

## 3 TILINGS

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### INTRODUCTION

Tilings of surfaces and packings of space have interested artisans and manufacturers throughout history; they are a means of artistic expression and lend economy and strength to modular constructions. Today scientists and mathematicians study tilings because they pose interesting mathematical questions and provide mathematical models for such diverse fields as the molecular anatomy of crystals, cell packings of viruses,  $n$ -dimensional algebraic codes, “nearest neighbor” regions for a set of discrete points, meshes for computational geometry, CW-complexes in topology, the self-assembly of nano-structures, and the study of aperiodic order.

The world of tilings is too vast to discuss in a chapter, or even in a gargantuan book. Even such basic questions as: *What bodies can tile space? In what ways do they tile?* are intractable unless the tiles and tilings are subject to constraints, and even then the subject is unmanageably large.

In this chapter, due to space limitations, we restrict ourselves, for the most part, to tilings of unbounded spaces. Be aware that this is a severe restriction. Tilings of the sphere and torus, for example, are also subtle and important.

In Section 3.1 we present some general results that are fundamental to the subject as a whole. Section 3.2 addresses tilings with congruent tiles. In Section 3.3 we discuss the classical subject of periodic tilings, which continues to be an active field of research. Section 3.4 concerns nonperiodic and aperiodic tilings. We conclude with a very brief description of some kinds of tilings not considered here.

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### 3.1 GENERAL CONSIDERATIONS

In this section we define terms that will be used throughout the chapter and state some basic results. Taken together, these results state that although there is no algorithm for deciding which bodies can tile, there are criteria for deciding the question in certain cases. We can obtain some quantitative information about their tilings in particularly well-behaved cases.

Like many ideas that seem simple in life, it is complicated to give a precise mathematical definition of the notion of tiling. In particular this relates to the hard problem of defining the general notion of shape. Beyond the simple examples such as squares and circles there are shapes with holes, or fractal boundary, the latter allowing a shape of finite volume to have an infinite number of holes. As many of these strange examples turn up as examples of tilings, the definition needs to include them. We therefore define a *body* to be a compact subset of a manifold  $S \subset \mathbb{E}^n$  that is the closure of its interior. A tiling is then a division of  $S$  into a countable (finite or infinite) number of bodies.

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**GLOSSARY**

**Body:** A compact subset (of a manifold  $S \subset \mathbb{E}^n$ ) that is the closure of its (nonempty) interior.

**Tiling (of  $S$ ):** A decomposition of  $S$  into a countable number of  $n$ -dimensional bodies whose interiors are pairwise disjoint. In this context, the bodies are also called  $n$ -cells and are the tiles of the tiling (see below). Synonyms: *tessellation*, *parquetry* (when  $n = 2$ ), *honeycomb* (for  $n \geq 2$ ).

**Tile:** A body that is an  $n$ -cell of one or more tilings of  $S$ . To say that a body *tiles* a region  $R \subseteq S$  means that  $R$  can be covered exactly by congruent copies of the body without gaps or overlaps.

**Locally finite tiling:** Every  $n$ -ball of finite radius in  $S$  meets only finitely many tiles of the tiling.

**Prototile set (for a tiling  $\mathcal{T}$  of  $S$ ):** A minimal subset of tiles in  $\mathcal{T}$  such that each tile in the tiling  $\mathcal{T}$  is the congruent copy of one of those in the prototile set. The tiles in the set are called prototiles and the prototile set is said to admit  $\mathcal{T}$ .

**$k$ -face (of a tiling):** An intersection of at least  $n - k + 1$  tiles of the tiling that is not contained in a  $j$ -face for  $j < k$ . (The 0-faces are the *vertices* and 1-faces the *edges*; the  $(n-1)$ -faces are simply called the *faces* of the tiling.)

**Cluster and Patch (in a tiling  $\mathcal{T}$ ):** The set of bodies in a tiling  $\mathcal{T}$  that intersect a compact subset of  $K \subset S$  is a *cluster*. The set is a *patch* if  $K$  can be chosen to be convex. See Figure 3.1.1.

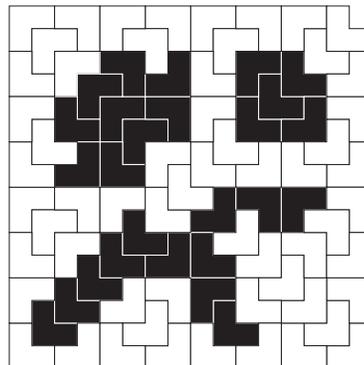


FIGURE 3.1.1

Three clusters in a tiling of the plane by “chairs.” Two of these clusters are patches.

**Normal tiling:** A tiling in which (i) each prototile is homeomorphic to an  $n$ -ball, and (ii) the prototiles are uniformly bounded (there exist  $r > 0$  and  $R > 0$  such that each prototile contains a ball of radius  $r$  and is contained in a ball of radius  $R$ ). It is technically convenient to include a third condition: (iii) the intersection of every pair of tiles is a connected set. (A normal tiling is necessarily locally finite.)

**Face-to-face tiling (by polytopes):** A tiling in which the faces of the tiling are also the  $(n-1)$ -dimensional faces of the polytopes. (A face-to-face tiling by convex polytopes is also  $k$ -face-to- $k$ -face for  $0 \leq k \leq n - 1$ .) In dimension 2, this is an *edge-to-edge* tiling by polygons, and in dimension 3, a face-to-face tiling by polyhedra.

**Dual tiling:** Two tilings  $\mathcal{T}$  and  $\mathcal{T}^*$  are dual if there is an incidence-reversing bijection between the  $k$ -faces of  $\mathcal{T}$  and the  $(n-k)$ -faces of  $\mathcal{T}^*$  (see Figure 3.1.2).

**Voronoi (Dirichlet) tiling:** A tiling whose tiles are the Voronoi cells of a discrete set  $\Lambda$  of points in  $S$ . The Voronoi cell of a point  $p \in \Lambda$  is the set of all points in  $S$  that are at least as close to  $p$  as to any other point in  $\Lambda$  (see Chapter 27).

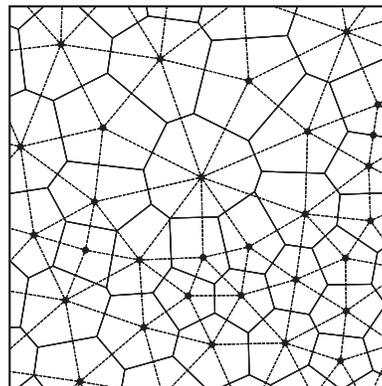


FIGURE 3.1.2

A Voronoi tiling (solid lines) and its Delaunay dual (dashed lines).

**Delaunay (or Delone) tiling:** A face-to-face tiling by convex circumscribable polytopes (i.e., the vertices of each polytope lie on a sphere).

**Isometry:** A distance-preserving self-map of  $S$ .

**Symmetry group (of a tiling):** The set of isometries of  $S$  that map the tiling to itself.

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## MAIN RESULTS

1. **The Undecidability Theorem.** There is no algorithm for deciding whether or not an arbitrary set of bodies admits a tiling of  $\mathbb{E}^2$  [Ber66]. This was initially proved by Berger in the context of Wang tiles, squares with colored edges (the colors can be replaced by shaped edges to give a result for tiles). Berger's result showed that any attempted algorithm would not work (would run forever giving no answer) for some set of Wang tiles. This set can be arbitrarily large. The result was improved by Ollinger [Oll09] to show that any attempted algorithm could be broken by a set of at most 5 polyominoes.
2. **The Extension Theorem (for  $\mathbb{E}^n$ ).** Let  $A$  be any finite set of bodies, each homeomorphic to a closed  $n$ -ball. If  $A$  tiles regions that contain arbitrarily large  $n$ -balls, then  $A$  admits a tiling of  $\mathbb{E}^n$ . (These regions need not be nested, nor need any of the tilings of the regions be extendable!) The proof for  $n = 2$  in [GS87] extends to  $\mathbb{E}^n$  with minor changes.
3. **The Normality Lemma (for  $\mathbb{E}^n$ ).** In a normal tiling, the ratio of the number of tiles that meet the boundary of a spherical patch to the number of tiles in the patch tends to zero as the radius of the patch tends to infinity. In fact, a stronger statement can be made: For  $s \in S$  let  $t(r, s)$  be the number of tiles in

the spherical patch  $P(r, s)$ . Then, in a normal tiling, for every  $x > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{t(r+x, s) - t(r, s)}{t(r, s)} = 0.$$

The proof for  $n = 2$  in [GS87] extends to  $\mathbb{E}^n$  with minor changes.

4. **Euler's Theorem for tilings of  $\mathbb{E}^2$ .** Let  $\mathcal{T}$  be a normal tiling of  $\mathbb{E}^2$ , and let  $t(r, s)$ ,  $e(r, s)$ , and  $v(r, s)$  be the numbers of tiles, edges, and vertices, respectively, in the circular patch  $P(r, s)$ . Then if one of the limits  $e(\mathcal{T}) = \lim_{r \rightarrow \infty} e(r, s)/t(r, s)$  or  $v(\mathcal{T}) = \lim_{r \rightarrow \infty} v(r, s)/t(r, s)$  exists, so does the other, and  $v(\mathcal{T}) - e(\mathcal{T}) + 1 = 0$ . Like Euler's Theorem for Planar Maps, on which the proof of this theorem is based, this result can be extended in various ways [GS87].
5. **Voronoi and Delaunay Duals.** Every Voronoi tiling has a Delaunay dual and conversely (see Figure 3.1.2) [Vor09].

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## OPEN PROBLEM

1. Is there an algorithm to decide whether any set of at most four bodies admits a tiling in  $\mathbb{E}^n$ ? The question can also be asked for other spaces. The tiling problem for the hyperbolic plane was shown to be undecidable independently by Kari and Morgenstern [Kar07, Mar08].

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## 3.2 TILINGS BY ONE TILE

To say that a body tiles  $\mathbb{E}^n$  usually means that there is a tiling all of whose tiles are congruent copies of this body. The artist M.C. Escher has demonstrated how intricate such tiles can be even when  $n = 2$ . But in higher dimensions the simplest tiles—for example, cubes—can produce surprises, as the counterexample to Keller's conjecture attests (see below).

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## GLOSSARY

**Monohedral tiling:** A tiling with a single prototile.

**$r$ -morphic tile:** A prototile that admits exactly  $r$  distinct (non-superposable) monohedral tilings. Figure 3.2.1 shows a 3-morphic tile and its three tilings, and Figure 3.2.2 shows a 1-morphic tile and its tiling.

**$k$ -rep tile:** A body for which  $k$  copies can be assembled into a larger, similar body. (Or, equivalently, a body that can be partitioned into  $k$  congruent bodies, each similar to the original.) More formally, a  $k$ -rep tile is a compact set  $A_1$  in  $S$  with nonempty interior such that there are sets  $A_2, \dots, A_k$  congruent to  $A_1$  that satisfy

$$\text{Int } A_i \cap \text{Int } A_j = \emptyset$$

for all  $i \neq j$  and  $A_1 \cup \dots \cup A_k = g(A_1)$ , where  $g$  is a similarity mapping.

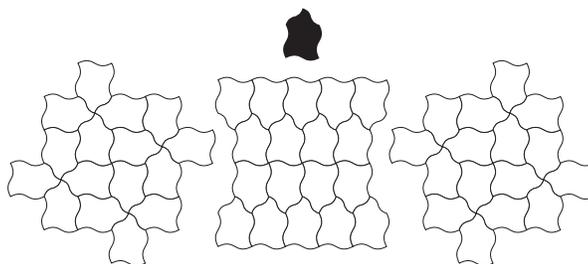


FIGURE 3.2.1  
A trimorphic tile.

**Transitive action:** A group  $G$  is said to act transitively on a nonempty set  $\{A_1, A_2, \dots\}$  if the set is an orbit for  $G$ . (That is, for every pair  $A_i, A_j$  of elements of the set, there is a  $g_{ij} \in G$  such that  $g_{ij}A_i = A_j$ .)

**Isohedral (tiling):** A tiling whose symmetry group acts transitively on its tiles.

**Anisohedral tile:** A prototile that admits monohedral tilings but no isohedral tilings. In Figure 3.2.2, the prototile admits a unique nonisohedral tiling; the black tiles are each surrounded differently, from which it follows that no isometry can map one to the other (while mapping the tiling to itself). This tiling is periodic, however; see Section 3.3.

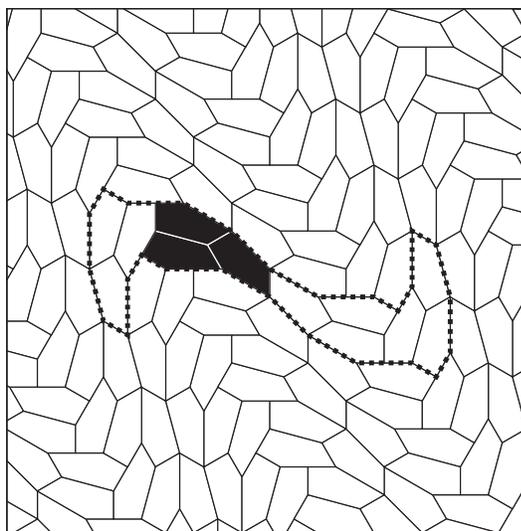


FIGURE 3.2.2  
An anisohedral tile. This tile was the fifteenth type of pentagon found to tile the plane [MMVD17].

**Corona (of a tile  $P$  in a tiling  $\mathcal{T}$ ):** Define  $C^0(P) = P$ . Then  $C^k(P)$ , the  $k$ th corona of  $P$ , is the set of all tiles  $Q \in \mathcal{T}$  for which there exists a path of tiles  $P = P_0, P_1, \dots, P_m = Q$  with  $m \leq k$  in which  $P_i \cap P_{i+1} \neq \emptyset$ ,  $i = 0, 1, \dots, m-1$ .

**Lattice:** The group of integral linear combinations of  $n$  linearly independent vectors in  $S$ . A point orbit of a lattice, often called a **point lattice**, is a particular case of a **regular system of points** (see Chapter 64).

**Translation tiling:** A monohedral tiling of  $S$  in which every tile is a translate of a fixed prototile. See Figure 3.2.3.

**Lattice tiling:** A monohedral tiling on whose tiles a lattice of translation vectors acts transitively. Figure 3.2.3 is not a lattice tiling since it is invariant by multiples of just one vector.

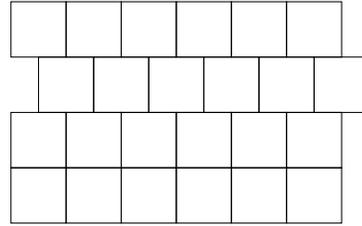


FIGURE 3.2.3

This translation non-lattice tiling is nonperiodic but not aperiodic.

***n*-parallelootope:** A convex  $n$ -polytope that tiles  $\mathbb{E}^n$  by translation.

***Belt (of an  $n$ -parallelootope):*** A maximal subset of parallel  $(n-2)$ -faces of a parallelootope in  $\mathbb{E}^n$ . The number of  $(n-2)$ -faces in a belt is its length.

***Center of symmetry (for a set  $A$  in  $\mathbb{E}^n$ ):*** A point  $a \in A$  such that  $A$  is invariant under the mapping  $x \mapsto 2a - x$ ; the mapping is called a ***central inversion*** and an object that has a center of symmetry is said to be ***centrosymmetric***.

***Stereohedron:*** A convex polytope that is the prototile of an isohedral tiling. A Voronoi cell of a regular system of points is a stereohedron.

***Linear expansive map:*** A linear transformation all of whose eigenvalues have modulus greater than one.

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## MAIN RESULTS

1. **The Local Theorem.** Let  $\mathcal{T}$  be a monohedral tiling of  $S$ , and for  $P \in \mathcal{T}$ , let  $S_i(P)$  be the subgroup of the symmetry group of  $P$  that leaves invariant  $C^i(P)$ , the  $i$ th corona of  $P$ .  $\mathcal{T}$  is isohedral if and only if there exists an integer  $k > 0$  for which the following two conditions hold: (a) for all  $P \in \mathcal{T}$ ,  $S_{k-1}(P) = S_k(P)$  and (b) for every pair of tiles  $P, P'$  in  $\mathcal{T}$ , there exists an isometry  $\gamma$  such that  $\gamma(P) = P'$  and  $\gamma(C^k(P)) = C^k(P')$ . In particular, if  $P$  is asymmetric, then  $\mathcal{T}$  is isohedral if and only if condition (b) holds for  $k = 1$  [DS98].
2. A convex polytope is a parallelootope if and only if it is centrosymmetric, its faces are centrosymmetric, and its belts have lengths four or six. First proved by Venkov, this theorem was rediscovered independently by McMullen [Ven54, McM80]. There are two combinatorial types of parallelotopes in  $\mathbb{E}^2$  and five in  $\mathbb{E}^3$ .
3. The number  $|F|$  of faces of a convex parallelotope in  $\mathbb{E}^n$  satisfies Minkowski's inequality,  $2n \leq |F| \leq 2(2^n - 1)$ . Both upper and lower bounds are realized in every dimension [Min97].
4. The number of faces of an  $n$ -dimensional stereohedron in  $\mathbb{E}^n$  is bounded. In fact, if  $a$  is the number of translation classes of the stereohedron in an isohedral tiling, then the number of faces is at most the Delaunay bound  $2^n(1 + a) - 2$  [Del61].
5. Anisohedral tiles exist in  $\mathbb{E}^n$  for every  $n \geq 2$  [GS80]. (The first example, given for  $n = 3$  by Reinhardt [Rei28], was the solution to part of Hilbert's 18th problem.) H. Heesch gave the first example for  $n = 2$  [Hee35] and R. Kershner the first convex examples [Ker68].
6. Every  $n$ -parallelootope admits a lattice tiling. However, for  $n \geq 3$ , there are nonconvex tiles that tile by translation but do not admit lattice tilings [SS94].

7. A lattice tiling of  $\mathbb{E}^n$  by unit cubes must have a pair of cubes sharing a whole face [Min07, Haj42]. However, a famous conjecture of Keller, which stated that for every  $n$ , any tiling of  $\mathbb{E}^n$  by congruent cubes must contain at least one pair of cubes sharing a whole face, is false: for  $n \geq 8$ , there are translation tilings by unit cubes in which no two cubes share a whole face [LS92].
8. Every linear expansive map that transforms the lattice  $\mathbb{Z}^n$  of integer vectors into itself defines a family of  $k$ -rep tiles; these tiles, which usually have fractal boundaries, admit lattice tilings [Ban91].

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## OPEN PROBLEMS

1. Which convex  $n$ -polytopes in  $\mathbb{E}^n$  are prototiles for monohedral tilings of  $\mathbb{E}^n$ ? This is unsolved for all  $n \geq 2$  (see [GS87, Sch78b, MMVD17] for the case  $n = 2$ ). In July 2017, M. Rao announced that the list of 15 convex pentagon types that tile is complete; the proof had not been verified at the time of writing [Rao17]. The most recent addition to the list is shown in Figure 3.2.2. For higher dimensions, little is known; it is not even known which tetrahedra tile  $\mathbb{E}^3$  [GS80, Sen81].
2. **Voronoi's conjecture:** Every convex parallelotope in  $\mathbb{E}^n$  is affinely equivalent to the Voronoi cell of a lattice in  $\mathbb{E}^n$ . The conjecture has been proved for  $n \leq 4$ , for the case when the parallelotope is a zonotope, and for certain other special cases [Erd99, Mag15].

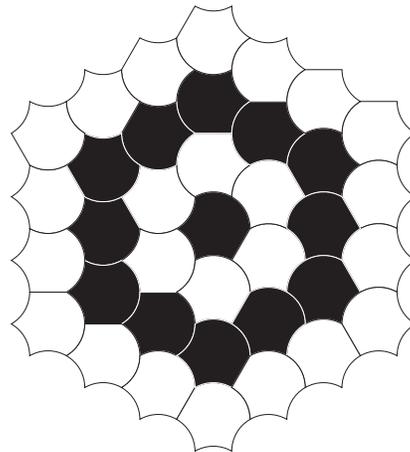


FIGURE 3.2.4  
A 3-corona tile. (It cannot be surrounded by a fourth corona.) 4-corona and 5-corona tiles also exist.

3. **Heesch's Problem.** Is there an integer  $k_n$ , depending only on the dimension  $n$  of the space  $S$ , such that if a body  $A$  can be completely surrounded  $k_n$  times by tiles congruent to  $A$ , then  $A$  is a prototile for a monohedral tiling of  $S$ ? ( $A$  is completely surrounded once if  $A$ , together with congruent copies that have nonempty intersection with  $A$ , tile a patch containing  $A$  in its interior.) When  $S = \mathbb{E}^2$ ,  $k_2 > 5$ . The body shown in Figure 3.2.4 can be completely surrounded three times but not four. William Rex Marshall and, independently, Casey Mann, found 4-corona tiles, and Mann 5-corona tiles [Man04]. Michael DeWeese has found hexagons with generalized matching rules having Heesch numbers 1 through 9 and 11 [MT16]. This problem is unsolved for all  $n \geq 2$ .

4. **Keller's conjecture** is true for  $n \leq 6$  and false for  $n \geq 8$  (see Result 7 above). The case  $n = 7$  is still open.
5. Find a good upper bound for the number of faces of an  $n$ -dimensional stereohedron. Delaunay's bound, stated above, is evidently much too high; for example, it gives 390 as the bound in  $\mathbb{E}^3$ , while the maximal known number of faces of a three-dimensional stereohedron (found by P. Engel [Eng81]) is 38.
6. For monohedral (face-to-face) tilings by convex polytopes there is an integer  $k_n$ , depending only on the dimension  $n$  of  $S$ , that is an upper bound for the constant  $k$  in the Local Theorem [DS98]. Find the value of this  $k_n$ . For the Euclidean plane  $\mathbb{E}^2$  it is known that  $k_2 = 1$  (convexity of the tiles is not necessary) [SD98], but for the hyperbolic plane,  $k_2 \geq 2$  [Mak92]. For  $\mathbb{E}^3$ , it is known that  $2 \leq k_3 \leq 5$ .

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### 3.3 PERIODIC TILINGS

Periodic tilings have been studied intensely, in part because their applications range from ornamental design to crystallography, and in part because many techniques (algebraic, geometric, and combinatorial) are available for studying them.

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#### GLOSSARY

**Periodic tiling of  $\mathbb{E}^n$ :** A tiling, not necessarily monohedral, whose symmetry group contains an  $n$ -dimensional lattice. This definition can be adapted to include "subperiodic" tilings (those whose symmetry groups contain  $1 \leq k < n$  linearly independent vectors) and tilings of other spaces (for example, cylinders). Tilings in Figures 3.2.1, 3.2.2, 3.3.1, and 3.3.3 are periodic.

**Fundamental domain (generating region) for a periodic tiling:** In the general case of a group acting on a space, a fundamental domain is a subset of the space containing exactly one point from each orbit. It generally requires that the subset be connected and have some restrictions on its boundary. In the case of tilings a fundamental domain can be a (usually connected) minimal subset of the set of tiles that generates the whole tiling under the symmetry group. For example, a fundamental domain may be a tile (Figure 3.2.1), a subset of a single tile (Figure 3.3.1), or a subset of tiles (three shaded tiles in Figure 3.2.2).

**Lattice unit (or translation unit) for a periodic tiling:** A (usually connected) minimal region of the tiling that generates the whole tiling under the translation subgroup of the symmetry group. A lattice unit can be a single tile or contain several tiles. Figure 3.3.1 has a 3-tile lattice unit; Figure 3.2.2 has a 12-tile lattice unit (outlined).

**Orbifold (of a tiling of  $S$ ):** An orbifold is a generalization of a manifold to allow singularities. They are usually formed by folding up a space by a discrete symmetry group, and they are therefore a powerful tool to study periodic patterns. For the plane this is described in [CBGS08].

**$k$ -isohedral (tiling):** A tiling whose tiles belong to  $k$  transitivity classes under the action of its symmetry group. Isohedral means 1-isohedral (Figures 3.3.1 and 3.3.3). The tiling in Figure 3.2.2 is 3-isohedral.

**Equitransitive (tiling by polytopes):** A tiling in which each combinatorial class of tiles forms a single transitivity class under the action of the symmetry group of the tiling.

**$k$ -isogonal (tiling):** A tiling whose vertices belong to  $k$  transitivity classes under the action of its symmetry group. Isogonal means 1-isogonal.

**$k$ -uniform (tiling of a 2-dimensional surface):** A  $k$ -isogonal tiling by regular polygons.

**Uniform (tiling for  $n > 2$ ):** An isogonal tiling with congruent edges and uniform faces.

**Flag of a tiling (of  $S$ ):** An ordered  $(n+1)$ -tuple  $(X_0, X_1, \dots, X_n)$ , with  $X_n$  a tile and  $X_k$  a  $k$ -face for  $0 \leq k \leq n-1$ , in which  $X_{i-1} \subset X_i$  for  $i = 1, \dots, n$ .

**Regular tiling (of  $S$ ):** A tiling  $\mathcal{T}$  whose symmetry group is transitive on the flags of  $\mathcal{T}$ . (For  $n > 2$ , these are also called regular honeycombs.) See Figure 3.3.3.

**$k$ -colored tiling:** A tiling in which each tile has a single color, and  $k$  different colors are used. Unlike the case of map colorings, in a colored tiling adjacent tiles may have the same color.

**Perfectly  $k$ -colored tiling:** A  $k$ -colored tiling for which each element of the symmetry group  $G$  of the uncolored tiling effects a permutation of the colors. The ordered pair  $(G, \Pi)$ , where  $\Pi$  is the corresponding permutation group, is called a  $k$ -color symmetry group.

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## CLASSIFICATION OF PERIODIC TILINGS

There is a variety of notations for classifying the different “types” of tilings and tiles. Far from being merely names by which to distinguish types, these notations tell us the investigators’ point of view and the questions they ask. Notation may tell us the global symmetries of the tiling, or how each tile is surrounded, or the topology of its orbifold. Notation makes possible the computer implementation of investigations of combinatorial questions about tilings.

Periodic tilings are classified by symmetry groups and, sometimes, by their skeletons (of vertices, edges, ...,  $(n-1)$ -faces). The groups are known as **crystallographic groups**; up to isomorphism, there are 17 in  $\mathbb{E}^2$ , 219 in  $\mathbb{E}^3$ , and 4894 in  $\mathbb{E}^4$ . For  $\mathbb{E}^2$  and  $\mathbb{E}^3$ , the most common notation for the groups has been that of the International Union of Crystallography (IUCr) [Hah83]. This is cross-referenced to earlier notations in [Sch78a]. Recently developed notations include Delaney-Dress symbols [Dre87] and orbifold notation for  $n = 2$  [Con92, CH02] and for  $n = 3$  [CDHT01, CBGS08].

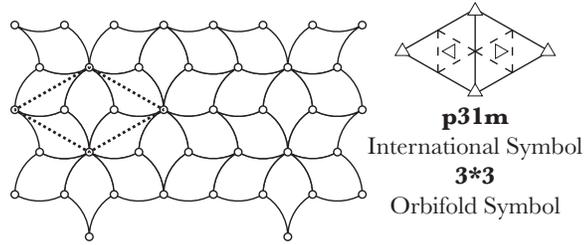
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## GLOSSARY

**International symbol (for periodic tilings of  $\mathbb{E}^2$  and  $\mathbb{E}^3$ ):** Encodes lattice type and particular symmetries of the tiling. In Figure 3.3.1, the lattice unit diagram at the right encodes the symmetries of the tiling and the IUCr symbol p31m indicates that the highest-order rotation symmetry in the tiling is 3-fold, that there is no mirror normal to the edge of the lattice unit, and that there is a mirror at  $60^\circ$  to the edge of the lattice unit. These symbols are augmented to denote symmetry groups of perfectly 2-colored tilings.

FIGURE 3.3.1

An isohedral tiling with standard IUCr lattice unit in dotted outline; a half-leaf is a fundamental domain. The classification symbols are for the symmetry group of the tiling.



**Delaney-Dress symbol (for tilings of Euclidean, hyperbolic, or spherical space of any dimension):** Associates an edge-colored and vertex-labeled graph derived from a *chamber system* (a formal barycentric subdivision) of the tiling. In Figure 3.3.2, the nodes of the graph represent distinct triangles  $A, B, C, D$  in the chamber system, and colored edges (dashed, thick, or thin) indicate their adjacency relations. Numbers on the nodes of the graph show the degree of the tile that contains that triangle and the degree of the vertex of the tiling that is also a vertex of that triangle.

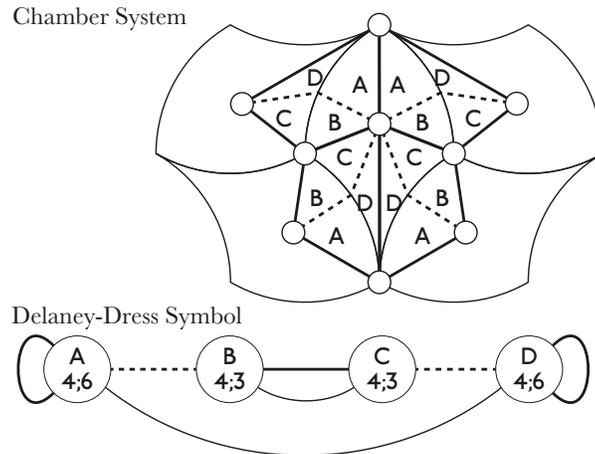


FIGURE 3.3.2

A chamber system of the tiling in Figure 3.3.1 determines the graph that is its Delaney-Dress symbol.

**Orbifold notation (for symmetry groups of tilings of 2-dimensional surfaces of constant curvature):** Encodes properties of the orbifold induced by the symmetry group of a periodic tiling of the Euclidean plane or hyperbolic plane, or a finite tiling of the surface of a sphere; introduced by Conway and Thurston. In Figure 3.3.1, the first 3 in the orbifold symbol  $3*3$  for the symmetry group of the tiling indicates there is a 3-fold rotation center (gyration point) that becomes a cone point in the orbifold, while  $*3$  indicates that the boundary of the orbifold is a mirror with a corner where three mirrors intersect.

See Table 3.3.1 for the IUCr and orbifold notations for  $\mathbb{E}^2$ .

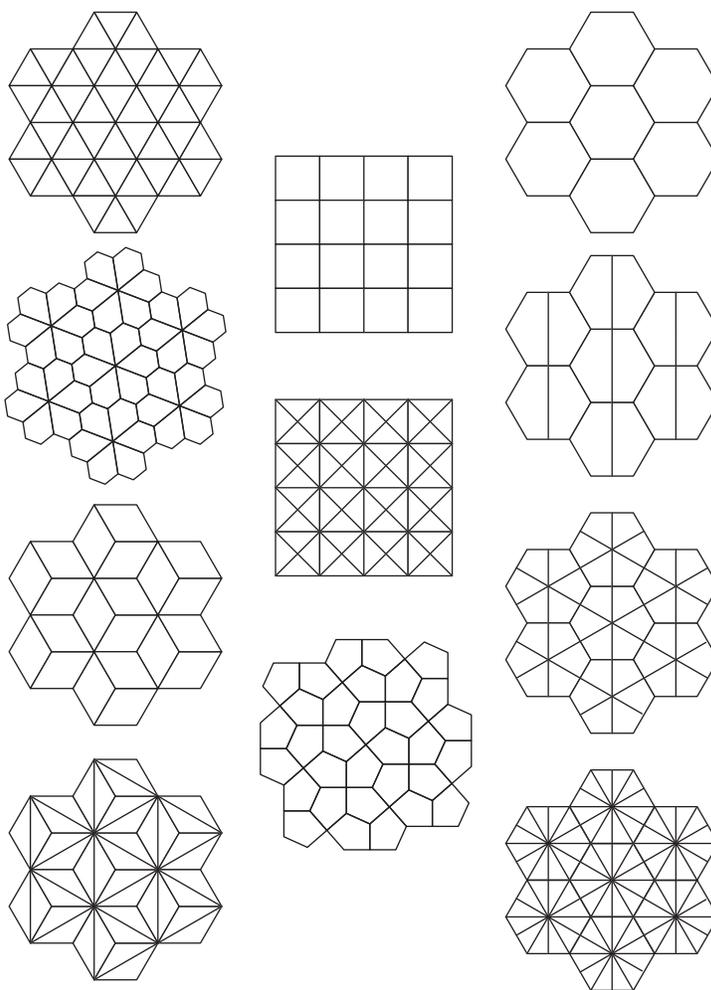
Isohedral tilings of  $\mathbb{E}^2$  fall into 11 combinatorial classes, typified by the Laves nets (Figure 3.3.3). The Laves net for the tiling in Figure 3.3.1 is [3.6.3.6]; this gives the vertex degree sequence for each tile. In an isohedral tiling, every tile is surrounded in the same way. Grünbaum and Shephard provide an incidence

TABLE 3.3.1 IUCr and orbifold notations for the 17 symmetry groups of periodic tilings of  $\mathbb{E}^2$ .

IUCr	ORBIFOLD	IUCr	ORBIFOLD
p1	o or o1	p3	333
pg	$\times\times$ or $1\times\times$	p31m	$3^*3$
cm	$^*\times$ or $1^*\times$	p3m1	$^*333$
pm	$^{**}$ or $1^{**}$	p4	442
p2	2222	p4g	$4^*2$
pgg	$22\times$	p4m	$^*442$
pmg	$22^*$	p6	632
cmm	$2^*22$	p6m	$^*632$
pmm	$^*2222$		

FIGURE 3.3.3

The 11 Laves nets. The three regular tilings of  $\mathbb{E}^2$  are at the top of the illustration.



symbol for each isohedral type by labeling and orienting the edges of each tile [GS79]. Figure 3.3.4 gives the incidence symbol for the tiling in Figure 3.3.1. The tile symbol  $a^+a^-b^+b^-$  records the cycle of edges of a tile and their orientations with respect to the (arrowed) first edge (+ indicates the same, - indicates opposite orientation). The adjacency symbol  $b^-a^-$  records for each different letter edge of a single tile, beginning with the first, the edge it abuts in the adjacent tile and their relative orientations (now - indicates same, + opposite).

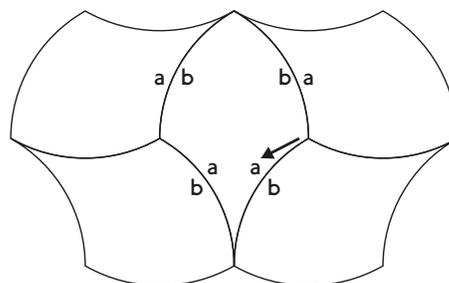


FIGURE 3.3.4

Labeling and orienting the edges of the isohedral tiling in Figure 3.3.1 determines its Grünbaum-Shephard incidence symbol.

Grünbaum-Shephard

Incidence Symbol

 $[a^+a^-b^+b^-;b^-a^-]$ 

These symbols can be augmented to adjacency symbols to denote  $k$ -color symmetry groups. Earlier, Heesch devised signatures for the 28 types of tiles that could be fundamental domains of isohedral tilings without reflection symmetry [HK63]; this signature system was extended in [BW94].

## MAIN RESULTS

1. If a finite prototile set of polygons admits an edge-to-edge tiling of the plane that has translational symmetry, then the prototile set also admits a periodic tiling [GS87].
2. The number of symmetry groups of periodic tilings in  $\mathbb{E}^n$  is finite (this is a famous theorem of Bieberbach [Bie10] that partially solved Hilbert's 18th problem: see also Chapter 64); the number of symmetry groups of corresponding tilings in hyperbolic  $n$ -space, for  $n = 2$  and  $n = 3$ , is infinite.
3. Using their classification by incidence symbols, Grünbaum and Shephard proved there are 81 classes of isohedral tilings of  $\mathbb{E}^2$ , 93 classes if the tiles are *marked* (that is, they have decorative markings to break the symmetry of the tile shape) [GS77]. There is an infinite number of classes of isohedral tilings of  $\mathbb{E}^n$ ,  $n > 2$ .
4. Every  $k$ -isohedral tiling of the Euclidean plane, hyperbolic plane, or sphere can be obtained from a  $(k-1)$ -isohedral tiling by a process of *splitting* (splitting an asymmetric prototile) and *gluing* (amalgamating two or more equivalent asymmetric tiles adjacent in the tiling into one new tile) [Hus93]; there are 1270 classes of normal 2-isohedral tilings and 48,231 classes of normal 3-isohedral tilings of  $\mathbb{E}^2$ .
5. Classifying isogonal tilings in a manner analogous to isohedral ones, Grünbaum and Shephard have shown [GS78] that there are 91 classes of normal isogonal tilings of  $\mathbb{E}^2$  (93 classes if the tiles are marked).

6. For every  $k$ , the number of  $k$ -uniform tilings of  $\mathbb{E}^2$  is finite. There are 11 uniform tilings of  $\mathbb{E}^2$  (also called *Archimedean*, or *semiregular*), of which 3 are regular. The Laves nets in Figure 3.3.3 are duals of these 11 uniform tilings [GS87]. There are 28 uniform tilings of  $\mathbb{E}^3$  [Grü94] and 20 2-uniform tilings of  $\mathbb{E}^2$  [Krö69]; see also [GS87]. In the hyperbolic plane, uniform tilings with vertex valence 3 and 4 have been classified [GS79].
7. There are finitely many regular tilings of  $\mathbb{E}^n$  (three for  $n = 2$ , one for  $n = 3$ , three for  $n = 4$ , and one for each  $n > 4$ ) [Cox63]. There are infinitely many normal regular tilings of the hyperbolic plane, four of hyperbolic 3-space, five of hyperbolic 4-space, and none of hyperbolic  $n$ -space if  $n > 4$  [Sch83, Cox54].
8. If two orbifold symbols for a tiling of the Euclidean or hyperbolic plane are the same except for the numerical values of their digits, which may differ by a permutation of the natural numbers, then the number of  $k$ -isohedral tilings for each of these orbifold types is the same [BH96].
9. There is a one-to-one correspondence between perfect  $k$ -colorings of a tiling whose symmetry group  $G$  acts on it freely and transitively, and the subgroups of index  $k$  of  $G$ . See [Sen79].

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## OPEN PROBLEMS

1. Conjecture: Every convex pentagon that tiles  $\mathbb{E}^2$  admits a  $k$ -isohedral tiling for some  $k \leq 3$ . Michaël Rao announced a proof of this conjecture in July 2017 depending on an exhaustive computer search; the proof had not been verified at the time of writing [Rao17].
2. Enumerate the uniform tilings of  $\mathbb{E}^n$  for  $n > 3$ .
3. Delaney-Dress symbols and orbifold notations have made progress possible on the classification of  $k$ -isohedral tilings in all three 2-dimensional spaces of constant curvature; extend this work to higher-dimensional spaces.

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## 3.4 NONPERIODIC TILINGS

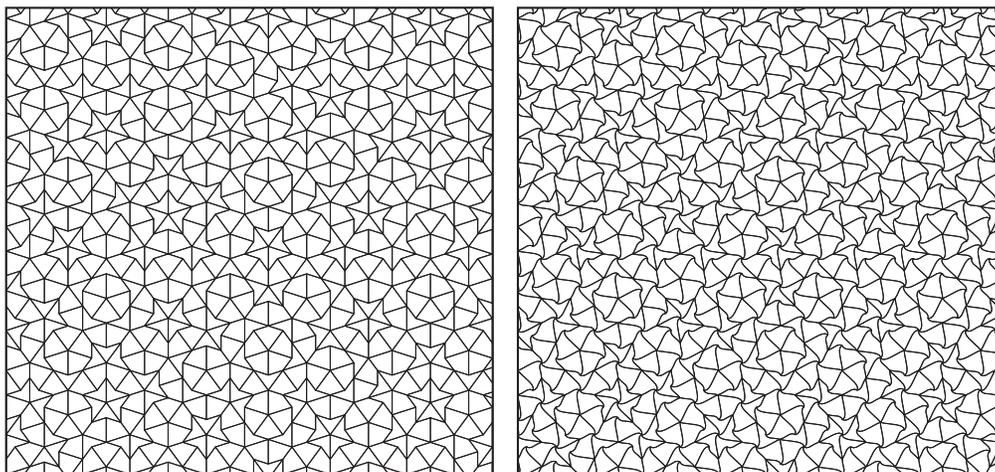
Nonperiodic tilings are found everywhere in nature, from cracked glazes to biological tissues to crystals. In a remarkable number of cases, such tilings exhibit strong regularities. For example, many nonperiodic tilings repeat on increasingly larger scales. An even larger class of tilings are those called repetitive, in which every bounded configuration appearing anywhere in the tiling is repeated infinitely many times throughout it.

Aperiodic prototype sets are particularly interesting. They were first introduced to prove the Undecidability Theorem (Section 3.1). Later, after Penrose found pairs of aperiodic prototiles (see Figure 3.4.1), they became popular in recreational mathematical circles.

The deep mathematical properties of Penrose tilings were first studied by Penrose, Conway, de Bruijn, and others. After the discovery of quasicrystals in 1984, aperiodic tilings became the focus of intense research. We present only the basic ideas of this rapidly developing subject here.

FIGURE 3.4.1

Patches of Penrose tilings of the plane. On the left the tiling by kites and darts, but note that these two unmarked tiles can tile the plane periodically. This is fixed on the right, where the matching rules that ensure nonperiodicity are enforced by the shapes of the edges. As a result the two shapes from the right-hand tiling constitute an aperiodic prototile set.




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## GLOSSARY

**Nonperiodic tiling:** A tiling with no translation symmetry.

**Aperiodic tiling:** A nonperiodic tiling that does not contain arbitrarily large periodic patches.

**Aperiodic prototile set:** A prototile set that admits only aperiodic tilings; see Figure 3.4.1.

**Relatively dense configuration:** A configuration  $C$  of tiles in a tiling for which there exists a radius  $r_C$  such that every ball of radius  $r_C$  in the tiling contains a copy of  $C$ .

**Repetitive:** A tiling in which every bounded configuration of tiles is relatively dense in the tiling.

**Local isomorphism class:** A family of tilings such that every bounded configuration of tiles that appears in any of them appears in all of the others. (For example, the uncountably many Penrose tilings with the same prototile set form a single local isomorphism class.)

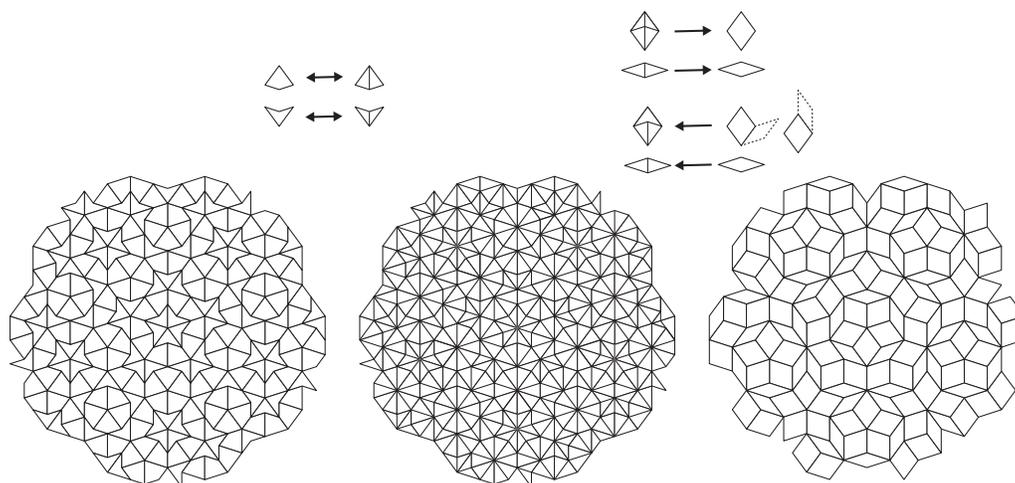
**Matching rules:** A list of rules for fitting together the prototiles of a given prototile set.

**Mutually locally derivable tilings:** Two tilings are mutually locally derivable if the tiles in either tiling can, through a process of decomposition into smaller tiles, or regrouping with adjacent tiles, or a combination of both processes, form the tiles of the other (see Figure 3.4.2).

The first set of aperiodic prototiles numbered 20,426. After Penrose found his set of two, the question naturally arose, does there exist an “einstein,” that is, a

FIGURE 3.4.2

The Penrose tilings by kites and darts and by rhombs are mutually locally derivable.



single tile that tiles only nonperiodically? Interesting progress was made by Socolar and Taylor [ST11], who found an einstein with several connected components. Rao's announcement of the classification of convex monohedral tiles (Section 3.2) implies that einsteins cannot be convex.

The next two sections discuss the best-known families of nonperiodic tilings.

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### 3.4.1 SUBSTITUTION TILINGS

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#### GLOSSARY

**Substitution tiling:** A tiling whose tiles can be composed into larger tiles, called *level-one* tiles, whose level-one tiles can be composed into level-two tiles, and so on *ad infinitum*. In some cases it is necessary to partition the original tiles before composition.

**Self-similar tiling:** A substitution tiling for which the larger tiles are copies of the prototiles (all enlarged by a constant expansion factor  $\lambda$ ). *k*-rep tiles are the special case when there is just one prototile (Figure 3.1.1).

**Unique composition property** A substitution tiling has the unique composition property if *j*-level tiles can be composed into (*j*+1)-level tiles in only one way ( $j = 0, 1, \dots$ ).

**Inflation rule (for a substitution tiling):** The equations  $T'_i = m_{i1}T_1 \cup \dots \cup m_{ik}T_k$ ,  $i = 1, \dots, k$ , that describe the numbers  $m_{ij}$  of each prototile  $T_j$  in the next higher level prototile  $T'_i$ . These equations define a linear map whose matrix has *i, j* entry  $m_{ij}$ .

**Pisot number:** A Pisot number is a real algebraic integer greater than 1 such that all its Galois conjugates are less than 1 in absolute value. A Pisot number is called a unit if its inverse is also an algebraic integer.

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## MAIN RESULTS

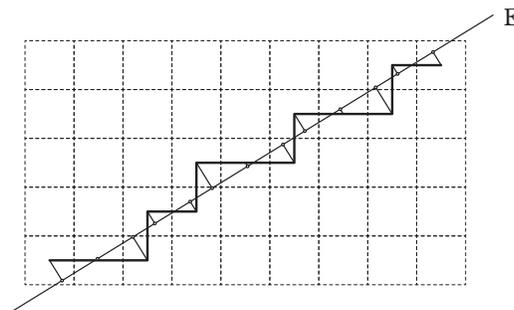
1. Tilings with the unique composition property are nonperiodic (the proof given in [GS87] for  $n = 2$  extends immediately to all  $n$ ). Conversely, nonperiodic self-similar tilings have the unique composition property [Sol98].
  2. Mutual local derivability is an equivalence relation on the set of all tilings. The existence or nonexistence of hierarchical structure and matching rules is a class property [KSB93].
  3. The prototile set of every substitution tiling can be equipped with matching rules that force the hierarchical structure [GS98].
- 

### 3.4.2 PROJECTED TILINGS

The essence of the “cut-and-project” method for constructing tilings is visible in the simplest example. Let  $L$  be the integer lattice in  $\mathbb{E}^2$  and  $\mathcal{T}$  the tiling of  $\mathbb{E}^2$  by squares whose vertices are the points of  $L$ . Let  $\mathcal{V}$  be the dual Voronoi tiling—in this example, the Voronoi tiles are squares centered at the points of  $L$ . Let  $E$  be any line in  $\mathbb{E}^2$  of slope  $\alpha$ , where  $0 < \alpha < \pi/2$ , and let  $F$  be a “face” of  $\mathcal{V}$  of dimension  $k$ ,  $k \in \{0, 1, 2\}$ . If  $E \cap F \neq \emptyset$ , project the corresponding  $2 - k$  face of  $\mathcal{T}$ . (Thus, if  $E$  cuts a face of  $\mathcal{V}$  of dimension 2, project the point  $x \in L$  at its center onto  $E$ ; if  $E$  cuts the edge of  $\mathcal{V}$  joining two faces with centers  $x, y$ , project the edge of  $\mathcal{T}$  with endpoints  $x$  and  $y$ .) This gives a tiling of  $E$  by line segments which are projections of a “staircase” whose runs and rises are consecutive edges of  $\mathcal{T}$ . The tiling is nonperiodic if and only if  $\alpha$  is irrational. Note that the construction does not guarantee that the tiling possesses substitution or matching rules, though for certain choices of  $L$  and  $\alpha$  it may.

FIGURE 3.4.3

*A lattice  $L$  with its Voronoi tiling. We connect successive lattice points whose cells are cut by the line  $E$  to create a “staircase.” Projecting the staircase onto the line, we get a tiling by line segments of two lengths (the projections of the horizontal and vertical line segments, respectively).*



## GLOSSARY

**Canonical projection method for tilings:** Let  $L$  be a lattice in  $\mathbb{E}^n$ ,  $\mathcal{V}$  its Voronoi tiling, and  $\mathcal{T}$  the dual Delaunay tiling. Let  $E$  be a translate of a subset of  $\mathbb{E}^n$  of dimensions  $m < n$ , and let  $F$  be a “face” of  $\mathcal{V}$  of dimension  $k$ ,  $k \in \{0, \dots, n\}$ . If  $E \cap F \neq \emptyset$ , project the corresponding  $n - k$  face of  $\mathcal{T}$  onto  $E$ . Thus, if  $E$  cuts a face of  $\mathcal{V}$ , project the point  $x \in L$  at its center onto  $E$ , and so forth.

**Cut-and-project method for tilings:** A more general projection method of which the canonical is a special case (see below).

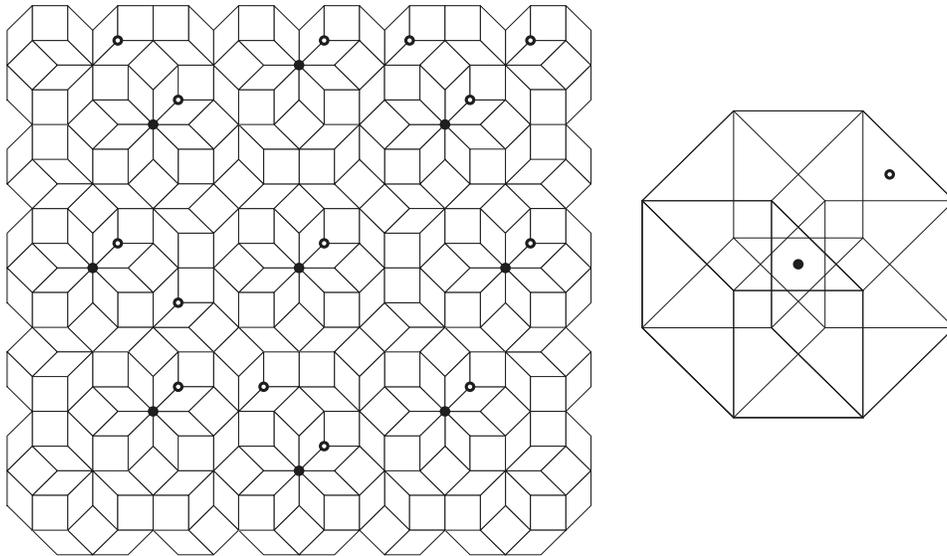
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**MAIN RESULTS**

- A canonically projected tiling is nonperiodic if and only if  $|E \cap L| \leq 1$ .
- The canonical projection method is equivalent to the following: Let  $L$  be a lattice in  $\mathbb{E}^n$  with Voronoi tiling  $\mathcal{V}$ , and let  $\mathcal{D}$  be the dual Delaunay tiling. Let  $E$  be a translate of a  $k$ -dimensional subspace and  $E^\perp$  its orthogonal complement, and  $\mathcal{V}^\perp$  be the projection of  $\mathcal{V}$  onto  $E^\perp$ . The elements of  $\mathcal{V}$  that are projected onto  $E$  are those for which the dual element projects into  $\mathcal{V}^\perp$ ,  $\mathcal{V}^\perp$  is the *window* of the projection.
- The canonical projection method is a special case of the more general cut-and-project method [BG13], in which the window and possibly  $E$  are modified in any of several possible ways (e.g., the window may be larger, smaller, discontinuous, fractal).
- In [Har04], Harriss determined which canonically projected tilings admit a substitution rule, and gives a method for constructing any substitution rule that generates the tiling.

FIGURE 3.4.4

An Ammann-Beenker tiling; the relative frequencies of the two marked vertex stars are the (relative) areas of the marked regions in the octagon.



Some of the best known projected nonperiodic tilings are listed in Table 3.4.1 (see [Sen96]). In all these cases, the lattice  $L$  is the standard integer lattice and the window  $\Omega$  is a projection of a hypercube (the Voronoi cell of  $L$ ). The subspace  $E$  is translated so that it does not intersect any faces of  $\mathcal{V}$  of dimension less than  $n - k$  (thus only a subset of the faces of the Delaunay tiling  $\mathcal{D}$  of dimensions  $0, 1, \dots, k$  will be projected). All of these tilings are substitution tilings.

TABLE 3.4.1 Canonically projected nonperiodic tilings.

TILING FAMILY	$L$	$E$
Fibonacci tiling	$I_2$	line with slope $1/\tau$ ( $\tau = (1 + \sqrt{5})/2$ )
Ammann-Beenker tiling	$I_4$	plane stable under 8-fold rotation
Danzer's Tetrahedra tiling	$D_6$	plane stable under 5-fold rotation
Danzer 3D tiling	$I_6$	3-space stable under icosahedral rotation group

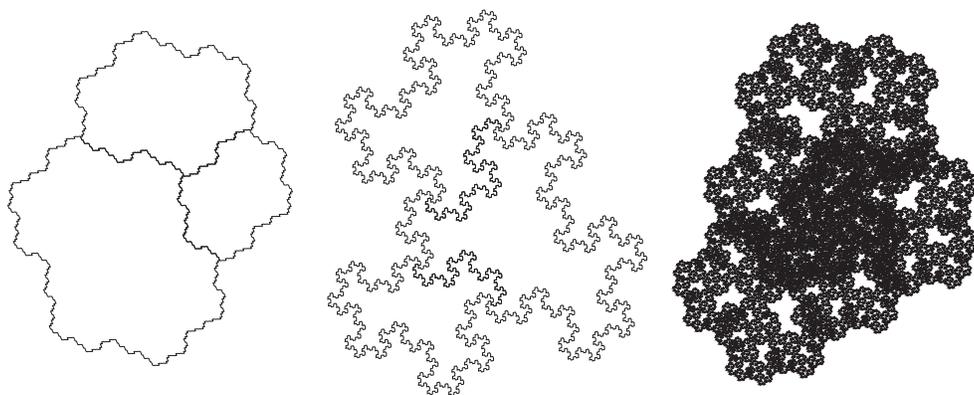
The relative frequencies of the vertex configurations of a canonically projected tiling are determined by the window: they are the ratios of volumes of the intersections of the projected faces of  $V(0)$  [Sen96]).

## OPEN PROBLEMS

The Pisot conjecture states that, for any  $1 - d$  substitution with a unit Pisot scaling factor, there is a window that makes it a projection tiling. This difficult conjecture can be formulated in many ways [AH14, KLS15]. The Rauzy fractals (see Figure 3.4.5) show how complex the windows can get (especially the right-hand window).

FIGURE 3.4.5

*Rauzy fractals, named for their discoverer [Rau82], give the projection windows for three-letter substitution rules, in this case  $(a \rightarrow ab, b \rightarrow ac, c \rightarrow a)$ ,  $(a \rightarrow bc, b \rightarrow c, c \rightarrow a)$ ,  $(a \rightarrow ab, b \rightarrow c, c \rightarrow a)$ . The windows can become very complicated, as seen on the right. This gives a sense of the complexity of the Pisot conjecture, as it states that these shapes will always be the closure of their interior.*



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## 3.5 OTHER TILINGS, OTHER METHODS

There is a vast literature on tilings (or dissections) of bounded regions (such as rectangles and boxes, spheres, polygons, and polytopes). This and much of the recreational literature focuses on tilings whose prototiles are of a particular type, such as rectangles, clusters of  $n$ -cubes (polyominoes—see Chapter 14—and polycubes) or  $n$ -simplices (polyiamonds in  $\mathbb{E}^2$ ), or tilings by recognizable animate figures. In the search for new ways to produce tiles and tilings, both mathematicians (such as P.A. MacMahon [Mac21]) and amateurs (such as M.C. Escher [Sch90]) have contributed to the subject. Recently the search for new shapes that tile a given bounded region  $S$  has produced knotted tiles, toroidal tiles, and twisted tiles. Kuperberg and Adams have shown that for any given knot  $K$ , there is a monohedral tiling of  $\mathbb{E}^3$  (or of hyperbolic 3-space, or of spherical 3-space) whose prototile is a solid torus that is knotted as  $K$ . Also, Adams has shown that, given any polyhedral submanifold  $M$  with one boundary component in  $\mathbb{E}^n$ , a monohedral tiling of  $\mathbb{E}^n$  can be constructed whose prototile has the same topological type as  $M$  [Ada95].

Other directions of research seek to broaden the definition of prototile set: in new contexts, the tiles in a tiling may be homothetic or topological (rather than congruent) images of tiles in a prototile set. A tiling of  $\mathbb{E}^n$  by polytopes in which every tile is combinatorially isomorphic to a fixed convex  $n$ -polytope (the combinatorial prototile) is said to be *monotypic*. It has been shown that in  $\mathbb{E}^2$ , there exist monotypic face-to-face tilings by convex  $n$ -gons for all  $n \geq 3$ ; in  $\mathbb{E}^3$ , every convex 3-polytope is the combinatorial prototile of a monotypic tiling [Sch84a]. Many (but not all) classes of convex 3-polytopes admit monotypic face-to-face tilings [DGS83, Sch84b].

Dynamical systems theory is perhaps the most powerful tool used to study tilings today, but it takes us beyond the scope of this chapter and we do not discuss it here. For more on this and further references see [BG13, KLS15, Sad08].

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## 3.6 SOURCES AND RELATED MATERIALS

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### SURVEYS

The following surveys are useful, in addition to the references below.

[GS87]: The definitive, comprehensive treatise on tilings of  $\mathbb{E}^2$ , state of the art as of the mid-1980's. All subsequent work (in any dimension) has taken this as its starting point for terminology, notation, and basic results. The Main Results of our Section 3.1 can be found here.

[BG13]: A survey of the rapidly growing field of aperiodic order, of which tilings are a major part.

The *Bielefeld Tilings Encyclopedia*, <http://tilings.math.uni-bielefeld.de/> gives details of all substitution tilings found in the literature.

[PF02]: A useful general reference for substitutions.

[Moo97]: The proceedings of the NATO Advanced Study Institute on the Mathematics of Aperiodic Order, held in Waterloo, Canada in August 1995.

[Sch93]: A survey of tiling theory especially useful for its accounts of monotypic and other kinds of tilings more general than those discussed in this chapter.

[Sen96]: Chapters 5–8 form an introduction to the theory of aperiodic tilings.

[SS94]: This book is especially useful for its account of tilings in  $\mathbb{E}^n$  by clusters of cubes.

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## RELATED CHAPTERS

Chapter 14: Polyominoes

Chapter 27: Voronoi diagrams and Delaunay triangulations

Chapter 64: Crystals, periodic and aperiodic

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