problems involving straight line arrangements. In some cases the question has been whether known results in the case of lines really depended on the *straightness* of the lines; for many (but not all) combinatorial results the answer has turned out to be negative. In other cases, generalization to pseudolines (or, equivalently, reformulations in terms of allowable sequences of permutations) has permitted the solution of a more general problem where none was known previously in the straight case. Finally, pseudolines have turned out to be more useful than lines for certain algorithmic applications; this will be discussed in Section 5.8.

For arrangements of straight lines, there is a rich history of combinatorial results, some of which will be summarized in Section 5.4. Much of this is discussed in [Grü72].

Line arrangements are often classified by isomorphism type. For (unlabeled) arrangements of five lines, for example, Figure 5.2.2 illustrates the four possible isomorphism types, only one of which is simple.

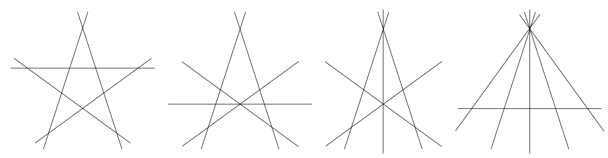

FIGURE 5.2.2
The 4 isomorphism types of arrangements of 5 lines.

There is a second classification of line arrangements, which has proven quite useful for certain problems. If a distinguished point not on any line of the arrangement is chosen to play the part of the "vertical point at infinity," we can think of the arrangement $\mathcal{A}$ as an arrangement of nonvertical lines in the Euclidean plane, and of P_∞ as the "upward direction." Rotating a directed line through P_∞ then amounts to sweeping a directed vertical line through $\mathcal{A}$ from left to right (say). We can then note the order in which this directed line cuts the lines of $\mathcal{A}$, and we arrive at a periodic sequence of permutations of $1, \ldots, n$, known as the ***circular sequence of permutations*** belonging to $\mathcal{A}$ (depending on the choice of P_∞ and the direction of rotation). This sequence is actually doubly infinite, since the rotation of the directed line through P_∞ can be continued in both directions. For the arrangement in Figure 5.2.3, for example, the circular sequence is

$$\mathcal{A}: \quad \ldots 12345 \overset{12,45}{\overline{\qquad}} 21354 \overset{135}{\overline{\qquad}} 25314 \overset{25,14}{\overline{\qquad}} 52341 \overset{234}{\overline{\qquad}} 54321 \ldots.$$

We have indicated the "moves" between consecutive permutations.

THEOREM 5.2.1 [GP84]

A circular sequence of permutations arising from a line arrangement has the following properties:

(i) *The move from each permutation to the next consists of the reversal of one or more nonoverlapping adjacent substrings;*

(ii) *After a move in which i and j switch, they do not switch again until every other pair has switched;*

(iii) $1, \ldots, n$ *do not all switch simultaneously with each other.*

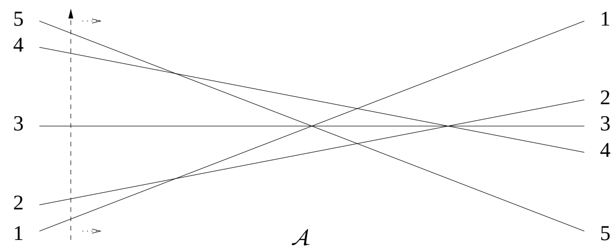

FIGURE 5.2.3
An arrangement of 5 lines.

If two line arrangements are isomorphic, they may have different circular sequences, depending on the choice of P_∞ (and the direction of rotation). We do have, however,

THEOREM 5.2.2 [GP84]

If $\mathcal{A}$ and $\mathcal{A}'$ are arrangements of lines in $\mathbf{P}^2$, and Σ and Σ' are any circular sequences of permutations corresponding to $\mathcal{A}$ and $\mathcal{A}'$, then $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic if and only if Σ and Σ' are locally equivalent.

CONFIGURATIONS OF POINTS

Under projective duality, arrangements of lines in $\mathbf{P}^2$ correspond to configurations of points. Some questions seem more natural in this setting of points, however, such as the Sylvester-Erdős problem about the existence of a line with only two points (ordinary line), and Scott's question whether n noncollinear points always determine at least $2\lfloor n/2 \rfloor$ directions. These problems are discussed in Chapter 1 of this Handbook.

Corresponding to the classification of line arrangements by isomorphism type, it turns out that the "dual" classification of point configurations is by order type.

THEOREM 5.2.3 [GP84]

If $\mathcal{A}$ and $\mathcal{A}'$ are arrangements of lines in $\mathbf{P}^2$ and $\mathcal{S}$ and $\mathcal{S}'$ the point sets dual to them, then $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic if and only if $\mathcal{S}$ and $\mathcal{S}'$ have the same (or opposite) order types.

From a configuration of points one also derives a circular sequence of permutations in a natural way, by projecting the points onto a rotating line. The sequence for the arrangement in Figure 5.2.3 comes from the configuration in Figure 5.2.4 in this way. Circular sequences yield a finer classification than order type; the order types of two point sets may be identical while their circular sequences are different.

It follows from projective duality that

THEOREM 5.2.4 [GP82b]

A sequence of permutations is realizable as the circular sequence of a set of points if and only if it is realizable as the sequence of an arrangement of lines.

The circular sequence of a point configuration can be reconstructed from the *set* of permutations obtained by projecting it in the directions that are not spanned by the points. The corresponding result in higher dimensions is useful (there the

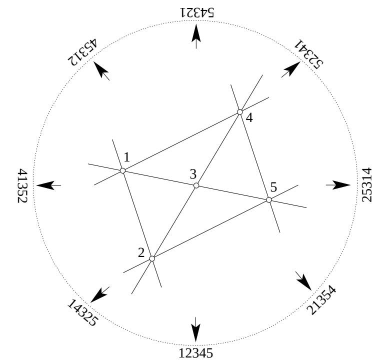

FIGURE 5.2.4
A configuration of 5 points and its circular sequence.

circular sequence generalizes to a somewhat unwieldy cell decomposition of a sphere with a permutation associated with every cell), since it means that all one really needs to know is the *set* of permutations; how they fit together can then be determined. Chapter 1 of this Handbook is concerned with results and some unsolved problems on point configurations.

GENERALIZED CONFIGURATIONS

Just as pseudoline arrangements generalize arrangements of straight lines, generalized configurations provide a generalization of configurations of points.

For example, a circular sequence for the generalized configuration in Figure 5.2.1, which is determined by the cyclic order in which the connecting pseudolines meet a distinguished pseudoline (in this case the "pseudoline at infinity"), is

$$\ldots 12345 \ ^{\underline{34}}\ 12435 \ ^{\underline{12}}\ 21435 \ ^{\underline{14}}\ 24135 \ ^{\underline{35}}\ 24513 \ ^{\underline{24}}\ 42513 \ ^{\underline{25}}\ 45213 \ ^{\underline{13}}\ 45231 \ ^{\underline{23}}\ 45321 \ ^{\underline{45}}\ 54321 \ldots$$

Another generalization of a point configuration is given by an abstraction of the order type. Intuitively an ***abstract order type*** prescribes an orientation clockwise, collinear, or counterclockwise, for triples of elements (points). The concept can be formalized by defining a (projective) abstract order type as a reorientation class of oriented matroids of rank 3. From the Folkman-Lawrence topological representation theorem it follows that abstract order types and pseudoline arrangements are the same. When working with abstract order types it is convenient to choose a line at infinity to get a Euclidean arrangement. In the Euclidean arrangement each triple of lines forms a subarrangement equivalent to one of the three arrangements shown in Figure 5.2.5; they tell whether the orientation of the corresponding triple of 'points' is clockwise, collinear or counterclockwise.

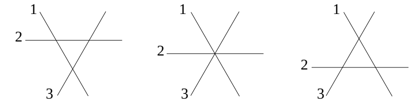

FIGURE 5.2.5
The Euclidean arrangements of 3 lines.

Knuth [Knu92] proposed an axiomatization of (Euclidean) abstract order types in general position (no collinearities) with five axioms. He calls a set together with a ternary predicate obeying the axioms a ***CC system***.

As in the case of circular sequences and order types we observe that generalized

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configurations yield a finer classification than abstract order types.

ALLOWABLE SEQUENCES

An allowable sequence of permutations is a combinatorial abstraction of the circular sequence of permutations associated with an arrangement of lines or a configuration of points. We can define, in a natural way, a number of geometric concepts for allowable sequences, such as *collinearity*, *betweenness*, *orientation*, *extreme point*, *convex hull*, *semispace*, *convex n-gon*, *parallel*, etc. [GP80a]. Not all allowable sequences are realizable, however, the smallest example being the sequence corresponding to Figure 5.2.1. A realization of this sequence would have to be a drawing of the bad pentagon of Figure 5.2.1 with straight lines, and it is not hard to prove that this is impossible; a proof of the nonrealizability of a larger class of allowable sequences can be found in [GP80a].

Allowable sequences provide a means of rephrasing many geometric problems about point configurations or line arrangements in combinatorial terms. For example, Scott's conjecture on the minimum number of directions determined by n lines has the simple statement: "Every allowable sequence of permutations of $1, \ldots, n$ has at least $2\lfloor n/2 \rfloor$ moves in a half-period." It was proved in this more general form by Ungar [Ung82], and the proof of the original Scott conjecture follows as a corollary; see also [Jam85], [BLS$^+$99, Section 1.11], and [AZ99, Chapter 9].

The Erdős-Szekeres problem (see Chapter 1 of this Handbook) looks as follows in this more general combinatorial formulation:

PROBLEM 5.2.5 *Generalized Erdős-Szekeres Problem* [GP81a]

What is the minimum n such that for every simple allowable sequence Σ on $1, \ldots, n$, there are k indices with the property that each occurs before the other $k-1$ in some term of Σ?

Allowable sequences arise from Euclidean pseudoline arrangements by sweeping a line across from left to right, just as with an arrangement of straight lines, and they arise as well from generalized configurations just as from configurations of points. In fact, the following theorem is just a restatement of Theorem 5.1.5.

THEOREM 5.2.6 [GP84]

Every allowable sequence of permutations can be realized both by an arrangement of pseudolines and by a generalized configuration of points.

Half-periods of allowable sequences correspond to Euclidean arrangements; they can be assumed to start with the identity permutation and to end with the reverse of the identity. If the sequence is simple it is the same as a ***reduced decomposition*** of the reverse of the identity in the Coxeter group of type A (symmetric group). Theorem 5.6.1 below was obtained in this context.

WIRING DIAGRAMS

Wiring diagrams provide the simplest "geometric" realizations of allowable sequences. To realize the sequence

$$\mathcal{A}: \quad \ldots 12345 \xrightarrow{45} 12354 \xrightarrow{12} 21354 \xrightarrow{135} 25314 \xrightarrow{25,14} 52341 \xrightarrow{23} 53241 \xrightarrow{24} 53421 \xrightarrow{34} 54321 \ldots$$

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for example, simply start with horizontal "wires" labeled $1, \ldots, n$ in (say) increasing order from bottom to top, and, for each move in the sequence, let the corresponding wires cross. This gives the wiring diagram of Figure 5.1.2, and at the end the wires have all reversed order. (It is then easy to extend the curves in both directions to the "line at infinity," thereby arriving at a pseudoline arrangement in $\mathbf{P}^2$.)

We have the following isotopy theorem for wiring diagrams.

THEOREM 5.2.7 [GP85]

If two wiring diagrams numbered $1, \ldots, n$ in order are isomorphic as labeled pseudoline arrangements, then one can be deformed continuously to the other (or to its reflection) through wiring diagrams isomorphic as pseudoline arrangements.

Two arrangements are related by a ***triangle-flip*** if one is obtained from the other by changing the orientation of a triangular face, i.e., moving one of the three pseudolines that form the face across the intersection of two others.

THEOREM 5.2.8 [Rin57]

Any two simple wiring diagrams numbered $1, \ldots, n$ in order can be obtained from each other with a sequence of triangle-flips.

This result has well-known counterparts in the terminology of *mutations* for oriented matroids and *Coxeter relations* for reduced decompositions, see [BLS+99, Section 6.4].

If $\mathcal{A}$ and $\mathcal{A}'$ are simple wiring diagrams and there are exactly t triples of lines such that the orientations of the induced subarrangement in $\mathcal{A}$ and $\mathcal{A}'$ differ, then it may require more than t triangle-flips to get from $\mathcal{A}$ to $\mathcal{A}'$; an example is given in [FZ01].

Wiring diagrams have also been considered in the bi-colored setting. Let L be a simple wiring diagram consisting of n blue and n red pseudolines, and call a vertex P *balanced* if P is the intersection of a blue and a red pseudoline such that the number of blue pseudolines strictly above P equals the number of red pseudolines strictly above P (and hence the same holds for those strictly below P as well).

THEOREM 5.2.9 [PP01]

A simple wiring diagram consisting of n blue and n red pseudolines has at least n balanced vertices, and this result is tight.

LOCAL SEQUENCES AND CLUSTERS OF STARS

The following theorem (proved independently by Streinu and by Felsner and Weil) solves the "cluster of stars" problem posed in [GP84]; we state it here in terms of local sequences of wiring diagrams, as in [FW01].

THEOREM 5.2.10 [Str97, FW01]

A set $(\alpha_i)_{i=1,\ldots,n}$ with each α_i a permutation of $\{1, \ldots, i-1, i+1, \ldots, n\}$, is the set of local sequences of unordered switches of a simple wiring diagram if and only if for all $i < j < k$ the pairs $\{i, j\}$, $\{i, k\}$, $\{j, k\}$ appear all in natural order or all in inverted order in α_k, α_j, α_i (resp.).

HIGHER DIMENSIONS

Just as isomorphism classes of pseudoline arrangements correspond to oriented matroids of rank 3, the corresponding fact holds for higher-dimensional arrangements, known as arrangements of pseudohyperplanes: they correspond to oriented matroids of rank $d+1$ (see Theorem 6.2.4 in Chapter 6 of this Handbook).

It turns out, however, that in higher dimensions, generalized configurations of points are (surprisingly) more restrictive than such oriented matroids; thus it is only in the plane that "projective duality" works fully in this generalized setting; see [BLS$^+$99, Section 5.3].

5.3 STRETCHABILITY

STRETCHABLE AND NONSTRETCHABLE ARRANGEMENTS

Stretchability can be described in either combinatorial or topological terms:

THEOREM 5.3.1 [BLS$^+$99, Section 6.3]

Given an arrangement $\mathcal{A}$ or pseudolines in $\mathbf{P}^2$, the following are equivalent.

(i) *The cell decomposition induced by $\mathcal{A}$ is isomorphic to that induced by some arrangement of straight lines;*

(ii) *Some homeomorphism of $\mathbf{P}^2$ to itself maps every $L_i \in \mathcal{A}$ to a straight line.*

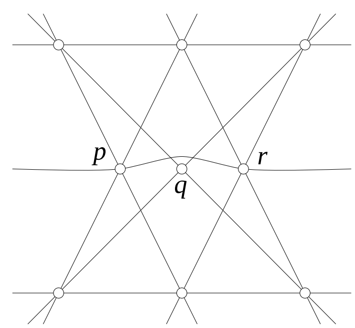

FIGURE 5.3.1
An arrangement that violates the theorem of Pappus.

Among the first examples observed of a nonstretchable arrangement of pseudolines was the non-Pappus arrangement of 9 pseudolines constructed by Levi: see Figure 5.3.1. Since Pappus's theorem says that points p, q, and r must be collinear if the pseudolines are straight, the arrangement in Figure 5.3.1 is clearly nonstretchable. A second example, involving 10 pseudolines, can be constructed similarly by violating Desargues's theorem.

Ringel showed how to convert the non-Pappus arrangement into a *simple* arrangement that was still nonstretchable. A symmetric drawing of it is shown in Figure 5.3.2.

Using allowable sequences, Goodman and Pollack proved the conjecture of

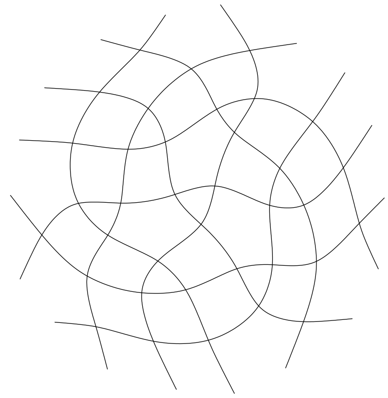

FIGURE 5.3.2
A simple nonstretchable arrangement of 9 pseudolines.

Grünbaum that the non-Pappus arrangement has the smallest size possible for a nonstretchable arrangement:

THEOREM 5.3.2 [GP80b]

Every arrangement of 8 or fewer pseudolines is stretchable.

In addition, Richter-Gebert proved that the non-Pappus arrangement is unique among simple arrangements of the same size.

THEOREM 5.3.3 [Ric89]

Every simple arrangement of 9 pseudolines is stretchable, with the exception of the simple non-Pappus arrangement.

The "bad pentagon" of Figure 5.2.1, with extra points inserted to "pin down" the intersections of the sides and corresponding diagonals, provides another example of a nonstretchable arrangement. The bad pentagon was generalized in [GP80a] yielding an infinite family of nonstretchable arrangements that were proved, by Bokowski and Sturmfels [BS89a], to be "minor-minimal." This shows that stretchability of simple arrangements cannot be guaranteed by the exclusion of a finite number of "forbidden" subarrangements. A similar example was found by Haiman and Kahn; see [BLS$^+$99, Section 8.3].

As for arrangements of more than 8 pseudolines, we have

THEOREM 5.3.4 [GPWZ94]

Let $\mathcal{A}$ be an arrangement of n pseudolines. If some face of $\mathcal{A}$ is bounded by at least $n-1$ pseudolines, then $\mathcal{A}$ is stretchable.

Finally, Shor shows in [Sho91] that even if a stretchable pseudoline arrangement has a symmetry, it may be impossible to realize this symmetry in any stretching.

THEOREM 5.3.5 [Sho91]

There exists a stretchable, simple pseudoline arrangement with a combinatorial symmetry such that no isomorphic arrangement of straight lines has the same combinatorial symmetry.

COMPUTATIONAL ASPECTS

Along with the Universality Theorem (Theorem 5.5.7 below), Mnëv proved that the problem of determining whether a given arrangement of n pseudolines is stretchable is NP-hard, in fact as hard as the problem of solving general systems of polynomial equations and inequalities over $\mathbb{R}$ (cf. Chapter 37 of this Handbook):

THEOREM 5.3.6 [Mnë85, Mnë88]

The stretchability problem for pseudoline arrangements is polynomially equivalent to the "existential theory of the reals" decision problem.

Shor [Sho91] presents a more compact proof of the NP-hardness result, by encoding a "monotone 3-SAT" formula in a family of suitably modified Pappus and Desargues configurations that turn out to be stretchable if and only if the corresponding formula is satisfiable. (See also [Ric96a].)

The following result provides an upper bound for the realizability problem.

THEOREM 5.3.7 [BLS+99, Sections 8.4, A.5]

The stretchability problem for pseudoline arrangements can be decided in singly exponential time and polynomial space in the Turing machine model of complexity. The number of arithmetic operations needed is bounded above by $2^{4n \log n + O(n)}$.

The NP-hardness does not mean, however, that it is pointless to look for algorithms to determine stretchability, particularly in special cases. Indeed, a good deal of work has been done on this problem by Bokowski, in collaboration with Guedes de Oliveira, Pock, Richter-Gebert, Scharnbacher, and Sturmfels. Four main algorithmic methods have been developed to test for the realizability (or nonrealizability) of an oriented matroid, i.e., in the rank 3 case, the stretchability (respectively nonstretchability) of a pseudoline arrangement:

(i) The ***inequality reduction method:*** this attempts to find a relatively small system of inequalities that still carries all the information about a given oriented matroid.

(ii) The ***solvability sequence method:*** this attempts to find an elimination order with special properties for the coordinates in a potential realization of an order type.

(iii) The ***final polynomial method:*** this attempts to find a bracket polynomial (cf. Chapter 60) whose existence will imply the *non*realizability of an order type.

(iv) Bokowski's's ***rubber-band method:*** an elementary heuristic that has proven surprisingly effective in finding realizations [Bok08].

Not every realizable order type has a solvability sequence, but it turns out that every nonrealizable one does have a final polynomial, and an algorithm due to Lombardi [Lom90] can be used to find one. Fukuda et al. [FMM13] have refined the techniques and used them successfully for the enumeration of realizable oriented matroids with small parameters.

All of these methods extend to higher dimensions. For details about the first three, see [BS89b].

DISTINCTIONS BETWEEN LINES AND PSEUDOLINES

The asymptotic number of pseudoline arrangements is strictly larger than the number of line arrangements; see Table 5.6.2 below. Nevertheless, we know only very few combinatorial properties or algorithmic problems that can be used to distinguish between line and pseudoline arrangements.

Combinatorial properties of this type are given by the number of triangles in nonsimple arrangements.

The first such example was given by Roudneff in [Rou88a]. He proved the following statement that had been conjectured of Grünbaum: An arrangement of n lines with only n triangles is simple. However, there exist nonsimple arrangements of n pseudolines with only n triangles. An example of such an arrangement is obtained by "collapsing" the central triangle in Figure 5.3.2.

Felsner and Kriegel [FK99] describe examples of nonsimple Euclidean pseudoline arrangements with only $2n/3$ triangles; see Figure 5.3.3. A theorem of Shannon [Sha79] (see also [Fel04, Thm. 5.18]), however, asserts that a Euclidean arrangement of n lines has at least $n-2$ triangles. Hence the examples from [FK99] are nonstretchable.

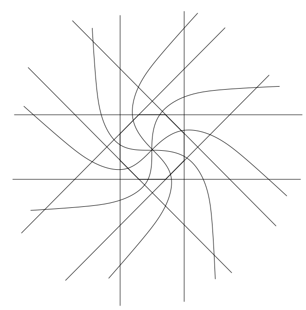

FIGURE 5.3.3
A nonsimple Euclidean arrangement of 12 pseudolines with 8 triangles.

Recently applications of the polynomial method to problems in incidence geometry have led to some breakthroughs [Gut16] The improved bounds apply only to line arrangements. This may open the door to a range of additional properties that distinguish between lines and pseudolines. Candidates could be the number of ordinary points or the constant for the strong Dirac conjecture; see [LPS14].

Another candidate problem where pseudolines and lines may behave differently is the maximum length of x-monotone paths, see Theorem 5.4.14.

Steiger and Streinu [SS94] consider the problem of x-sorting line or pseudoline intersections, i.e., determining the order of the x-coordinates of the intersections of the lines or pseudolines in a Euclidean arrangement. They prove that in comparison-based sorting the vertices of a simple arrangement of n lines can be x-sorted with $O(n^2)$ comparisons while the x-sorting of vertices of a simple arrangement of n pseudolines requires at least $\Omega(n^2 \log n)$ comparisons. The statement for pseudolines is a corollary of Theorem 5.6.1, i.e., based on the number of possible x-sortings.

5.4 COMBINATORIAL RESULTS

In this section we survey combinatorial results. This includes several results that update Grünbaum's comprehensive 1972 survey [Grü72]. For a discussion of *levels in arrangements* (dually, *k-sets*), see Chapters 28 and 1, respectively. Erdős and Purdy [EP95] survey related material with an emphasis on extremal problems.

GLOSSARY

Simplicial arrangement: An arrangement of lines or pseudolines in which every cell is a triangle.

Near-pencil: An arrangement with all but one line (or pseudoline) concurrent.

Projectively unique: A line arrangement $\mathcal{A}$ with the property that every isomorphic line arrangement is the image of $\mathcal{A}$ under a projective transformation.

RELATIONS AMONG NUMBERS OF VERTICES, EDGES, AND FACES

THEOREM 5.4.1 *Euler*

If $f_i(\mathcal{A})$ is the number of faces of dimension i in the cell decomposition of $\mathbf{P}^2$ induced by an arrangement $\mathcal{A}$, then $f_0(\mathcal{A}) - f_1(\mathcal{A}) + f_2(\mathcal{A}) = 1$.

In addition to ***Euler's formula***, the following inequalities are satisfied for arbitrary pseudoline arrangements (here, $n(\mathcal{A})$ is the number of pseudolines in the arrangement $\mathcal{A}$). We state the results for projective arrangements. Similar inequalities for Euclidean arrangements can easily be derived.

THEOREM 5.4.2 [Grü72, SE88]

(i) $1 + f_0(\mathcal{A}) \le f_2(\mathcal{A}) \le 2f_0(\mathcal{A}) - 2$, *with equality on the left for precisely the simple arrangements, and on the right for precisely the simplicial arrangements;*

(ii) $n(\mathcal{A}) \le f_0(\mathcal{A}) \le \binom{n(\mathcal{A})}{2}$, *with equality on the left for precisely the near-pencils, and on the right for precisely the simple arrangements;*

(iii) *For $n \gg 0$, every f_0 satisfying $n^{3/2} \le f_0 \le \binom{n}{2}$, with the exceptions of $\binom{n}{2} - 3$ and $\binom{n}{2} - 1$, is the number of vertices of some arrangement of n pseudolines (in fact, of straight lines);*

(iv) $2n(\mathcal{A}) - 2 \le f_2(\mathcal{A}) \le \binom{n(\mathcal{A})}{2} + 1$, *with equality on the left for precisely the near-pencils, and on the right for precisely the simple arrangements;*

(v) $f_2(\mathcal{A}) \ge 3n(\mathcal{A}) - 6$ *if $\mathcal{A}$ is not a near-pencil.*

There are gaps in the possible values for $f_2(\mathcal{A})$, as shown by Theorem 5.4.3. This proves a conjecture of Grünbaum which was generalized by Purdy, it refines Theorem 5.4.2(iv).

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THEOREM 5.4.3 [Mar93]

There exists an arrangement $\mathcal{A}$ of n pseudolines with $f_2(\mathcal{A}) = f$ if and only if, for some integer k with $1 \leq k \leq n-2$, we have $(n-k)(k+1) + \binom{k}{2} - \min(n-k, \binom{k}{2}) \leq f \leq (n-k)(k+1) + \binom{k}{2}$. Moreover, if $\mathcal{A}$ exists, it can be chosen to consist of straight lines.

Finally, the following result (proved in the more general setting of geometric lattices) gives a complete set of inequalities for the flag vectors $(n(\mathcal{A}), f_0(\mathcal{A}), i(\mathcal{A}))$. The component $i(\mathcal{A})$ is the number of vertex–pseudoline incidences in the arrangement $\mathcal{A}$.

THEOREM 5.4.4 [Nym04]

The closed convex set generated by all flag vectors of arrangements of pseudolines is characterized by the following set of inequalities: $n \geq 3$, $f_0 \geq n$, $i \geq 2f_0$, $i \leq 3f_0 - 3$, and $(k-1)i - kn - (2k-3)f_0 + \binom{k+1}{2} \geq 0$, for all $k \geq 3$. Moreover, this set of inequalities is minimal.

THE NUMBER OF CELLS OF DIFFERENT SIZES

It is easy to see by induction that a *simple* arrangement of more than 3 pseudolines must have at least one nontriangular cell. This observation leads to many questions about numbers of cells of different types in both simple and nonsimple arrangements, some of which have not yet been answered satisfactorily.

On the minimum number of triangles, we have a classical result of Levi.

THEOREM 5.4.5 [Lev26]

In any arrangement of $n \geq 3$ pseudolines, every pseudoline borders at least 3 triangles. Hence every arrangement of n pseudolines determines at least n triangles.

This minimum is achieved by the "cyclic arrangements" of lines generated by regular polygons, as in Figure 5.4.1.

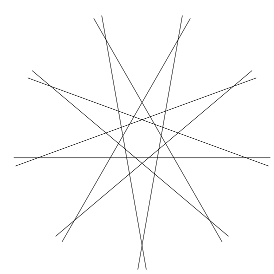

FIGURE 5.4.1
A cyclic arrangement of 9 lines.

Grünbaum [Grü72] asked for the maximum number of triangles and provided the upper bound of $\lfloor n(n-1)/3 \rfloor$ for the simple case. Harborth [Har85] introduced the doubling method to construct infinite families of simple arrangements of pseudolines attaining the upper bound. Roudneff [Rou96] proved that the upper bound also holds for nonsimple arrangements. Forge and Ramírez Alfonsín [FR98] constructed infinite families of line arrangements attaining the upper bound.

Already Grünbaum knew that his upper bound could not be attained for all values of n. Building on results from [BBL08], Blanc proved

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THEOREM 5.4.6 [Bla11]

If $\mathcal{A}$ is a simple arrangement of n pseudolines, with $n \geq 4$ and $p_3(\mathcal{A})$ is the number of triangles in $\mathcal{A}$, then

$$p_3(\mathcal{A}) \leq \begin{cases} (n(n-1))/3 & \text{if } n \equiv 0,4 \pmod 6 \\ (n(n-2)-2)/3 & \text{if } n \equiv 1 \pmod 6 \\ (n(n-1)-5)/3 & \text{if } n \equiv 2 \pmod 6 \\ (n(n-2))/3 & \text{if } n \equiv 3,5 \pmod 6 \end{cases}$$

Furthermore, each of these bounds is tight for infinitely many integers n, and for $n \not\equiv 2 \pmod 6$ a family of straight line arrangements attaining the bound is known.

For arrangements in the Euclidean plane $\mathbb{R}^2$, on the other hand, we have

THEOREM 5.4.7 [FK99]

(i) *Every simple arrangement of n pseudolines in $\mathbb{R}^2$ contains at least $n-2$ triangles, with equality achieved for all $n \geq 3$.*

(ii) *Every arrangement of n pseudolines in $\mathbb{R}^2$ contains at least $2n/3$ triangles, with equality achieved for all $n \equiv 0 \pmod 3$.*

Regarding the maximum number of triangles in simple Euclidean arrangements of n pseudolines Blanc proves a theorem similar to Theorem 5.4.6 that improves on the upper bound of $n(n-2)/3$ depending on the residue modulo six.

The following result disproved a conjecture of Grünbaum:

THEOREM 5.4.8 [LRS89]

There is a simple arrangement of straight lines containing no two adjacent triangles.

The proof involved finding a pseudoline arrangement with this property, then showing (algebraically, using Bokowski's "inequality reduction method"—see Section 5.3) that the arrangement, which consists of 12 pseudolines, is stretchable.

The following general problem was posed by Grünbaum [Grü72].

PROBLEM 5.4.9

What is the maximum number of k-sided cells in an arrangement of n pseudolines, for $k > 3$?

Some results about quadrilaterals have been obtained.

THEOREM 5.4.10 [Grü72, Rou87, FR01]

(i) *Every arrangement of $n \geq 5$ pseudolines contains at most $n(n-3)/2$ quadrilaterals. For straight line arrangements, this bound is achieved by a unique arrangement for each n.*

(ii) *A pseudoline arrangement containing $n(n-3)/2$ quadrilaterals must be simple.*

There are infinitely many simple pseudoline arrangements with no quadrilaterals, contrary to what was once believed. The following result implies, however, that there must be many quadrilaterals or pentagons in *every* simple arrangement.

THEOREM 5.4.11 [Rou87]

Every pseudoline in a simple arrangement of $n > 3$ pseudolines borders at least 3 quadrilaterals or pentagons. Hence, if p_4 is the number of quadrilaterals and p_5 the number of pentagons in a simple arrangement, we have $4p_4 + 5p_5 \geq 3n$.

Leaños et al. [LLM+07] study simple Euclidean arrangements of pseudolines without faces of degree ≥ 5. They show that they have exactly $n-2$ triangles and $(n-2)(n-3)/2$ quadrilaterals. Moreover, all these arrangements are stretchable.

SIMPLICIAL ARRANGEMENTS

Simplicial arrangements were first studied by Melchior [Mel41]. Grünbaum [Grü71] listed 90 "sporadic" examples of simplicial arrangements together with the following three infinite families:

THEOREM 5.4.12 [Grü72]

Each of the following arrangements is simplicial:

(i) *the near-pencil of n lines;*

(ii) *the sides of a regular n-gon, together with its n axes of symmetry;*

(iii) *the arrangement in* (ii)*, together with the line at infinity, for n even.*

In [Grü09] Grünbaum presented his collection from 1971 with corrections and in a more user-friendly way. Subsequently, Cuntz [Cun12] did an exhaustive computer enumeration of all simplicial arrangements with up to 27 pseudolines. As a result he found four additional sporadic examples.

On the other hand, additional infinite families of (nonstretchable) simplicial arrangements of pseudolines are known, which are constructible from regular polygons by extending sides, diagonals, and axes of symmetry and modifying the resulting arrangement appropriately. More recently Berman [Ber08] elaborated an idea of Eppstein to generate simplicial arrangements of pseudolines with rotational symmetry. The key idea is to collect segments contributed by the pseudolines of an orbit in a wedge of symmetry, they can be interpreted as (pseudo)light-beams between two mirrors. The method, also described in [Grü13], was used by Lund et al. [LPS14] to construct counterexamples for the strong Dirac conjecture. Figure 5.4.2 shows an example.

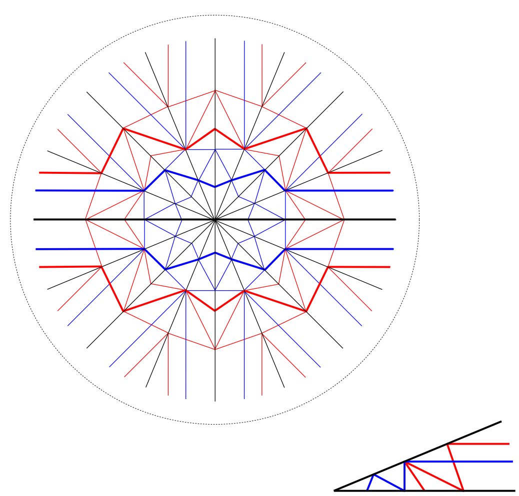

FIGURE 5.4.2
A simplicial arrangement of 25 pseudolines (including line at infinity). Each pseudoline is incident to ≤ 10 vertices.

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One of the most important problems on arrangements is the following.

PROBLEM 5.4.13 [Grü72]

Classify all simplicial arrangements of pseudolines. Which of these are stretchable? In particular, are there any infinite families of simplicial line arrangements besides the three of Theorem 5.4.12*?*

It has apparently not been disproved that every (pseudo)line arrangement is a subarrangement of a simplicial (pseudo)line arrangement.

PATHS IN PSEUDOLINE ARRANGEMENTS

A ***monotone path*** π of a Euclidean arrangement $\mathcal{A}$ is a collection of edges and vertices of $\mathcal{A}$ such that each vertical line contains exactly one point of π. The length of π is defined as the number of vertices where π changes from one support line to another. Edelsbrunner and Guibas [EG89] have shown that the maximum length of a monotone path in an arrangement of n pseudolines can be computed via a topological sweep. They also asked for the maximum length λ_n of a monotone path where the maximum is taken over all arrangements of n lines. Matoušek [Mat91] proved that λ_n is $\Omega(n^{5/3})$. This was raised to $\Omega(n^{7/4})$ by Radoičić and Tóth [RT03], they also showed $\lambda_n \leq 5n^2/12$.

THEOREM 5.4.14 [BRS$^+$05]

The maximum length λ_n of a monotone path in an arrangement of n lines is $\Omega(n^2/2^{c\sqrt{\log n}})$, for some constant $c > 0$.

Dumitrescu [Dum05] studied arrangements with a restricted number of slopes. One of his results is that if there are at most 5 slopes, then the length of a monotone path is $O(n^{5/3})$. Together with the construction of Matoušek we thus know that the maximum length of a monotone path in arrangements with at most 5 slopes is in $\Theta(n^{5/3})$.

Matoušek [Mat91] showed that in the less restrictive setting of arrangements of pseudolines there are arrangements with n pseudolines which have a monotone path of length $\Omega(n^2/\log n)$.

For related results on k-levels in arrangements, see Chapter 28.

A ***dual path*** of an arrangement is a path in the dual of the arrangement, i.e., a sequence of cells such that any two consecutive cells of the sequence share a boundary edge. A dual path in an arrangement with lines colored red and blue is an *alternating dual path* if the path alternatingly crosses red and blue lines. Improving on work by Aichholzer et al. [ACH$^+$14], the following has been obtained by Hoffmann, Kleist, and Miltzow.

THEOREM 5.4.15 [HKM15]

Every arrangement of n lines has a dual path of length $n^2/3 - O(n)$ and there are arrangements with a dual path of length $n^2/3 + O(n)$.

Every bicolored arrangement has an alternating dual path of length $\Omega(n)$ and there are arrangements with $3k$ red and $2k$ blue lines where every alternating dual path has length at most $14k$.

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COMPLEXITY OF SETS OF CELLS IN AN ARRANGEMENT

The ***zone*** of a line ℓ in $\mathcal{A}$ is the collection of all faces that have a segment of ℓ on their boundary. The complexity of the zone of ℓ is the sum of the degrees of all faces in the zone of ℓ.

THEOREM 5.4.16 *Zone Theorem* [BEPY91, Pin11]

The sum of the numbers of sides in all the cells of an arrangement of $n+1$ pseudolines that are supported by one of the pseudolines is at most $19n/2-3$; this bound is tight.

For general sets of faces, on the other hand, Canham proved

THEOREM 5.4.17 [Can69]

If $F_1,\ldots,F_k$ are any k distinct faces of an arrangement of n pseudolines, then $\sum_{i=1}^{k} p(F_i) \le n+2k(k-1)$, where $p(F)$ is the number of sides of a face F. This is tight for $2k(k-1)\le n$.

For $2k(k-1) > n$, this was improved by Clarkson et al. to the following result, with simpler proofs later found by Székely and by Dey and Pach; the tightness follows from a result of Szemerédi and Trotter, proved independently by Edelsbrunner and Welzl.

THEOREM 5.4.18 [ST83, EW86, CEG+90, Szé97, DP98]

The total number of sides in any k distinct cells of an arrangement of n pseudolines is $O(k^{2/3}n^{2/3}+n)$. This bound is (asymptotically) tight in the worst case.

There are a number of results of this kind for arrangements of objects in the plane and in higher dimensions; see Chapter 28, as well as [CEG+90] and [PS09].

5.5 TOPOLOGICAL PROPERTIES

GLOSSARY

Topological projective plane: $\mathbf{P}^2$, with a distinguished family $\mathcal{L}$ of pseudolines (its "lines"), is a topological projective plane if, for each $p,q \in \mathbf{P}^2$, exactly one $L_{p,q} \in \mathcal{L}$ passes through p and q, with $L_{p,q}$ varying continuously with p and q.

Isomorphism of topological projective planes: A homeomorphism that maps "lines" to "lines."

Universal topological projective plane: One containing an isomorphic copy of every pseudoline arrangement.

Topological sweep: If $\mathcal{A}$ is a pseudoline arrangement in the Euclidean plane and $L \in \mathcal{A}$, a topological sweep of $\mathcal{A}$ "starting at L" is a continuous family of pseudolines including L, each compatible with $\mathcal{A}$, which forms a partition of the plane.

Basic semialgebraic set: The set of solutions to a finite number of polynomial equations and strict polynomial inequalities in $\mathbb{R}^d$. (This term is sometimes used even if the inequalities are not necessarily strict.)

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GRAPH AND HYPERGRAPH PROPERTIES

An arrangement of pseudolines can be interpreted as a graph drawn crossing free in the projective plane. Hence, it is natural to study graph-theoretic properties of arrangements.

THEOREM 5.5.1 [FHNS00]

The graph of a simple projective arrangement of $n \geq 4$ pseudolines is 4-connected.

Using wiring diagrams, the same authors prove

THEOREM 5.5.2 [FHNS00]

Every projective arrangement with an odd number of pseudolines can be decomposed into two edge-disjoint Hamiltonian paths (plus two unused edges), and the decomposition can be found efficiently.

Harborth and Möller [HM02] identified a projective arrangements of 6 lines that admits no decomposition into two Hamiltonian cycles.

Bose et al. [BCC+13] study the maximum size of independent sets and vertex cover as well as the chromatic numbers of three hypergraphs defined by arrangements. They consider the hypergraphs given by the line-cell, vertex-cell and cell-zone incidences, respectively.

THEOREM 5.5.3 [BCC+13, APP+14]

The lines of every arrangement of n pseudolines in the plane can be colored with $O(\sqrt{n/\log n})$ colors so that no face of the arrangement is monochromatic. There are arrangements requiring at least $\Omega(\frac{\log n}{\log\log n})$ colors.

Let P be a set of points in the plane and let $\alpha(P)$ be the maximum size of a subset S of P such that S is in general position. Let $\alpha_4(n)$ be the minimum of $\alpha(P)$ taken over all sets P of n points in the plane with no four points on a line. Any improvement of the upper bound in the above theorem would imply an improvement on the known lower bound for $\alpha_4(n)$. The study of $\alpha_4(n)$ was initiated by Erdős, see [APP+14] for details.

EMBEDDING IN LARGER STRUCTURES

In [Grü72], Grünbaum asked a number of questions about extending pseudoline arrangements to more complex structures. The strongest result known about such extendibility is the following, which extends results of Goodman, Pollack, Wenger, and Zamfirescu [GPWZ94].

THEOREM 5.5.4 [GPW96]

There exist uncountably many pairwise nonisomorphic universal topological projective planes.

In particular, this implies the following statements, all of which had been conjectured in [Grü72].

(i) Every pseudoline arrangement can be extended to a topological projective plane.

(ii) There exists a universal topological projective plane.

(iii) There are nonisomorphic topological projective planes such that every arrangement in each is isomorphic to some arrangement in the other.

Theorem 5.5.4 also implies the following result, established earlier by Snoeyink and Hershberger (and implicitly by Edmonds, Fukuda, and Mandel; see [BLS+99, Section 10.5]).

THEOREM 5.5.5 *Sweeping Theorem* [SH91]

A pseudoline arrangement $\mathcal{A}$ in the Euclidean plane can be swept by a pseudoline, starting at any $L \in \mathcal{A}$.

PROBLEM 5.5.6 [Grü72]

Which arrangements are present (up to isomorphism) in every *topological projective plane?*

MOVING FROM ONE ARRANGEMENT TO ANOTHER

In [Rin56], Ringel asked whether an arrangement $\mathcal{A}$ of straight lines could always be moved continuously to a given isomorphic arrangement $\mathcal{A}'$ (or to its reflection) so that all intermediate arrangements remained isomorphic. This question, which became known as the "isotopy problem" for arrangements, was eventually solved by Mnëv, and (independently, since news of Mnëv's results had not yet reached the West) by White in the nonsimple case, then by Jaggi and Mani-Levitska in the simple case [BLS+99]. Mnëv's results are, however, by far stronger.

THEOREM 5.5.7 *Mnëv's Universality Theorem* [Mnë85]

If V is any basic semialgebraic set defined over $\mathbb{Q}$, there is a configuration $\mathcal{S}$ of points in the plane such that the space of all configurations of the same order type as $\mathcal{S}$ is stably equivalent to V. If V is open in some $\mathbb{R}^n$, then there is a simple configuration $\mathcal{S}$ with this property.

From this it follows that the space of line arrangements isomorphic to a given one may have the homotopy type of *any* semialgebraic variety, and in particular may be disconnected, which gives a (very strongly) negative answer to the isotopy question. For a further generalization of Theorem 5.5.7, see [Ric96a].

The line arrangement of smallest size known for which the isotopy conjecture fails consists of 14 lines in general position and was found by Suvorov [Suv88]; see also [Ric96b]. Special cases where the isotopy conjecture *does* hold include:

(i) every arrangement of 9 or fewer lines in general position [Ric89], and
(ii) an arrangement of n lines containing a cell bounded by at least $n-1$ of them.

There are also results of a more combinatorial nature about the possibility of transforming one pseudoline arrangement to another. Ringel [Rin56, Rin57] proved

THEOREM 5.5.8 *Ringel's Homotopy Theorem*

If $\mathcal{A}$ and $\mathcal{A}'$ are simple arrangements of pseudolines, then $\mathcal{A}$ can be transformed to $\mathcal{A}'$ by a finite sequence of steps each consisting of moving one pseudoline continuously across the intersection of two others (triangle flip). If $\mathcal{A}$ and $\mathcal{A}'$ are simple arrangements of lines, this can be done within the space of line arrangements.

The second part of Theorem 5.5.8 has been generalized by Roudneff and Sturmfels [RS88] to arrangements of planes; the first half is still open in higher dimensions.

Ringel also observed that the isotopy property does hold for pseudoline arrangements.

THEOREM 5.5.9 [Rin56]

If $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic arrangements of pseudolines, then $\mathcal{A}$ can be deformed continuously to $\mathcal{A}'$ through isomorphic arrangements.

Ringel did not provide a proof of this observation, but one method of proving it is via Theorem 5.2.7, together with the following isotopy result.

THEOREM 5.5.10 [GP84]

Every arrangement of pseudolines can be continuously deformed (through isomorphic arrangements) to a wiring diagram.

5.6 COMPLEXITY ISSUES

THE NUMBER OF ARRANGEMENTS

Various exact values, as well as bounds, are known for the number of equivalence classes of the structures discussed in this chapter. Early work in this direction is documented in [Grü72, GP80a, Ric89, Knu92, Fel97, BLS$^+$99]. Table 5.6.1 shows the values for $n \leq 11$, additional values are known for rows 3, 6, and 7.

TABLE 5.6.1 Exact numbers known for low n.

	3	4	5	6	7	8	9	10	11	
Arr's of n lines	1	2	4	17	143	4890	460779			[FMM13]
Simple arr's of n lines	1	1	1	4	11	135	4381	312114	41693377	[AK07]
Simplicial arr's of n lines	1	1	1	2	2	2	2	4	2	[Grü09]
Arr's of n pseudolines	1	2	4	17	143	4890	461053	95052532		[Fin]
Simple arr's of n p'lines	1	1	1	4	11	135	4382	312356	41848591	[AK07]
Simplicial arr's of n p'lines	1	1	1	2	2	2	2	4	2	[Cun12]
Simple Eucl. arr's of n p'lines	2	8	62	908	24698	1232944	112018190	18410581880	5449192389984	[KSYM11]
Simple Eucl. config's	1	2	3	16	135	3315	158817	14309547	2334512907	[AAK01]
Simple Eucl. gen'd config's	1	2	3	16	135	3315	158830	14320182	2343203071	[AAK01]

The only exact formula known for arbitrary n follows from Stanley's formula:

THEOREM 5.6.1 [Sta84]

The number of simple allowable sequences on $1, \ldots, n$ containing the permutation $123 \ldots n$ is

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}.$$

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For n arbitrary, Table 5.6.2 indicates the known asymptotic bounds.

TABLE 5.6.2 Asymptotic bounds for large n (all logarithms are base 2).

EQUIVALENCE CLASS	LOWER BOUND	UPPER BOUND	
Isom classes of simple (labeled) arr's of n p'lines	$2^{.1887n^2}$	$2^{.6571n^2}$	[FV11]
Isom classes of (labeled) arr's of n p'lines	"	$2^{1.0850n^2}$	[BLS+99, p. 270]
Order types of (labeled) n pt configs (simple or not)	$2^{4n\log n+\Omega(n)}$	$2^{4n\log n+O(n)}$	[GP93, p. 122]
Comb'l equiv classes of (labeled) n pt configs	$2^{7n\log n}$	$2^{8n\log n}$	[GP93, p. 123]

CONJECTURE 5.6.2 [Knu92]

The number of isomorphism classes of simple pseudoline arrangements is $2^{\binom{n}{2}+o(n^2)}$.

Consider simple Euclidean arrangements with $n+1$ pseudolines. Removing pseudoline $n+1$ from such an arrangement yields an arrangement of n pseudolines, the *derived* arrangement. Call $\mathcal{A}$ and $\mathcal{A}'$ equivalent if they have the same derived arrangement. Let γ_n be the maximum size of an equivalence class of arrangements with $n+1$ pseudolines. Dually, γ_n is the maximum number of extensions of an arrangement of n pseudolines. Knuth proved that $\gamma_n \leq 3^n$ and suggested that $\gamma_n \leq n2^n$ might be true. This inequality would imply Conjecture 5.6.2. In the context of social choice theory, $\gamma_n \leq n2^n$ was also conjectured by Fishburn and by Galambos and Rainer. This conjecture was disproved by Ondřej Bílka in 2010. The current bounds on γ_n are $2.076^n \leq \gamma_n \leq 4n\, 2.487^n$, see [FV11].

HOW MUCH SPACE IS NEEDED TO SPECIFY AN ARRANGEMENT?

Allowable sequences of $1,\dots,n$ can be encoded with $O(n^2 \log n)$ bits. This can be done via the *balanced tableaux* of Edelman and Greene or by encoding the ends of the interval that is reversed between consecutive permutations with $\leq 2\log n$ bits.

Given their asymptotic number, wiring diagrams should be encodable with $O(n^2)$ bits. The next theorem shows that this is indeed possible.

THEOREM 5.6.3 [Fel97]

Given a wiring diagram $\mathcal{A} = \{L_1,\dots,L_n\}$, let $t^i_j = 1$ if the jth crossing along L_i is with L_k for $k > i$, 0 otherwise. Then the mapping that associates to each wiring diagram $\mathcal{A}$ the binary $n \times (n-1)$ matrix (t^i_j) is injective.

The number of stretchable pseudoline arrangements is much smaller than the total number, which suggests that it could be possible to encode these more compactly. The following result of Goodman, Pollack, and Sturmfels (stated here for the dual case of point configurations) shows, however, that the "naive" encoding, by coordinates of an integral representative, is doomed to be inefficient.

THEOREM 5.6.4 [GPS89]

For each configuration $\mathcal{S}$ of points (x_i, y_i), $i = 1,\dots,n$, in the integer grid $\mathbb{Z}^2$, let

$$\nu(\mathcal{S}) = \min\max\{|x_1|,\dots,|x_n|,|y_1|,\dots,|y_n|\},$$

the minimum being taken over all configurations $\mathcal{S}'$ of the same order type as $\mathcal{S}$, and *let* $\nu^*(n) = \max \nu(\mathcal{S})$ *over all* n*-point configurations. Then, for some* $c_1, c_2 > 0$*,*

$$2^{2^{c_1 n}} \leq \nu^*(n) \leq 2^{2^{c_2 n}}.$$

5.7 APPLICATIONS

Planar arrangements of lines and pseudolines, as well as point configurations, arise in many problems of computational geometry. Here we describe several such applications involving pseudolines in particular.

GLOSSARY

Pseudoline graph: Given a Euclidean pseudoline arrangement Γ and a subset E of its vertices, the graph $G = (\Gamma, E)$ whose vertices are the members of Γ, with two vertices joined by an edge whenever the intersection of the corresponding pseudolines belongs to E.

Extendible set of pseudosegments: A set of Jordan arcs, each chosen from a different pseudoline belonging to a simple Euclidean arrangement.

TOPOLOGICAL SWEEP

The original idea behind what has come to be known as *topologically sweeping an arrangement* was applied, by Edelsbrunner and Guibas, to the case of an arrangement of straight lines. In order to construct the arrangement, rather than using a line to sweep it, they used a pseudoline, and achieved a saving of a factor of $\log n$ in the time required, while keeping the storage linear.

THEOREM 5.7.1 [EG89]

The cell complex of an arrangement of n *lines in the plane can be computed in* $O(n^2)$ *time and* $O(n)$ *working space by sweeping a pseudoline across it.*

This result can be applied to a number of problems, and results in an improvement of known bounds on each: minimum area triangle spanned by points, visibility graph of segments, and (in higher dimensions) enumerating faces of a hyperplane arrangement and testing for degeneracies in a point configuration.

The idea of a topological sweep was then generalized, by Snoeyink and Hershberger, to sweeping a pseudoline across an arrangement of *pseudolines*; they prove the possibility of such a sweep (Theorem 5.5.5), and show that it can be performed in the same time and space as in Theorem 5.7.1. They also apply this result to finding a short Boolean formula for a polygon with curved edges.

The topological sweep method was also used by Chazelle and Edelsbrunner [CE92] to report all k-segment intersections in an arrangement of n line segments in (optimal) $O(n \log n + k)$ time, and has been generalized to higher dimensions.

APPLICATIONS OF DUALITY

Theorem 5.1.6, and the algorithm used to compute the dual arrangement, are used by Agarwal and Sharir to compute incidences between points and pseudolines and to compute a subset of faces in a pseudoline arrangement [AS05]. An additional application is due to Sharir and Smorodinsky. Define a ***diamond*** in a Euclidean arrangement as two pairs $\{l_1, l_2\}, \{l_3, l_4\}$ of pseudolines such that the intersection of one pair lies above each member of the second and the intersection of the other pair below each member of the first.

THEOREM 5.7.2 [SS03]

Let Γ be a simple Euclidean pseudoline arrangement, E a subset of vertices of Γ, and $G = (\Gamma, E)$ the corresponding pseudoline graph. Then there is a drawing of G in the plane, with the edges constituting an extendible set of pseudosegments, such that for any two edges e, e' of G, e and e' form a diamond if and only if their corresponding drawings cross.

Conversely, for any graph $G = (V, E)$ drawn in the plane with its edges constituting an extendible set of pseudosegments, there is a simple Euclidean arrangement Γ of pseudolines and a one-to-one mapping ϕ from V onto Γ with each edge $uv \in E$ mapped to the vertex $\phi(u) \cap \phi(v)$ of Γ, such that two edges in E cross if and only if their images are two vertices of Γ forming a diamond.

This can then be used to provide a simple proof of the Tamaki-Tokuyama theorem:

THEOREM 5.7.3 [TT03]

Let Γ and G be as in Theorem 5.7.2. *If Γ is diamond-free, then G is planar, and hence $|E| \leq 3n - 6$.*

PSEUDOTRIANGULATIONS

A ***pseudotriangle*** is a simple polygon with exactly three convex vertices, and a ***pseudotriangulation*** is a tiling of a planar region into pseudotriangles. Pseudotriangulations first appeared as subdivisions of a polygon obtained by adding a collection of noncrossing geodesic paths between vertices of the polygon. The name pseudotriangulation was coined by Pocchiola and Vegter [PV94]. They discovered them when studying the visibility graph of a collection of pairwise disjoint convex obstacles.

Pseudotriangulations have an interesting connection with arrangements of pseudolines: A pseudotriangle in $\mathbb{R}^2$ has a unique interior tangent parallel to each direction. Dualizing the supporting lines of these tangents of a pseudotriangle we obtain the points of an x-monotone curve. Now consider a collection of pseudotriangles with pairwise disjoint interiors (for example those of a pseudotriangulation). Any two of them have exactly one common interior tangent, so the dual pseudolines form a pseudoline arrangement. Every line arrangement can be obtained with this construction. Pocchiola and Vegter also show that some nonstretchable arrangements can be obtained.

Streinu [Str05] studied minimal pseudotriangulations of point sets in the context of her algorithmic solution of the *Carpenter's Rule problem* previously settled existentially by Connelly, Demaine, and Rote [CDR00]. She obtained the following characterization of minimal pseudotriangulations

THEOREM 5.7.4 [Str05]

The following properties are equivalent for a geometric graph T on a set P of n vertices.

(i) *T is a pseudotriangulation of P with the minimum possible number of edges, i.e., a minimal pseudotriangulation.*

(ii) *Every vertex of T has an angle of size $> \pi$, i.e., T is a* pointed *pseudotriangulation.*

(iii) *T is a pseudotriangulation of P with $2n-3$ edges.*

(iv) *T is noncrossing, pointed, and has $2n-3$ edges.*

There has been a huge amount of research related to pseudotriangulations. Aichholzer et al. proved Conjecture 5.7.7 from the previous edition of this chapter, see [RSS08, Thm. 3.7]. A detailed treatment and an ample collection of references can be found in the survey article of Rote, Santos, and Streinu [RSS08]. We only mention some directions of research on pseudotriangulations:

- Polytopes of pseudotriangulations.
- Combinatorial properties, flips of pseudotriangulations and enumeration.
- Connections with rigidity theory.
- Combinatorial pseudotriangulations and their stretchability.

Pilaud and Pocciola [PP12] extended the correspondence between pseudoline arrangements and pseudotriangulations to some other classes of geometric graphs. The ***k*-kernel** of a Euclidean arrangement of pseudolines is obtained by deleting the first k and the last k levels of the arrangement. If the arrangement is given by a wiring diagram this corresponds to deleting the first k and the last k horizontal lines.

Call an arrangement $\mathcal{B}$ of $n-2k$ pseudolines a ***k*-descendant** of an arrangement $\mathcal{A}$ of n pseudolines if $\mathcal{B}$ can be drawn on the lines of the k-kernel of $\mathcal{A}$ such that at a crossing of the k-kernel the lines of $\mathcal{B}$ either cross or touch each other, Figure 5.7.1 shows an example.

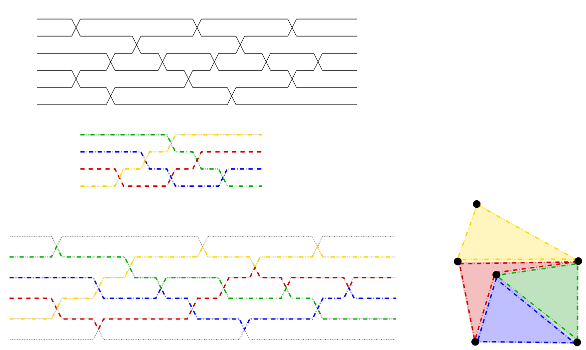

FIGURE 5.7.1
Arrangements $\mathcal{A}$ and $\mathcal{B}$ of 6 and 4 pseudolines, resp., and a drawing of $\mathcal{B}$ on the 1-kernel of $\mathcal{A}$. It corresponds to the pointed pseudotriangulation on the dual point set of $\mathcal{A}$.

THEOREM 5.7.5 [PP12]

(i) *For a point set P and its dual arrangement $\mathcal{A}_P$ there is a bijection between pointed pseudotriangulations of P and 1-descendants of $\mathcal{A}_P$.*

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(ii) *For a set C_n in convex position and its dual arrangement $\mathcal{C}_n$ (a cyclic arrangement) there is a bijection between k-triangulations of C_n and k-descendants of $\mathcal{C}_n$.*

Both parts of the theorem can be proven by showing that the objects on both sides of the bijection have corresponding flip-structures. Pilaud and Pocciola also consider the k-descendants of the dual arrangement of nonconvex point sets; their duals are *k-pseudotriangulations*.

Another generalization relates pseudotriangulations of the free space of a set of disjoint convex bodies in the plane with k-descendants of arrangements of *double pseudolines*. This is based on the duality between sets of disjoint convex bodies and arrangements of double pseudolines in the projective plane. A thorough study of this duality can be found in [HP13].

PSEUDOPOLYGONS

A polygon is a cyclic sequence of vertices and noncrossing edges on a configuration of points. Similarly a ***pseudopolygon*** is based on vertices and edges taken from generalized configuration of points (see Section 5.2), O'Rourke and Streinu studied pseudopolygons in the context of visibility problems. They prove

THEOREM 5.7.6 [OS96]

There is a polynomial-time algorithm to decide whether a graph is realizable as the vertex-edge pseudo-visibility graph of a pseudopolygon.

The recognition problem for vertex-edge visibility graphs of polygons is likely to be hard. So this is another instance where relaxing a problem from straight to pseudo helps.

5.8 SOURCES AND RELATED MATERIAL

FURTHER READING

[BLS+99]: A comprehensive account of oriented matroid theory, including a great many references; most references not given explicitly in this chapter can be traced through this book.

[Ede87]: An introduction to computational geometry, focusing on arrangements and their algorithms.

[Fel04] Covers combinatorial aspects of arrangements. Also includes a chapter on pseudotriangulations.

[GP91, GP93]: Two surveys on allowable sequences and order types and their complexity.

[Grü72]: A monograph on planar arrangements and their generalizations, with excellent problems (many still unsolved) and a very complete bibliography up to 1972.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 4: Helly-type theorems and geometric transversals
Chapter 6: Oriented matroids
Chapter 28: Arrangements
Chapter 37: Computational and quantitative real algebraic geometry

REFERENCES

[AAK01] O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small point sets with applications. *Order* 19:265–281, 2002. See also `www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes`

[ACH+14] O. Aichholzer, J. Cardinal, T. Hackl, F. Hurtado, M. Korman, A. Pilz, R.I. Silveira, R. Uehara, P. Valtr, B. Vogtenhuber, and E. Welzl. Cell-paths in mono- and bichromatic line arrangements in the plane. *Disc. Math. & Theor. Comp. Sci.*, 16:317–332, 2014.

[AK07] O. Aichholzer and H. Krasser. Abstract order type extension and new results on the rectilinear crossing number. *Comput. Geom.* 36:2–15, 2007.

[APP+14] E. Ackerman, J. Pach, R. Pinchasi, R. Radoičić, and G. Tóth. A note on coloring line arrangements. *Electr. J. Comb.*, 21:P2.23, 2014.

[AS05] P.K. Agarwal and M. Sharir. Pseudo-line arrangements: duality, algorithms, and applications. *SIAM J. Comput.*, 34:526–552, 2005.

[AZ99] M. Aigner and G.M. Ziegler. *Proofs from THE BOOK*, 5th edition. Springer, Berlin, 2014.

[BBL08] N. Bartholdi, J. Blanc, and S. Loisel. On simple arrangements of lines and pseudo-lines in $\mathbb{P}^2$ and $\mathbb{R}^2$ with the maximum number of triangles. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, pages 105–116, AMS, Providence, 2008.

[BCC+13] P. Bose, J. Cardinal, S. Collette, F. Hurtado, M. Korman, S. Langerman, and P. Taslakian. Coloring and guarding arrangements. *Disc. Math. Theor. Comp. Sci.*, 15:139–154, 2013.

[BEPY91] M. Bern, D. Eppstein, P. Plassmann, and F. Yao. Horizon theorems for lines and polygons. In J.E. Goodman, R. Pollack, and W. Steiger, editors, *Discrete and Computational Geometry: Papers from the DIMACS Special Year*, pages 45–66, AMS, Providence, 1991.

[Ber08] L.W. Berman. Symmetric simplicial pseudoline arrangements. *Electr. J. Comb.*, 15:R13, 2008.

[Bla11] J. Blanc. The best polynomial bounds for the number of triangles in a simple arrangement of n pseudo-lines. *Geombinatorics*, 21:5–17, 2011.

[BLS+99] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler. *Oriented Matroids*. 2nd edition, vol. 46 of *Encyclopedia of Mathematics*, Cambridge University Press, 1999.

[Bok08] J. Bokowski. On heuristic methods for finding realizations of surfaces. In Bobenko et al., editors, *Discrete Differential Geometry*, vol. 38 of *Oberwolfach Seminars*, pages 255–260, Birkhäuser, 2008.

[BRS+05] J. Balogh, O. Regev, C. Smyth, W. Steiger, and M. Szegedy. Long monotone paths in line arrangements. *Discrete Comput. Geom.*, 32:167–176, 2005.

[BS89a] J. Bokowski and B. Sturmfels. An infinite family of minor-minimal nonrealizable 3-chirotopes. *Math. Zeitschrift*, 200:583–589, 1989.

[BS89b] J. Bokowski and B. Sturmfels. *Computational Synthetic Geometry*. Vol. 1355 of *Lecture Notes in Math.*, Springer, Berlin, 1989.

[Can69] R.J. Canham. A theorem on arrangements of lines in the plane. *Israel J. Math.*, 7:393–397, 1969.

[CDR00] R. Connelly, E.D. Demaine, and G. Rote. Straightening polygonal arcs and convexifying polygonal cycles. *Proc. 41st IEEE Sympos. Found. Comput. Sci.*, pages 432–442, 2000.

[CE92] B. Chazelle and H. Edelsbrunner. An optimal algorithm for intersecting line segments in the plane. *J. ACM* 39:1–54, 1992.

[CEG+90] K.L. Clarkson, H. Edelsbrunner, L.J. Guibas, M. Sharir, and E. Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.*, 5:99–160, 1990.

[Cun12] M. Cuntz. Simplicial arrangements with up to 27 lines. *Discrete Comput. Geom.*, 48:682–701, 2012.

[DP98] T.K. Dey and J. Pach. Extremal problems for geometric hypergraphs. *Discrete Comput. Geom.*, 19:473–484, 1998.

[Dum05] A. Dumitrescu. Monotone paths in line arrangements with a small number of directions. *Discrete Comput. Geom.*, 33:687–697, 2005.

[Ede87] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer, Berlin, 1987.

[EG89] H. Edelsbrunner and L.J. Guibas. Topologically sweeping an arrangement. *J. Comput. System Sci.*, 38:165–194, 1989; Corrigendum: 42:249–251, 1991.

[EP95] P. Erdős and G. Purdy. Extremal problems in combinatorial geometry. In Grötschel et al., editors, *Handbook of Combinatorics*, vol. 1, Elsevier, Amsterdam, 1995.

[Epp14] D. Eppstein. Drawing arrangement graphs in small grids, or how to play planarity. *J. Graph Algorithms Appl.*, 18:211–231, 2014.

[EW86] H. Edelsbrunner and E. Welzl. On the maximal number of edges of many faces in arrangements. *J. Combin. Theory Ser. A*, 41:159–166, 1986.

[Fel97] S. Felsner. On the number of arrangements of pseudolines. *Discrete Comput. Geom.*, 18:257–267, 1997.

[Fel04] S. Felsner. *Geometric Graphs and Arrangements*. Vieweg, Wiesbaden, 2004.

[FHNS00] S. Felsner, F. Hurtado, M. Noy, and I. Streinu. Hamiltonicity and colorings of arrangement graphs. *Discrete Appl. Math.*, 154:2470–2483, 2006.

[Fin] L. Finschi. Homepage of Oriented Matroids. `http://www.om.math.ethz.ch`.

[FK99] S. Felsner and K. Kriegel. Triangles in Euclidean arrangements. *Discrete Comput. Geom.*, 22:429–438, 1999.

[FMM13] K. Fukuda, H. Miyata, and S. Moriyama. Complete enumeration of small realizable oriented matroids. *Discrete Comput. Geom.*, 49:359–381, 2013.

[FR98] D. Forge and J.L. Ramírez Alfonsín. Straight line arrangements in the real projective plane. *Discrete Comput. Geom.*, 20:155–161, 1998.

[FR01] D. Forge and J.L. Ramírez Alfonsín. On counting the k-face cells of cyclic arrangements. *European. J. Combin.*, 22:307–312, 2001.

[FV11] S. Felsner and P. Valtr. Coding and counting arrangements of pseudolines. *Discrete Comput. Geom.*, 46:405–416, 2011.

[FW01] S. Felsner and H. Weil. Sweeps, arrangements, and signotopes. *Discrete Appl. Math.*, 109:67–94, 2001.

[FZ01] S. Felsner and G.M. Ziegler. Zonotopes associated with higher Bruhat orders. *Discrete Math.*, 241:301–312, 2001.

[Goo80] J.E. Goodman. Proof of a conjecture of Burr, Grünbaum, and Sloane. *Discrete Math.*, 32:27–35, 1980.

[GP80a] J.E. Goodman and R. Pollack. On the combinatorial classification of nondegenerate configurations in the plane. *J. Combin. Theory Ser. A*, 29:220–235, 1980.

[GP80b] J.E. Goodman and R. Pollack. Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines. *J. Combin. Theory Ser. A*, 29:385–390, 1980.

[GP81a] J.E. Goodman and R. Pollack. A combinatorial perspective on some problems in geometry. *Congressus Numerantium*, 32:383–394, 1981.

[GP81b] J.E. Goodman and R. Pollack. Three points do not determine a (pseudo-) plane. *J. Combin. Theory Ser. A*, 31:215–218, 1981.

[GP82a] J.E. Goodman and R. Pollack. Helly-type theorems for pseudoline arrangements in P^2. *J. Combin. Theory Ser. A*, 32:1–19, 1982.

[GP82b] J.E. Goodman and R. Pollack. A theorem of ordered duality. *Geometriae Dedicata*, 12:63–74, 1982.

[GP84] J.E. Goodman and R. Pollack. Semispaces of configurations, cell complexes of arrangements. *J. Combin. Theory Ser. A*, 37:257–293, 1984.

[GP85] J.E. Goodman and R. Pollack. A combinatorial version of the isotopy conjecture. *Ann. New York Acad. Sci.*, 440:31–33, 1985.

[GP91] J.E. Goodman and R. Pollack. The complexity of point configurations. *Discrete Appl. Math.*, 31:167–180, 1991.

[GP93] J.E. Goodman and R. Pollack. Allowable sequences and order types in discrete and computational geometry. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, vol. 10 of *Algorithms Combin.*, pages 103–134, Springer, Berlin, 1993.

[GPS89] J.E. Goodman, R. Pollack, and B. Sturmfels. Coordinate representation of order types requires exponential storage. *Proc. 21th Sympos. Comput. Geom.*, pages 405–410, ACM Press, 1989.

[GPW96] J.E. Goodman, R. Pollack, and R. Wenger. There are uncountably many universal topological planes. *Geom. Dedicata*, 59:157–162, 1996.

[GPWZ94] J.E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. *Amer. Math. Monthly*, 101:866–878, 1994.

[Grü71] B. Grünbaum. Arrangements of hyperplanes. *Congressus Numerantium*, 3:41–106, 1971.

[Grü72] B. Grünbaum. *Arrangements and Spreads.* Volume 10 of *CBMS Regional Conf. Ser. in Math.* AMS, Providence, 1972.

[Grü09] B. Grünbaum. A catalogue of simplicial arrangements in the real projective plane. *Ars Math. Contemp.*, 2:1–25, 2009.

[Grü13] B. Grünbaum. Simplicial arrangements revisited. *Ars Math. Contemp.*, 6:419–433, 2013.

[Gut16] L. Guth. *Polynomial Methods in Combinatorics.* AMS, Providence, 2016.

[Har85] H. Harborth. Some simple arrangements of pseudolines with a maximum number of triangles. *Ann. New York Acad. Sci.*, 440:31–33, 1985.

[HKM15] U. Hoffmann, L. Kleist, and T. Miltzow. Upper and lower bounds on long dual paths in line arrangements. In *Proc. 40th Sympos. Math. Found. Comp. Sci.*, vol. 9235 of *LNCS*, pages 407–419, Springer, Berlin, 2015.

[HM02] H. Harborth and M. Möller. An arrangement graph without a Hamiltonian cycle decomposition. *Bull. Inst. Combin. Appl.*, 34:7–9, 2002.

[HP13] L. Habert and M. Pocchiola. LR characterization of chirotopes of finite planar families of pairwise disjoint convex bodies. *Discrete Comput. Geom.*, 50:552–648, 2013.

[Jam85] R.E. Jamison. A survey of the slope problem. *Ann. New York Acad. Sci.*, 440: 34–51, 1985.

[Knu92] D.E. Knuth. *Axioms and Hulls.* Vol. 606 of *LNCS*, Springer, Berlin, 1992.

[KSYM11] J. Kawahara, T. Saitoh, R. Yoshinaka, and S. Minato. Counting primitive sorting networks by πDDs. Technical Report, Hokkaido University, TCS-TR-A-11-54, 2011. `http://www-alg.ist.hokudai.ac.jp/~thomas/TCSTR/tcstr_11_54/tcstr_11_54.pdf`

[LLM+07] J. Leaños, M. Lomelí, C. Merino, G. Salazar, and J. Urrutia. Simple Euclidean arrangements with no (≥ 5)-gons. *Discrete Comput. Geom.*, 38:595–603, 2007.

[Lev26] F. Levi. Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade. *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss.*, 78:256–267, 1926.

[Lom90] H. Lombardi. Nullstellensatz réel effectif et variantes. *C. R. Acad. Sci. Paris, Sér. I*, 310:635–640, 1990.

[LPS14] B. Lund, G.B. Purdy, and J.W. Smith. A pseudoline counterexample to the strong dirac conjecture. *Electr. J. Comb.*, 21:P2.31, 2014.

[LRS89] D. Ljubić, J.-P. Roudneff, and B. Sturmfels. Arrangements of lines and pseudolines without adjacent triangles. *J. Combin. Theory Ser. A*, 50:24–32, 1989.

[Mar93] N. Martinov.Classification of arrangements by the number of their cells. *Discrete Comput. Geom.*, 9:39–46, 1993.

[Mat91] J. Matoušek. Lower bounds on the length of monotone paths in arrangements. *Discrete Comput. Geom.*, 6:129–134, 1991.

[Mel41] E. Melchior. Über Vielseite der projektiven Ebene. *Deutsche Math.*, 5:461–475, 1941.

[Mnë85] N.E. Mnëv. On manifolds of combinatorial types of projective configurations and convex polyhedra. *Soviet Math. Dokl.*, 32:335–337, 1985.

[Mnë88] N.E. Mnëv. The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In O.Y. Viro, editor, *Topology and Geometry—Rohlin Seminar*, vol. 1346 of *Lecture Notes in Math.*, pages 527–544, Springer, Berlin, 1988.

[Nym04] K. Nyman. Linear inequalities for rank 3 geometric lattices. *Discrete Comput. Geom.*, 31:229–242, 2004.

[OS96] J. O'Rourke and I. Streinu. Pseudo-visibility graphs in pseudo-polygons: Part II. Tech. Report 042, Dept. Comput. Sci., Smith College, Northampton, 1996.

[Pin11] R. Pinchasi. The zone theorem revisited. Manuscript, `http://www2.math.technion.ac.il/~room/ps_files/zonespl.pdf`, 2011.

[PP01] J. Pach and R. Pinchasi. On the number of balanced lines. *Discrete Comput. Geom.*, 25:611–628, 2001.

[PP12] V. Pilaud and M. Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. *Discrete Comput. Geom.*, 48:142–191, 2012.

[PS09] J. Pach and M. Sharir. *Combinatorial Geometry and its Algorithmic Applications: The Alcalá Lectures.* vol. 152 of *Mathematical Surveys and Monographs*, AMS, Providence, 2009.

[PV94] M. Pocchiola and G. Vegter. Order types and visibility types of configurations of disjoint convex plane sets. Extended abstract, Tech. Report 94-4, Labo. d'Inf. de l'ENS, Paris, 1994.

[Ric89] J. Richter. Kombinatorische Realisierbarkeitskriterien für orientierte Matroide. *Mitt. Math. Sem. Gießen*, 194:1–112, 1989.

[Ric96a] J. Richter-Gebert. *Realization Spaces of Polytopes.* Vol. 1643 of *Lecture Notes in Math.*, Springer, Berlin, 1996.

[Ric96b] J. Richter-Gebert. Two interesting oriented matroids. *Documenta Math.*, 1:137–148, 1996.

[Rin56] G. Ringel. Teilungen der Ebene durch Geraden oder topologische Geraden. *Math. Z.*, 64:79–102, 1956.

[Rin57] G. Ringel. Über Geraden in allgemeiner Lage. *Elem. Math.*, 12:75–82, 1957.

[Rou87] J.-P. Roudneff. Quadrilaterals and pentagons in arrangements of lines. *Geom. Dedicata*, 23:221–227, 1987.

[Rou88a] J.-P. Roudneff. Arrangements of lines with a minimal number of triangles are simple. *Discrete Comput. Geom.*, 3:97–102, 1988.

[Rou88b] J.-P. Roudneff. Tverberg-type theorems for pseudoconfigurations of points in the plane. *European J. Combin.*, 9:189–198, 1988.

[Rou96] J.-P. Roudneff. The maximum number of triangles in arrangements of (pseudo-) lines. *J. Combin. Theory Ser. B*, 66:44–74, 1996.

[RS88] J.-P. Roudneff and B. Sturmfels. Simplicial cells in arrangements and mutations of oriented matroids. *Geom. Dedicata*, 27:153–170, 1988.

[RSS08] G. Rote, F. Santos, and I. Streinu. Pseudo-triangulations–a survey. In J.E. Goodman J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, pages 343–410, AMS, Providence, 2008.

[RT03] R. Radoičić and G. Tóth. Monotone paths in line arrangements. *Comput. Geom.*, 24:129–134, 2003.

[SE88] P. Salamon and P. Erdős. The solution to a problem of Grünbaum. *Canad. Math. Bull.*, 31:129–138, 1988.

[SH91] J. Snoeyink and J. Hershberger. Sweeping arrangements of curves. In J.E. Goodman, R. Pollack, and W. Steiger, editors, *Discrete and Computational Geometry: Papers from the DIMACS Special Year*, vol. 6 of *DIMACS Series in Discrete Math. Theor. Comp. Sci.*, pages 309–349, AMS, Providence, 1991.

[Sha79] R.W. Shannon. Simplicial cells in arrangements of hyperplanes. *Geom. Dedicata*, 8:179–187, 1979.

[Sho91] P. Shor. Stretchability of pseudolines is NP-hard. In P. Gritzmann and B. Sturmfels, editors, *Applied Geometry and Discrete Mathematics—The Victor Klee Festschrift*, volume 4 of *DIMACS Series in Discrete Math. Theor. Comp. Sci.*, pages 531–554, AMS, Providence, 1991.

[SS94] W. Steiger and I. Streinu. A pseudo-algorithmic separation of lines from pseudo-lines. *Inf. Process. Lett.*, 53:295–299, 1995.

[SS03] M. Sharir and S. Smorodinsky. Extremal configurations and levels in pseudoline arrangements. In *Proc. 8th Workshop on Algorithms and Data Structures*, vol. 2748 of *LNCS*, pages 127–139, Springer, Berlin, 2003.

[ST83] E. Szemerédi and W.T. Trotter. Extremal problems in discrete geometry. *Combinatorica*, 3:381–392, 1983.

[Sta84] R.P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5:359–372, 1984.

[Str97] I. Streinu. Clusters of stars. *Proc. 13th Sympos. on Comput. Geom.*, pages 439–441, ACM Press, 1997.

[Str05] I. Streinu. Pseudo-triangulations, rigidity and motion planning. *Discrete Comput. Geom.*, 34:587–635, 2005.

[Suv88] P.Y. Suvorov. Isotopic but not rigidly isotopic plane systems of straight lines. In O.Ya. Viro and A.M. Varshik, editors, *Topology and Geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 545–556, Springer, Berlin, 1988.

[Szé97] L.A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 6:353–358, 1997.

[TT03] H. Tamaki and T. Tokuyama. A characterization of planar graphs by pseudo-line arrangements. *Algorithmica*, 35:269–285, 2003.

[Ung82] P. Ungar. $2N$ noncollinear points determine at least $2N$ directions. *J. Combin. Theory Ser. A*, 33:343–347, 1982.

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