# 41 RAY SHOOTING AND LINES IN SPACE

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## INTRODUCTION

The geometry of lines in 3-space has been a part of the body of classical algebraic geometry since the pioneering work of Plücker. Interest in this branch of geometry has been revived by several converging trends in computer science. The discipline of computer graphics (Chapter 52) has pursued the task of rendering realistic images by simulating the flow of light within a scene according to the laws of elementary optical physics. In these models light moves along straight lines in 3-space and a computational challenge is to find efficiently the intersections of a very large number of rays with the objects comprising the scene. In robotics (Chapters 50 and 51) the chief problem is that of moving 3D objects without collisions. Effects due to the edges of objects have been studied as a special case of the more general problem of representing and manipulating lines in 3-space. Computational geometry (whose core is better termed "design and analysis of geometric algorithms") has moved in the nineties from the realm of planar problems to tackling directly problems that are specifically 3D. The new and sometimes unexpected computational phenomena generated by lines (and segments) in 3-space have emerged as a main focus of research.

In this chapter we will survey the present state of the art on lines and ray shooting in 3-space from the point of view of computational geometry. The emphasis is on provable nontrivial bounds for the time and storage used by algorithms for solving natural problems on lines, rays, and polyhedra in 3-space. We start by mentioning different possible choices of coordinates for lines (Section 41.1. This is an essential initial step because different coordinates highlight different properties of the lines in their interaction with other geometric objects. Here a special role is played by *Plücker coordinates* [Plu65], which represent the starting point for many results. Then we consider how lines interact with each other (Section 41.2). We are given a finite set of lines $L$ that act as obstacles and we will define other (infinite) sets of lines induced by $L$ that capture some of the important properties of visibility and motion problems. We show bounds on the storage required for a complete description of such sets. Then we move a step forward by considering the same sets of lines when the obstacles are polyhedral sets, more commonly encountered in applications. We arrive in Section 41.3 at the ray-shooting problem and its variants (on-line, off-line, arbitrary direction, fixed direction, and shooting with objects other than rays). Again, the obstacles are usually polyhedral objects, but in one case we are able to report a ray-shooting result on spheres.

Section 41.4 is devoted to the problem of collision-free movements (arbitrary or translation only) of lines among obstacles. This problem arises, for example, when lines are used to model radiation or light beams (e.g., lasers). In Section 41.5 we define a few notions of distance among lines, and as a consequence we have several natural proximity problems for lines in 3-space. Finding the closest pair in a set of lines is the most basic of such problems.



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In Section 41.6 we survey what is known about the "dominance" relation among lines. This relation is central for many visibility problems in graphics. It is used, for example, in the painter's algorithm for hidden surface removal (Chapter 52). Another direction of research has explored the relation between lines in 3-space and their orthogonal projections. A central topic here is realizability: Given a set of planar lines together with a relation, does there exist a corresponding set of lines in 3-space whose dominance is consistent with the given relation?

## 41.1 COORDINATES OF LINES

### GLOSSARY

***Homogeneous coordinates:*** A point $(x, y, z)$ in Cartesian coordinates has homogeneous coordinates $(x_0, x_1, x_2, x_3)$, where $x = x_1/x_0$, $y = x_2/x_0$, and $z = x_3/x_0$.

***Oriented lines:*** A line may have two distinct orientations. A line and an orientation form an oriented line.

***Unoriented line:*** A line for which an orientation is not distinguished.

**(I) Canonical coordinates by pairs of planes.** The intersection of two planes with equations $y = az+b$ and $x = cz+d$ is a nonhorizontal line in 3-space, uniquely defined by the four parameters $(a, b, c, d)$. Thus these parameters can be taken as coordinates of such lines. In fact, the space of nonhorizontal lines is homeomorphic to $\mathbb{R}^4$. Results on ray shooting among boxes and some lower bounds on stabbing are obtained using these coordinates.

**(II) Canonical coordinates by pairs of points.** Given two parallel horizontal planes, $z = 1$ and $z = 0$, the intersection points of a nonhorizontal line $l$ with the two planes uniquely define that line. If $(x_0, y_0, 0)$ and $(x_1, y_1, 1)$ are two such points for $l$, then the quadruple $(x_0, y_0, x_1, y_1)$ can be used as coordinates of $l$. Results on sets of horizontal polygons are obtained using these coordinates.

Although four is the minimum number of coordinates needed to represent an *unoriented* line, such parametrizations have proved useful only in special cases. Many interesting results have been derived using instead a five-dimensional parametrization for *oriented* lines, called ***Plücker coordinates***.

**(III) Plücker coordinates of lines**. An oriented line in 3-space can be given by the homogeneous coordinates of two of its points. Let $l$ be a line in 3-space and let $a = (a_0, a_1, a_2, a_3)$ and $b = (b_0, b_1, b_2, b_3)$ be two distinct points in homogeneous coordinates on $l$. We can represent the line $l$, oriented from $a$ to $b$, by the matrix

$$l = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}, \quad \text{with } a_0, b_0 > 0.$$

By taking the determinants of the six $2 \times 2$ submatrices of the above $2 \times 4$ matrix we obtain the ***homogeneous Plücker coordinates*** of the line:

$$p(l) = (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{12}, \xi_{31}, \xi_{23}), \quad \text{with } \xi_{ij} = det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}.$$

The six numbers $\xi_{ij}$ are interpreted as homogeneous coordinates of a point in 5-space. For a given line $l$ the six numbers are unique modulo a positive multiplicative factor, and they do not depend on the particular distinct points $a$ and $b$ that we have chosen on $l$. We call $p(l)$ the ***Plücker point*** of $l$ in real projective 5-dimensional space $\mathbb{P}^5$.

We also define the ***Plücker hyperplane*** of the line $l$ to be the hyperplane in $\mathbb{P}^5$ with vector of coefficients $v(l) = (\xi_{23}, \xi_{31}, \xi_{12}, \xi_{03}, \xi_{02}, \xi_{01})$. So the Plücker hyperplane is:

$$h(l) = \{p \in \mathbb{P}^5 \mid v(l) \cdot p = 0\}.$$

For each Plücker hyperplane we have a positive and a negative halfspace given by $h^+(l) = \{p \in \mathbb{P}^5 \mid v(l) \cdot p \geq 0\}$ and $h^-(l) = \{p \in \mathbb{P}^5 \mid v(l) \cdot p \leq 0\}$. Not every tuple of 6 real numbers corresponds to a line in 3-space since the Plücker coordinates must satisfy the condition

$$\xi_{01}\xi_{23} + \xi_{02}\xi_{31} + \xi_{03}\xi_{12} = 0. \tag{41.1.1}$$

The set of points in $\mathbb{P}^5$ satisfying Equation 41.1.1 forms the so-called ***Plücker hypersurface*** $\Pi$; it is also called the ***Klein quadric*** or the ***Grassmannian*** (manifold). The converse is also true: every tuple of six real numbers satisfying Equation 41.1.1 is the Plücker point of some line in 3-space. Given two lines $l$ and $l'$, they intersect or are parallel (i.e., they intersect at infinity) when the four defining points are coplanar. In this case the determinant of the $4\times4$ matrix formed by the 16 homogeneous coordinates of the four points is zero. In terms of Plücker coordinates we have the following basic lemmas.

### LEMMA 41.1.1

*Lines $l$ and $l'$ intersect or are parallel (meet at infinity) if and only if $p(l) \in h(l')$.*

Note that Equation 41.1.1 states in terms of Plücker coordinates the fact that any line always meets itself.

### LEMMA 41.1.2

*Let $l$ be an oriented line and $t$ a triangle in Cartesian 3-space with vertices $(p_0, p_1, p_2)$. Let $l_i$ be the oriented line through $(p_i, p_{i+1})$ (indices mod 3). Then $l$ intersects $t$ if and only if either $p(l) \in h^+(l_0) \cap h^+(l_1) \cap h^+(l_2)$ or $p(l) \in h^-(l_0) \cap h^-(l_1) \cap h^-(l_2)$.*

These two lemmas allow us to map combinatorial and algorithmic problems involving lines (and polyhedral sets) in 3-space into problems involving sets of hyperplanes and points in projective 5-space (Plücker space). The main advantage is that we can use the rich collection of results on the combinatorics of high dimensional arrangements of hyperplanes (see Chapter 28). The main drawback is that we are using five (nonhomogeneous) parameters, instead of four which is the minimum number necessary. This choice has a potential for increasing the time bounds of line algorithms. We are rescued by the following theorem:

### THEOREM 41.1.3 [APS93]

*Given a set $H$ of $n$ hyperplanes in 5-dimensional space, the complexity of the cells of the arrangement $\mathcal{A}(H)$ intersected by the Plücker hypersurface $\Pi$ (also called the **zone** of $\Pi$ in $\mathcal{A}(H)$) is $O(n^4 \log n)$.*

Although the entire arrangement $\mathcal{A}(H)$ can be of complexity $\Theta(n^5)$, if we are working only with Plücker points we can limit our constructions to the zone of $\Pi$, the complexity of which is one order of magnitude smaller. Theorem 41.1.3 is especially useful for deriving ray-shooting results.

The list of coordinatizations discussed in this section is by no means exhaustive. Other parametrizations are used, for example, in [Ame92] and [AAS97].

### A TYPICAL EXAMPLE

A typical example of the use of Plücker coordinates in 3D problems is the result for fast ray shooting among polyhedra (see Table 41.3.1). We triangulate the faces of the polyhedra and extend each edge to a full line. Each such line is mapped to a Plücker hyperplane. Lemma 41.1.2 guarantees that each cell in the resulting arrangement of Plücker hyperplanes contains Plücker points that pass through the same set of triangles. Thus to answer a ray-shooting query, we first locate the query Plücker point in the arrangement, and then search the list of triangles associated with the retrieved cell. This final step is accomplished using a binary search strategy when the polyhedra are disjoint. Theorem 41.1.3 guarantees that we need to build a point-location structure only for the zone of the Plücker hypersurface, thus saving an order of magnitude over general point-location methods for arrangements (see Sections 28.7 and 38.3).

## 41.2 SETS OF LINES IN 3-SPACE

With Plücker coordinates (III) to represent oriented lines, we can use the topology induced by the standard topology of 5-dimensional projective space $\mathbb{P}^5$ on $\Pi$ as a natural topology on sets of oriented lines. Using the four-dimensional coordinatizations (I) or (II), we can impose the standard topology of $\mathbb{R}^4$ on the set of nonhorizontal unoriented lines. Thus we can define the concepts of "neighborhood," "continuous path," "open set," "closed set," "boundary," "path-connected component," and so on, for the set $\mathcal{L}$ of lines in 3-space. The distinction between oriented lines and unoriented lines is mainly technical and the complexity bounds hold in either case.

### FAMILIES OF LINES INDUCED BY A FINITE SET OF LINES

### GLOSSARY

***Semialgebraic set:*** The set of all points that satisfy a Boolean combination of a finite number of algebraic constraints (equalities and inequalities) in the Cartesian coordinates of $\mathbb{R}^d$. See Chapter 37.

***Path-connected component:*** A maximal set of lines that can be connected by a path of lines, a continuous function from the interval $[0,1]$ to the space of lines.

***Positively-oriented lines:*** Oriented lines $l'_1$ and $l'_2$ on the $xy$-plane are posi-

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tively-oriented if the triple scalar product of vectors parallel to $l'_1$, $l'_2$, and the positive $z$-axis is positive.

***Consistently-oriented lines:*** An oriented line $l$ in 3-space is oriented consistently with a 3D set $L$ of oriented lines if the projection $l'$ of $l$ onto the $xy$-plane is positively-oriented with the projection of every line in $L$.

A finite set $L$ of $n$ lines in 3-space can be viewed as an obstacle to the free movement of other lines in 3-space. Many applications lead to defining families of lines with some special properties with respect to the fixed lines $L$. The resources used by algorithms for these applications are often bounded by the "complexity" of such families.

The boundary of a semialgebraic set in $\mathbb{R}^4$ is partitioned into a finite number of faces of dimension 0, 1, 2, and 3, each of which is also a semialgebraic set. The number of faces on the boundary of a semialgebraic set is the ***complexity*** of that set. The families of lines that we consider are represented in $\mathbb{R}^4$ by semialgebraic sets, with the coefficients of the corresponding algebraic constraints a function of the given finite set of lines $L$.

The set Miss($L$) consists of lines that do not meet any line in $L$. The sets Vert($L$) and Free($L$) consists of lines that may be translated to infinity without collision with lines in $L$. The basic complexities displayed in Table 41.2.1 are derived from [CEG+96, Pel94b, Aga94].

TABLE 41.2.1 Complexity of families of lines defined by lines.

| SET OF LINES | DEFINITION | COMPLEXITY |
|---|---|---|
| Miss($L$) | do not meet any line in $L$ | $\Theta(n^4)$ |
| 1 component of Miss($L$) | 1 path-connected component | $\Theta(n^2)$ |
| Vert($L$) | can be translated vertically to $\infty$ | $\Theta(n^3)$ |
| Free($L$) | can be translated to $\infty$ in some direction | $\Omega(n^3)$,$O(n^3 \log n)$ |
| VertCO($L$) | above $L$ and oriented consistently with $L$ | $\Theta(n^2)$ |

## MEMBERSHIP TESTS

Given $L$, we can build a data structure during a preprocessing phase so that when presented with a new (query) line $l$, we can decide efficiently whether $l$ is in one of the sets defined in the previous section. Such an algorithm implements a membership test for a group of lines. Table 41.2.2 shows the main results.

TABLE 41.2.2 Membership tests for families of lines defined by lines.

| SET OF LINES | QUERY TIME | PREPROC/STORAGE | SOURCE |
|---|---|---|---|
| Miss($L$) | $O(\log n)$ | $O(n^{4+\epsilon})$ | [Pel93b, AM93] |
| 1 component of Miss($L$) | $O(\log n)$ | $O(n^{2+\epsilon})$ | [Pel91] |
| Vert($L$), VertCO($L$) | $O(\log n)$ | $O(n^{2+\epsilon})$ | [CEG+96] |
| Free($L$) | $O(\log n)$ | $O(n^{3+\epsilon})$ | [Pel94b] |

## FAMILIES OF LINES INDUCED BY POLYHEDRA

### GLOSSARY

***ϵ*:** A positive real number, which we may choose arbitrarily close to zero for each algorithm or data structure. A caveat is that the multiplicative constant implicit in the big-$O$ notation depends on $\epsilon$ and its value increases when $\epsilon$ tends to zero.

***α*( · ):** The inverse of Ackermann's function. $\alpha(n)$ grows very slowly and is at most 4 for any practical value of $n$. See Section 28.10.

***β*( · ):** $\beta(n) = 2^{c\sqrt{\log n}}$ for a constant $c$. $\beta(\cdot)$ is a function that is smaller than any polynomial but larger than any polylogarithmic factor. Formally we have that for every $a, b > 0$, $\log^a n \leq \beta(n) \leq n^b$ for any $n \geq n_0(a, b)$.

***Polyhedral set P*:** A region of 3-space bounded by a collection of interior-disjoint vertices, segments, and planar polygons. We denote by $n$ the total number of vertices, edges, and faces.

***Star-shaped polyhedron*:** A polyhedron $P$ for which there exists a point $o \in P$ such that for every point $p \in P$, the open segment $op$ is contained in $P$.

***Terrain*:** When the star-shaped polyhedron is unbounded and $o$ is at infinity we obtain a terrain, a monotone surface (cf. Section 30.1).

***Horizontal polygons*:** Convex polygons contained in planes parallel to the $xy$-plane.

A collection of polyhedra in 3-space may act as obstacles limiting the collision-free movements of lines. Following the blueprint of the previous section, the complexity of some interesting families of lines induced by polyhedra are displayed in Table 41.2.3 (see [HS94, Pel94b, Aga94] and [AAKS04, GL10, Rub12]).

TABLE 41.2.3 Complexity of families of lines defined by polyhedra and spheres.

| SET OF LINES | DEFINITION | COMPLEXITY |
|---|---|---|
| Miss($P$) | do not meet polyhedron $P$ | $\Theta(n^4)$ |
| Vert($P$) | can be translated vertically to $\infty$ | $\Omega(n^3)$, $O(n^3\beta(n))$ |
| Free($P$) | can be translated to $\infty$ in some direction | $\Omega(n^3)$ |
| Miss($Q$), Free($Q$) | $Q$ star-shaped polyhedron or a terrain | $\Omega(n^2\alpha(n))$, $O(n^3 \log n)$ |
| Miss($U$) | $U$ a set of $n$ unit balls | $O(n^{3+\epsilon})$ |
| Miss($B$) | $B$ a set of $n$ balls | $\Omega(n^3)$, $O(n^{3+\epsilon})$ |
| Miss($H$) | $H$ a set of horizontal polygons with $n$ edges | $\Omega(n^2)$, $O(n^4)$ |

Similarly, we can define families of *3D segments* defined by polyhedra in 3D. The set of relatively open segments that miss $P$ is also a semialgebraic set, known as the ***3D Visibility skeleton*** (see [DDP97, Dur99] and [DDP02, Zha09]). Its combinatorial complexity is $\Theta(n^4)$. The visibility skeleton induced by $k$ possibly intersecting convex polyhedra of total size $n$ has $\Theta(n^2k^2)$ connected components [BDD$^+$07]. The Visibility skeleton induced by $n$ uniformly distributed unit spheres has linear expected complexity [DDE$^+$03].

## OPEN PROBLEMS

1. Find an almost cubic upper bound on the complexity of the group of lines Free($P$) for a polyhedron $P$.
2. Close the gap between the quadratic lower and the cubic upper bound for the group Free($T$) induced by a terrain $T$ (Table 41.2.3).

## SETS OF STABBING LINES

## GLOSSARY

***Stabber:*** A line $l$ that intersects every member of a collection $\mathcal{P} = \{P_1, ..., P_k\}$ of polyhedral sets. The sum of the sizes of the polyhedral sets in $\mathcal{P}$ is $n$. The set of lines stabbing $\mathcal{P}$ is denoted $S(\mathcal{P})$. Stabbers are also called *line transversals.*

***Box:*** A parallelepiped each of whose faces is orthogonal to one of the three Cartesian axes.

***c-oriented:*** Convex polyhedra whose face normals come from a set of $c$ fixed directions.

Table 41.2.4 lists the worst-case complexity of the set $S(\mathcal{P})$ and the time to find a witness stabbing line.

TABLE 41.2.4 Complexity of the set of stabbing lines and detection time.

| OBJECTS | COMPLEXITY OF $S(\mathcal{P})$ | FIND TIME | SOURCES |
|---|---|---|---|
| Convex polyhedra | $\Omega(n^3)$, $O(n^3 \log n)$ | $O(n^3\beta(n))$ | [PS92, Pel93a, Aga94] |
| $k$ polyhedra | $O(n^2k^{1+\epsilon})$ | $O(n^2k^{1+\epsilon})$ | [KRS10] |
| Boxes | $O(n^2)$ | $O(n)$ | [Ame92, Meg91] |
| $c$-oriented polyhedra | $O(n^2)$ | $O(n^2)$ | [Pel91] |
| Horiz. polygons | $\Theta(n^2)$ | $O(n)$ | [Pel91] |

Note that in some cases (boxes, parallel polygons) a stabbing line can be found in linear time, even though the best bound known for the complexity of the stabbing set is quadratic. These results are obtained using linear programming techniques (Chapter 49).

We can determine whether a given line $l$ is a stabber for a preprocessed set $\mathcal{P}$ of convex polyhedra in time $O(\log n)$, using data structures of size $O(n^{2+\epsilon})$ that can be constructed in time $O(n^{2+\epsilon})$ [PS92].

For an *oriented* stabber $l$ and a set $\mathcal{O}$ of $k$ *disjoint* convex bodies in $\mathbb{R}^d$, the order of the intersection of the objects along $l$ is called a ***geometric permutation*** (cf. Chapter 4). A result of Cheong et al. [CGN05] shows that for $k$ disjoint balls of unit radius in $\mathbb{R}^3$ and $k > 9$, there are at most two geometric permutations, while for $3 \leq k \leq 9$ there are at most three geometric permutations. For $k$ disjoint balls of any radius in $\mathbb{R}^3$ there are $\Theta(k^2)$ geometric permutations [SMS00]. If the ratio of the largest radius to the smallest radius of the spheres in the collection is a constant $c$, then there are at most $O(c^{\log c})$ geometric permutations. For $k$ rectangular boxes in $\mathbb{R}^d$ there are at most $2^{d-1}$ geometric permutations, which is tight (see also [O'R01]). There are at most $n$ geometric permutations of $n$ line segments in $\mathbb{R}^3$ [BEL$^+$05]; while for arbitrary disjoint convex sets there are $O(n^3 \log n)$ geometric permutations [RKS12].

### OPEN PROBLEMS

1. Can linear programming techniques yield a linear-time algorithm for $c$-oriented polyhedra?
2. The lower bound for $S(\mathcal{P})$ for a set of pairwise *disjoint* convex polyhedra is only $\Omega(n^2)$ [PS92]. Close the gap between this and the cubic upper bound.

## 41.3 RAY SHOOTING

Ray shooting is an important operation in computer graphics and a primitive operation useful in several geometric computations (e.g., hidden surface removal, and detecting and computing intersections of polyhedra). The problem is defined as follows. Given a large collection $\mathcal{P}$ of simple polyhedral objects, we want to know, for a given point $p$ and direction $\vec{d}$, the first object in $\mathcal{P}$ intersected by the ray defined by the pair $p, \vec{d}$. A single polyhedron with many faces can be represented without loss of generality by the collection of its faces, each treated as a separate polygon.

### ON-LINE RAY SHOOTING IN AN ARBITRARY DIRECTION

Here we consider the on-line model in which the set $\mathcal{P}$ is given in advance and a data structure is produced and stored. Afterward we are given the query rays one-by-one and the answer to one query must be produced before the next query is asked.

Table 41.3.1 summarizes the known complexity bounds on this problem. For a given class of objects we report the query time, the storage, and the preprocessing time of the method with the best bound. In this table and in the following ones we omit the big-$O$ symbols. Again, $n$ denotes the sum of the sizes of all the polyhedra in $\mathcal{P}$. The main references on ray shooting (Table 41.3.1) are in [Pel93b, BHO$^+$94] (boxes), [AM93, AM94, Pel93b, BHO$^+$94, AS93] (polyhedra), [Pel96] (horizontal polygons), [AAS97, MS94] (spheres), [DK85, AS96] (convex polyhedra), [ABG08] (fat convex polyhedra), and [Kol04] (semialgebraic sets of constant complexity).

### GLOSSARY

***Fat horizontal polygons***: Convex polygons contained in planes parallel to the $xy$-plane, with a constant lower bound on the ratio of the radius of the maximum inscribed circle over the radius of the minimum enclosing circle.

***Curtains***: Polygons in 3-space bounded by one segment and by two vertical rays from the endpoints of the segment.

***Axis-oriented curtains***: Curtains hanging from a segment parallel to the $x$- or $y$-axis.

***c-fat polyhedra***: A convex polyhedron $P$ in $\mathbb{R}^3$ is c-fat, for $0 \le c \le 1$ if, for any ball $b$ whose center lies in $P$ and which does not completely contain $P$, we have: $vol(b \cap P) \ge c \cdot vol(b)$.

TABLE 41.3.1 On-line ray shooting in an arbitrary direction.

| OBJECTS | QUERY | STORAGE | PREPROCESSING |
|---|---|---|---|
| Boxes, terrains, curtains | $\log n$ | $n^{2+\epsilon}$ | $n^{2+\epsilon}$ |
| Boxes | $n^{1+\epsilon}/m^{1/2}$ | $n \le m \le n^2$ | $m^{1+\epsilon}$ |
| Polyhedra | $\log n$ | $n^{4+\epsilon}$ | $n^{4+\epsilon}$ |
| Polyhedra | $n^{1+\epsilon}/m^{1/4}$ | $n \le m \le n^4$ | $m^{1+\epsilon}$ |
| Fat horiz. polygons | $\log n$ | $n^{2+\epsilon}$ | $n^{2+\epsilon}$ |
| Horiz. polygons | $\log^3 n$ | $n^{3+\epsilon} + K$ | $n^{3+\epsilon} + K \log n$ |
| Spheres | $\log^4 n$ | $n^{3+\epsilon}$ | $n^{3+\epsilon}$ |
| 1 convex polyhedron | $\log n$ | $n$ | $n \log n$ |
| $s$ convex polyhedra | $\log^2 n$ | $n^{2+\epsilon} s^2$ | $n^{2+\epsilon} s^2$ |
| $c$-fat convex polyhedra | $(n/m^{1/2}) \log^2 n$ | $n^{1+\epsilon} \le m \le n^{2+\epsilon}$ | $m^{1+\epsilon}$ |
| semialgebraic sets of $O(1)$ complexity | $n \log n$ | $n^{4+\epsilon}$ | $n^{4+\epsilon}$ |

When we drop the fatness assumption for horizontal polygons we obtain bounds that depend on $K$, the actual complexity of the set of lines missing the *edges* of the polygons (see Section 41.2).

Most of the data structures for ray shooting mentioned in Table 41.3.1 can be made dynamic (under insertion and deletion of objects in the scene) by using general dynamization techniques (see [Meh84, AEM92]).

## ON-LINE RAY SHOOTING IN A FIXED DIRECTION

We can usually improve on the general case if the direction of the rays is fixed a priori, while the source of the ray can lie anywhere in $\mathbb{R}^3$. See Table 41.3.2; here $k$ is the number of vertices, edges, faces, and cells of the arrangement of the (possibly intersecting) polyhedra. References for ray shooting in a fixed direction (Table 41.3.2) are [Ber93, BGH94] and [BG08, KRS09].

## OFF-LINE RAY SHOOTING IN AN ARBITRARY DIRECTION

In the previous section we considered the on-line situation when the answer to the query must be generated before the next question is asked. In many situations we do not need such strict requirements. For example, we might know all the queries from the start and are interested in minimizing the total time needed to answer all of the queries (the ***off-line*** situation). In this case there are simpler algorithms that improve on the storage bounds of on-line algorithms:

TABLE 41.3.2 On-line ray shooting in a fixed direction.

| OBJECTS | QUERY TIME | STORAGE | PREPROCESSING |
|---|---|---|---|
| Boxes | $\log n$ | $n^{1+\epsilon}$ | $n^{1+\epsilon}$ |
| Boxes | $\log n(\log\log n)^2$ | $n\log n$ | $n\log^2 n$ |
| Axis-oriented curtains | $\log n$ | $n\log n$ | $n\log n$ |
| Polyhedra | $\log^2 n$ | $n^{2+\epsilon}+k$ | $n^{2+\epsilon}+k\log n$ |
| Polyhedra | $n^{1+\epsilon}/m^{1/3}$ | $n\le m\le n^3$ | $m^{1+\epsilon}$ |
| c-fat convex constant size polyhedra | $\log^2 n$ | $n\log^2 n$ | $n\log^2 n$ |
| h polyhedra | $\log^2 n$ | $nh^2\log^2 n$ | $nh^2\log^2 n$ |

### THEOREM 41.3.1

*Given a polyhedral set $\mathcal{P}$ with $n$ vertices, edges, and faces, and given $m$ rays* off-line, *we can answer the $m$ ray-shooting queries in time $O(m^{0.8}n^{0.8+\epsilon}+m\log^2 n+n\log n\log m)$ using $O(n+m)$ storage.*

One of the most interesting applications of this result is the current asymptotically fastest algorithm for detecting whether two nonconvex polyhedra in 3-space intersect, and to compute their intersection. See Table 41.3.3; here $k$ is the size of the intersection.

TABLE 41.3.3 Detection and computation of intersection among polyhedra.

| OBJECTS | DETECTION | COMPUTATION | SOURCES |
|---|---|---|---|
| Polyhedra | $n^{1.6+\epsilon}$ | $n^{1.6+\epsilon}+k\log^2 n$ | [Pel93b] |
| Terrains | $n^{4/3+\epsilon}$ | $n^{4/3+\epsilon}+k^{1/3}n^{1+\epsilon}+k\log^2 n$ | [CEGS94, Pel94b] |

For a set of *convex* polyhedra in 3D with a total of $n$ vertices, the number $h$ of intersecting pairs of polyhedra can be computed in time $O(n^{1.6+\epsilon}+h)$ [ABHP+02]. Given $s$ convex polyhedra of total complexity $n$, we can pre-process them in linear time and storage so that any pair of them can be tested for intersection in time $O(\log n)$ per pair [BL15]. Lower bounds on off-line ray-shooting and intersection problems in 3D are difficult to prove. It has been shown in [Eri95] that many such problems are at least as hard as Hopcroft's incidence problem in the appropriate ambient space (see Chapter 40).

## RAY-SHOOTING IN SIMPLICIAL COMPLEXES

If we have a subdivision of the free space $\mathbb{R}^3\setminus\mathcal{P}$ into a simplicial complex we can answer ray-shooting queries by locating the tetrahedron containing the source of the ray and tracing the ray in the complex at cost $O(1)$ for each visited face of the complex. There are scenes $\mathcal{P}$ for which any simplicial complex has some line meeting $\Omega(n)$ faces of the complex. The average time for tracing a ray in a simplicial complex is proportional to the sum of the areas of all faces in the complex. It is possible to find a complex of total surface area within a constant multiplicative factor of the minimum, with $O(n^3 \log n)$ simplices in time $O(n^3 \log n)$ for general $\mathcal{P}$. For $\mathcal{P}$, a point set or a single polyhedron $O(n^2 \log n)$ time suffices (see [AAS95, AF99, CD99]). These results are obtained via a generalization of Eppstein's method for two-dimensional Minimum Weighted Steiner Triangulation (2D-MWST) of a point set [Epp94]. In the 3D context the weight is the surface area of the 2D faces of the complex. Starting from the set $\mathcal{P}$ of polyhedral obstacles in $\mathbb{R}^3$, an oct-tree-based decomposition of $\mathbb{R}^3$ is produced which is "balanced" and "smooth." It is then proved, via a charging argument, that the sum of the surface areas of all the boxes in the decomposition is within a constant factor of the surface area of any Minimum Surface Steiner Simplicial Complex compatible with $\mathcal{P}$. From the oct-tree the final complex is derived within just a constant factor increase in the total surface.

## EXTENSIONS AND ALTERNATIVE METHODS

Some ray-shooting results of Agarwal and Matoušek are obtained from the observation that a ray is traced by a family of segments $\rho(t)$, where one endpoint is the ray source and the second endpoint lies on the ray at distance $t$ from the source. Using *parametric search* techniques Agarwal and Matoušek compute the first value of $t$ for which $\rho(t)$ intersects $\mathcal{P}$, and thus answer the ray-shooting query.

An interesting extension of the concept of shooting rays against obstacles is obtained by shooting triangles and more generally simplices. We consider a family of simplices $s(t)$, indexed by real parameter $t \in \mathbb{R}^+$, where $t$ is the volume of the simplex $s(t)$, such that the simplices form a chain of inclusions: $t_1 \leq t_2 \Rightarrow s(t_1) \subset s(t_2)$, Intuitively we grow a simplex until it first meets one of the obstacles. Surprisingly, when the obstacles are general polyhedra, shooting simplices is not harder than shooting rays.

**THEOREM 41.3.2** [Pel94a]

*Given a set of polyhedra $\mathcal{P}$ with $n$ edges we can preprocess it in time $O(m^{1+\epsilon})$ into a data structure of size $m$, such that the following queries can be answered in time $O(n^{1+\epsilon}/m^{1/4})$: Given a simplex $s$, does $s$ avoid $\mathcal{P}$? Given a family of simplices $s(t)$ as above, which is the first value of $t^*$ for which $s(t^*)$ intersects $\mathcal{P}$?*

When the polyhedra are convex and c-fat, fixed simplex intersection queries can be answered in time $O((n/m^{1/3}) \log n)$ [ABG08] with $O(m^{1+\epsilon})$ storage. Thus by applying parametric search to this data structure one can also solve the shooting simplices problem within the same time/storage bound for this class of polyhedra. Computing the interaction between beams and polyhedral objects is a central problem in radio-therapy and radio-surgery (see, e.g., [SAL93, For99, CHX00]).

Other popular methods for solving ray-shooting problems are based binary space partitions, kD-trees, solid modeling schemes, etc. These methods, although important in practice, are usually not fully analyzable a priori using algorithmic analysis. In [ABCC06] Aronov et al. propose techniques that give a posteriori estimates of the cost of ray shooting.

## OPEN PROBLEMS

1. Find time and storage bounds for ray-shooting among general polyhedra that

are sensitive to the actual complexity of a group of lines (as opposed to the worst-case bound on such a complexity).

2. For a collection of $s$ convex polyhedra there is a wide gap in storage and preprocessing requirements for ray-shooting between the special case $s = 1$ and the case for general $s$. It would be interesting to obtain a bound that depends smoothly on $s$.

3. No lower bound on time or storage required for ray shooting is known.

## 41.4 MOVING LINES AMONG OBSTACLES

### ARBITRARY MOTIONS

So far we have treated lines as static objects. In this section we consider moving lines. A laser beam in manufacturing or a radiation beam in radiation therapy can be modeled as lines in 3-space moving among obstacles. The main computational problem is to decide whether a source line $l_1$ can be moved continuously until it coincides with a target line $l_2$ while avoiding a set of obstacles $\mathcal{P}$. We consider the following situation where the set of obstacles $\mathcal{P}$ is given in advance and preprocessed to obtain a data structure. When the query lines $l_1$ and $l_2$ are given the answer is produced before a new query is accepted. We have the results shown in Table 41.4.1, where $K$ is the complexity of the set of lines missing the edges of a set of horizontal polygons (cf. Section 41.2). The result on moving lines among polyhedral obstacles is extended in [Kol05] to moving a line segment, with the same complexity.

TABLE 41.4.1 On-line collision-free movement of lines among obstacles.

| OBJECTS | QUERY TIME | STORAGE | PREPROC | SOURCES |
|---|---|---|---|---|
| Polyhedra | $\log n$ | $n^{4+\epsilon}$ | $n^{4+\epsilon}$ | [Pel93b] |
| Horiz. polygons | $\log^3 n$ | $n^{3+\epsilon} + K$ | $n^{3+\epsilon} + K \log n$ | [Pel96] |

### OPEN PROBLEMS

It is not known how to trade off storage and query time, or whether better bounds can be obtained in an off-line situation.

### TRANSLATIONS

We now restrict the type of motion and consider only translations of lines. The first result is negative: there are sets of lines which cannot be split by any collision-free translation. There exists a set $L$ of 9 lines such that, for all directions $v$ and all subsets $L_1 \subset L$, $L_1$ cannot be translated continuously in direction $v$ without collisions with $L \setminus L_1$ [SS93]. Positive results are displayed in Table 41.4.2.

## GLOSSARY

***Towering property:*** Two sets of lines $L_1$ and $L_2$ are said to satisfy the towering property if we can translate simultaneously all lines in $L_1$ in the vertical direction without any collision with any lines in $L_2$.

***Separation property:*** Two sets of lines satisfy the separation property if they satisfy the towering property in some direction (not necessarily vertical).

TABLE 41.4.2 Separating lines by translations.

| PROPERTY | TIME TO CHECK PROPERTY | SOURCES |
|---|---|---|
| Towering | $O(n^{4/3+\epsilon})$ | [CEG+96] |
| Separation | $O(n^{3/2+\epsilon})$ | [Pel94b] |

# 41.5 CLOSEST PAIR OF LINES

## GLOSSARY

***Distance between lines:*** The Euclidean distance between two lines $l_1$ and $l_2$ in 3-space is the length of the shortest segment with one endpoint on $l_1$ and the other on $l_2$.

***Vertical distance between lines (segments):*** The length of the shortest vertical segment with one endpoint on line $l_1$ (resp. segment $s_1$) and one endpoint on line $l_2$ (resp. segment $s_2$). If no vertical segment joins two lines (resp. segments) the vertical distance is undefined.

TABLE 41.5.1 Closest and farthest pair of lines and segments.

| PROBLEM | OBJECTS | TIME | SOURCES |
|---|---|---|---|
| Smallest distance | lines | $O(n^{8/5+\epsilon})$ | [CEGS93] |
| Smallest vertical distance | lines, segments | $O(n^{8/5+\epsilon})$ | [Pel94a] |
| Largest vertical distance | lines, segments | $O(n^{4/3+\epsilon})$ | [Pel94a] |

Note that when some of the lines/segments are co-planar on a vertical plane, the problem of finding the smallest/largest vertical distance among them degenerates to a simpler distance problem relative to a planar arrangements of lines (or segments). Any centrally symmetric convex polyhedron $C$ in 3D defines a metric. If $C$ has constant combinatorial complexity, then the complexity of the Voronoi diagram of $n$ lines in 3-space is $O(n^2\alpha(n)\log n)$ [CKS+98]. For Euclidean distance the best bound is $O(n^{3+\epsilon})$.

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## OPEN PROBLEM

1. Finding an algorithm with subquadratic time complexity for the smallest distance among segments (and more generally, among polyhedra) is a notable open question.
2. Close the gap between the complexity of Voronoi diagrams of lines induced by polyhedral metrics and the Euclidean metric.

# 41.6 DOMINANCE RELATION AND WEAVINGS

## GLOSSARY

***Dominance relation:*** Given a finite set $L$ of nonvertical disjoint lines in $\mathbb{R}^3$, define a dominance relation $\prec$ among lines in $L$ as follows: $l_1 \prec l_2$ if $l_2$ lies above $l_1$, i.e., if, on the vertical line intersecting $l_1$ and $l_2$, the intersection with $l_1$ has a smaller $z$-coordinate than does the intersection with $l_2$.

***Weaving:*** A weaving is a pair $(L', \prec')$ where $L'$ is a set of lines on the plane and $\prec'$ is an anti-symmetric nonreflexive binary relation $\prec' \subset L' \times L'$ among the lines in $L'$.

***Realizable:*** A weaving is realizable if there exists a set of lines $L$ in 3-space such that $L'$ is the projection of $L$ and $\prec'$ is the image of the dominance relation $\prec$ for $L$.

***Elementary cycle:*** A cycle in the dominance relation such that the projections of the lines in the cycle bound a cell of the arrangement of projected lines.

***Perfect:*** A weaving $(L', \prec')$ is perfect if each line $l$ alternates below and above the other lines in the order they cross $l$ (see Figure 41.6.1a).

***Bipartite weaving:*** Two families of segments in 3-space such that, when projecting on the $xy$-plane, each segment does not meet segments from its own family and meets all the segments from the other family in the same order. (A bipartite weaving of size $4 \times 4$ is shown in Figure 41.6.1b.)

***Perfect bipartite weaving:*** Every segment alternates above and below the segments of the other family (see Figure 41.6.1b).

The dominance relation is possibly cyclic, that is, there may be three lines such that $l_1 \prec l_2 \prec l_3 \prec l_1$. Some results in [CEG+92, PPW93, BOS94, Sol98] [HPS01, BDP05, ABGM08, AS16] related to dominance are the following:

1. *How fast can we generate a consistent linear extension if the relation $\prec$ is acyclic?* $O(n^{4/3+\epsilon})$ time is sufficient for the case of lines. This result has been extended to the case of segments and of polyhedra. If an ordering is given as input, it is possible to *verify* that it is a linear extension of $\prec$ in time $O(n^{4/3+\epsilon})$.

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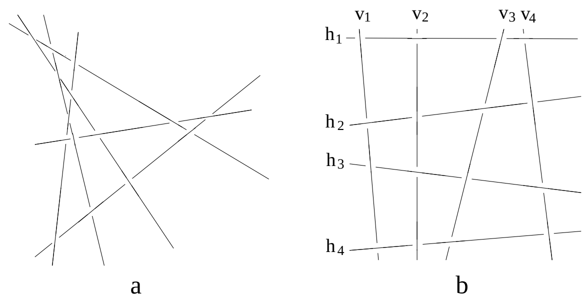


FIGURE 41.6.1
(a) *A perfect weaving;*
(b) *a perfect bipartite weaving.*

2. *How many elementary cycles in the dominance relation can $n$ lines define?* In the case of bipartite weavings, the dominance relation has $O(n^{3/2})$ elementary cycles and there is a family of bipartite weavings attaining the lower bound $\Omega(n^{4/3})$. For general weavings there is a construction attaining $\Omega(n^{3/2})$.

3. *If we cut the segments to eliminate cycles, how many "cuts" are necessary to eliminate all cycles?* From the previous result we have that sometimes $\Omega(n^{4/3})$ cuts are necessary since a single cut can eliminate only one elementary cycle. In order to eliminate all cycles (including the nonelementary ones) in any weaving, $O(n^{3/2}\text{polylog}(n))$ cuts are always sufficient.

4. *How fast can we find those cuts?* There are algorithms to find cuts in bipartite weavings in time $O(n^{9/5}\log n)$, and in time $O(n^{11/6+\epsilon})$ for general weavings. In a general weaving, calling $\mu$ is the minimum number of cuts, there is an algorithm to cut all cycles in time $O(n^{4/3+\epsilon}\mu^{1/3})$ that produces $O(n^{1+\epsilon}\mu^{1/3})$ cuts. Finding the minimum number $\mu$ of cuts is an NP-complete problem, and there is a polynomial-time approximation algorithm producing a set of cuts of size within a factor $O(\log\mu\log\log\mu)$ of the optimal.

5. The fraction of realizable weavings over all possible weavings of $n$ lines tends to 0 exponentially as $n$ tends to $\infty$. This result holds also when we generalize lines into semi-algebraic curves defined coordinate-wise by polynomials of constant degree.

6. A perfect weaving of $n \geq 4$ lines is not realizable.

7. Perfect bipartite weavings are realizable if and only if one of the families has fewer than four segments.

## 41.7 SOURCES AND RELATED MATERIAL

### FURTHER READING

[Som51, HP52, Jes69]: Extensive book-length treatments of the geometry of lines in space.

[Sto89, Sto91]: Algorithmic aspects of computing in projective spaces.

[BR79, Shi78]: Uses of the geometry of lines in robotics. For uses in graphics see [FDFH90].

[Ber93] [PS09, Chapter 7]: A detailed description of many ray-shooting results.

[Spe92, Dur99, Hav00]: Pointers to the vast related literature on pragmatic aspects of ray shooting.

[Goa10]: A survey on line transversals for sets of balls.

## RELATED CHAPTERS

Chapter 28: Arrangements
Chapter 38: Point location
Chapter 40: Range searching
Chapter 42: Geometric intersection
Chapter 50: Algorithmic motion planning
Chapter 52: Computer graphics

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